

POPULATION MOMENTUM

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ABSTRACT

Population momentum is the ratio of a population's ultimate size after a demographic transition to its initial size before the transition. For an instantaneous drop to replacement fertility, Nathan Keyfitz found a simple expression M_K for the momentum. However, as Keyfitz pointed out, 'no one thinks that any country will drop immediately to stationary reproduction patterns'. We present results concerning the momentum of a population whose fertility (in level and shape) declines to replacement level within a finite time γ . First, we provide an exact analytical formula for such a population's momentum. Second, we apply our formula to rapid fertility transitions, those that are completed in fewer than α years (α is the youngest fertile age, usually 15 years). For such cases, using a general description of fertility change, we present a simple exact expression for population momentum that reduces to Keyfitz's M_K if the transition is instantaneous. Next we present a more general formula for momentum that is exact for fertility transitions that are completed in fewer than 2α years, i.e., about 30 years. We show by example that these two formulae are accurate approximations to population momentum for transitions that take as long as 100 years. Finally, we show that the speed of fertility decline makes a substantial difference to population momentum.

Mortality rates around the world began a notable downward trend early in this century, and in many countries fertility only began to decline decades after a substantial reduction in mortality had occurred. The resulting rise in world population has been the subject of considerable debate. In recent years some countries have experienced historically rapid declines to replacement levels of fertility. In China, Japan and South Korea, such fertility decline was accomplished in two decades or less. Other countries, such as India, have begun a more gradual fertility decline that should achieve replacement fertility in about 40 years. Any population with a history of high fertility has an age distribution with a high proportion of young people. If such a population begins a fertility transition, a high birth rate will persist until the proportion of people in the child-bearing ages begins to reflect the decline in fertility rates. To capture this inertial effect, Keyfitz (1971) introduced the aptly named population momentum, defined as the ratio of a population's ultimate size after a demographic transition to its initial size before the transition. For the special case of an instantaneous drop in fertility to a replacement level, Keyfitz found a simple closed-form expression M_K for the momentum. He showed that for many high-fertility countries the value of M_K was in the range of 1.5 to 1.7. Thus the initial, skewed, age distribution was responsible for the addition of 50 percent or more to the ultimate population size.

In practice, 'no one thinks that any country will drop immediately to stationary reproduction patterns (Keyfitz 1971),' and a moment's reflection makes it clear that a gradual fertility decline must lead to a higher population momentum. How much will the momentum increase? This paper answers this question, showing how the population momentum may be calculated for populations whose fertility declines to replacement level in finite time.

Classical demography describes population change in terms of age-specific fertility and mortality rates applied to the female population. Suppose that a population's fertility starts to change at $t = 0$ and stops changing at $t = \gamma$, upon reaching a level that ultimately results in a stationary population. Over time, the population size will ultimately converge to some number $P(\infty)$. If the population size before the transition began was $P(0)$, the population momentum M is defined to be

$$M = \frac{P(\infty)}{P(0)}. \quad (1)$$

Keyfitz (1971) considered the special case in which this transition is achieved by an instantaneous drop in the level (but not the shape) of age-specific fertility rates. Assuming an initially stable population with growth rate r , Keyfitz obtained the simple expression

$$M_K = \frac{b_0 e_0 (R_0 - 1)}{r \mu_0 R_0}. \quad (2)$$

Here R_0 is the initial net reproductive rate and b_0 the initial crude birth rate; μ_0 is the average age of childbearing (in this case, for both initial and ultimate population); e_0 is the expectation of life at birth (which is the same before and after transition).

Mitra (1976) extended Keyfitz's result to allow for a change in the age-pattern of fertility rates, but retaining the assumption of an instantaneous drop, to obtain

$$M_M = \frac{b_0 e_0}{r \mu_0} (1 - \phi_F[r]), \quad (3)$$

where $\phi_F[r] = \int_{\alpha}^{\beta} dx e^{-rx} \phi_F(x)$ is the Laplace transform of the ultimate maternity function $\phi_F(x)$. Here α is the youngest age (typically about 15 years) of childbearing and β is the oldest age (typically about 45 years) of childbearing.

The assumption of instantaneous change used by Keyfitz and Mitra avoids the difficulties of dealing with a time-dependent population renewal equation. Instant fertility decline makes it possible to use the Lotka equation (e.g., Polard, 1973) with constant fertility and mortality rates. Previous attempts to remove Keyfitz's assumption of instant change have dealt with particular time-patterns of change in vital rates. Cerone (1996) described fertility decline by a sum of exponential functions, and obtained an iterative algorithm that may in principle be used to estimate the momentum. Mitra (1987) studied a different explicit polynomial trajectory for change in fertility rates. As far as we know, no general analysis has been carried out.

In this paper we suppose that vital rates follow some general time-trajectory as they decline to replacement levels. To analyze population change in this case, we need to grapple with a general time-dependent renewal equation. We have developed a solution to such equations (Li and Tuljapurkar 1997). Using this solution, we obtain below a general formula for population momentum which applies to arbitrary changes in fertility and mortality that are

completed in a finite time γ . Next we obtain simpler, but more immediately useful, expressions for the momentum that apply to a rather general model for fertility transitions. For a fertility transition that is completed in a time $\gamma \leq \alpha$, and for which the time and age variables are separable, we express the population momentum as a simple closed form M_1 . We then turn to slower transitions and present an expression for population momentum M_2 that is exact when $\gamma < 2\alpha$. Numerical examples show that M_1 is an accurate approximation to momentum for γ as high as 40 years, while M_2 is an accurate approximation for γ as high as 100 years. We end the paper with a discussion of the implications of our results.

BIRTHS PRODUCED BY TIME-VARYING VITAL RATES

Consider a population with time-varying fertility and mortality rates. We represent these rates in terms of a maternity function $\phi(x, t)$ of age x and time t , such that $\phi(x, t)$ is the rate of births to an individual of age x at time t . Births into the population are described by a time-dependent version of Lotka's population model,

$$B(t) = \int_{\alpha}^t dx \phi(x, t)B(t-x) + g_0(t), \quad (4)$$

where $B(t)$ is the time-rate of births for time $t > 0$, $\phi(x, t)$ is the maternity function, and $g_0(t)$ is the rate of births produced by survivors of the initial population present at $t = 0$. Note that $g_0(t) = 0$ for $t > \beta$, because all individuals present at time $t = 0$ will have completed reproduction by then.

We assume that the vital rates undergo a transition to ultimate values in a finite time γ , so that

$$\phi(x, t) = \phi_F(x), \quad \text{for } t \geq \gamma. \quad (5)$$

Given equation (5), Li and Tuljapurkar (1997) have shown that equation (4) may be solved by an iterative procedure that terminates in a finite number of steps. Here we summarize the relevant results (readers may verify these by iterating equation (4)). To begin, we define for integers $n \geq 1$ the functions

$$g_n(t) = \int_{\alpha}^t dx \phi(x, t)g_{n-1}(t-x), \quad n \geq 1. \quad (6)$$

Next define the integer

$$N = \text{int}(\gamma/\alpha), \quad (7)$$

where the notation $\text{int}(x)$ denotes the largest integer that is less than or equal to the number x . For instance, if $\gamma < \alpha$ we have $N = 0$, while if $\gamma < 2\alpha$ we have $N = 1$. For a given γ and the corresponding N , define the function

$$\tilde{g}(t) = g_0(t) + \int_{\alpha}^t dx [\phi(x, t) - \phi_F(x)] \sum_{n=0}^N g_n(t-x). \quad (8)$$

Note that the sum on the right side of the above equation will terminate at the finite N determined by equation (7).

Using the function $\tilde{g}(t)$, equation (4) may be rewritten as an equation in which the renewal term (i.e., the first term on the right side of (4)) involves only the final maternity function ϕ_F ,

$$B(t) = \int_{\alpha}^t dx \phi_F(x) B(t-x) + \tilde{g}(t), \quad (9)$$

This rewriting allows us to derive the population momentum.

GENERAL FORMULA FOR POPULATION MOMENTUM

We suppose that after a transition (i.e., for times $t > \gamma$) the vital rates are such that they will lead to a stationary population. Equation (9) is in the form of a standard renewal equation, and the function $\tilde{g}(t)$ is zero for $t > \max(\gamma, \beta)$. Using well-known results about such equations described in Feller (1966, Chapter XI), we conclude that the birth rate $B(t)$ is asymptotically given by

$$B(\infty) = \lim_{t \rightarrow \infty} B(t) = \frac{1}{\mu_0} \int_0^{\delta} dt \tilde{g}(t), \quad \text{where } \delta = \max\{\beta, \gamma\}. \quad (10)$$

Here μ_0 is the average age at childbearing determined by the final maternity function,

$$\mu_0 = \int_{\alpha}^{\beta} dx x \phi_F(x). \quad (11)$$

The population momentum M for any change in vital rates is obtained by employing equation (10) in equation (1). The initial population has an age-distribution of the form $C u_0(x)$ for some function u_0 and some constant C ; this constant is just a scaling factor that will cancel out of the ratio $P(\infty)/P(0)$. We ignore any constant multiplier. Let $b_0 = (1/\int dx u_0(x))$ be the initial crude birth rate, note that $P(\infty) = e_0 B(\infty)$ where e_0 is the expectation of life at birth after the transition is complete, and use equation (1) to find

$$M = \frac{e_0 b_0}{\mu_0} \int_0^\delta dt \tilde{g}(t). \quad (12)$$

Equation (12) is perfectly general, but as written affords little understanding and insight. These call for a more explicit and comprehensible result.

EXPLICIT FORMULA FOR RAPID TRANSITIONS

Keyfitz's M_K is simple to use and interpret, because it is known explicitly in terms of a few summary properties of the vital rates. We now obtain a similarly explicit result for the situation in which fertility completes a transition to replacement level in finite time (mortality is assumed not to change here). We make three key assumptions. The first of these, that the initial population (at the start of the transition) is stable, is the least restrictive of Keyfitz's assumptions, and it does lead to the simplest formulae.

The second assumption is that the transition in vital rates may be described in separable form, so that the maternity function can be written as

$$\phi(x, t) = \phi_F(x) + f(t)\phi_1(x). \quad (13)$$

Since mortality is not changing here, we can also divide every term in equation (13) by the survivorship $\ell(x)$ to obtain an equation in terms of fertility rates. The time-factor f in equation (13) is a smooth function that starts at the value unity and reaches the value zero at the end of the transition,

$$f(0) = 1, \quad f(\gamma) = 0. \quad (14)$$

The initial maternity function is $\phi_0(x) = \phi(x, 0) = \phi_F(x) + \phi_1(x)$, while the final maternity function is simply $\phi_F(x)$. Thus our assumptions allow for

quite general shifts in the age-pattern and level of fertility, which may be estimated from a time-series of age-specific fertility rates using a statistical procedure such as a singular-value decomposition. An example of such a decomposition for fertility change is given by Lee (1993). Our result does not require that $f(t)$ be a decreasing function of time for the entire period $0 \leq t \leq \gamma$. For example, we can compute momentum in cases where $f(t)$ has some periods of increase, so long as it reaches zero by the end of the transition.

The third, most restrictive, assumption is that the transition in vital rates is completed in a time γ less than α . Since α is typically about 15 years in modern human populations, our assumption covers transitions that are completed in less than 15 years. It is a happy fact that the individual drop from high fertility to replacement levels has, in many cases, occurred this rapidly. Examples of such rapid fertility declines are found among Southeast Asian countries after the Second World War, including Japan, South Korea, and China.

To calculate momentum under these assumptions we employ equation (8). The assumption $\gamma < \alpha$ implies, in equation (7), that the quantity $N = 0$; in equation (8) the term involving the sum over N drops out, with the result that $\tilde{g}(t) = g_0(t)$.

Let $u_0(x)$ be the age-distribution of individuals in the initial population, and let $\ell(x)$ be the survival rate from birth to age x at $t = 0$. As noted in the remarks preceding equation (12), we may ignore any constant factors that multiply $u_0(x)$. Then, with $\gamma < \alpha$, equation (8) yields

$$\begin{aligned} \int_0^\beta dt \tilde{g}(t) &= \int_0^\beta dt g_0(t), \\ &= \int_0^\beta dt \int_0^\beta dx \frac{u_0(x)\phi(x+t,t)}{\ell(x)}. \end{aligned} \quad (15)$$

For an initially stable population, $u_0(x) = \ell(x)e^{-rx}$ where the initial population growth rate is the positive solution of

$$\int_0^\beta \phi(x,0)e^{-rx} dx = 1. \quad (16)$$

Inserting the stable u_0 and the separable maternity function of equation (13)

into the last line of equation (15), and doing some algebra, we obtain

$$\int_0^\beta dt \tilde{g}(t) = -\frac{1 - \phi_F[r]}{r} \int_0^\gamma f'(t) e^{rt} dt, \quad (17)$$

where

$$\phi_F[r] = \int_0^\beta \phi_F(x) e^{-rx} dx, \quad (18)$$

is the Laplace transform of the final maternity function ϕ_F .

Putting equation (17) into equation (12), and recalling Mitra's equation (3), we obtain our formula for population momentum,

$$M_1 = -M_M \int_0^\gamma \left(\frac{df}{dt}\right) e^{rt} dt. \quad (19)$$

The negative sign on the right of equation (19) should not alarm readers: recall that over the transition, $f(t)$ must decrease from its initial value of one to a final value of zero.

Equation (19) applies to perfectly general trajectories $f(t)$ of fertility decline, for example $f(t)$ may be taken to decline linearly with time or as a logistic. As stated earlier, the decline need not be monotonic. It is straightforward to show that

$$\frac{dM_1}{d\gamma} = -M_M \left(\frac{df}{dt}\right)_{t=\gamma} e^{r\gamma} > 0. \quad (20)$$

Thus momentum increases when the fertility transition takes longer, in perfect accord with intuition.

Although M_1 is the exact momentum when $\gamma < \alpha$, equation (19) may also be used as a possible approximation to momentum for any value of γ . To do so, simply compute the value of M_1 for any γ from equation (19). As shown later, values of M_1 computed in this way accurately approximate the momentum for transitions that take as long as 40 years.

The Mitra and Keyfitz results are recovered easily from equation (19). If fertility drops instantaneously to its final value, equation (19) reduces to Mitra's equation (3). If there is no change in the shape of fertility, but only an instantaneous drop in the net reproductive rate (NRR) from R_0 to unity, we have $\phi_F[r] = 1 - (1/R_0)$ and equation (19) reduces to the classic Keyfitz result equation (2).

MOMENTUM FOR SLOWER TRANSITIONS

The population momentum is exactly M_1 if the fertility transition is completed in time γ less than α , or about 15 years. For longer (slower) transitions, we may extend our analysis in steps of length α . Thus, suppose that the fertility transition is completed in time γ with $\gamma < 2\alpha$, or about 30 years. Retaining our assumptions of an initially stable population and the separation of time and age in equation (13), we apply our analytical method and find that the exact population momentum in this case is

$$M_2 = M_1 - M_M \int_{\alpha}^{\gamma} dt r e^{rt} f(t) \int_{\alpha}^t dx [1 - f(t-x)] \phi_1(x) e^{-rx}. \quad (21)$$

The two terms here have a simple demographic interpretation. The term M_1 describes the effect of fertility decline on births produced by the zeroth generation – individuals present at the start of the transition. The second term of M_2 describes how births produced by the first generation – individuals born after the transition begins – are in their turn affected by fertility decline. With each successive generation, fertility rates approach ever closer to their ultimate stationary level, so that successive generations make ever-smaller contributions to the population momentum. In accord with this interpretation, it is possible to prove that the magnitude of the second term in equation (21) is always smaller than the first.

Just as with equation (19) for M_1 , we may compute the value of M_2 in equation (21) for any value of γ . Numerical examples, presented in the next section, show that M_2 is a very accurate approximation to the true momentum even if γ is much larger than 2α .

Our iterative analysis may be carried onward to compute the exact momentum for longer transitions; the next step would be to consider $\gamma < 3\alpha$, for transitions that are complete in about 45 years. The generational argument given above shows that, whenever $f(t)$ is a monotonically decreasing function, higher order terms in the momentum will be progressively less important. Given that M_1 and M_2 are exact for transitions taking as long as 30 years, and are (as shown below) accurate approximations for much slower transitions, we stop with M_2 .

ACCURACY OF FORMULAE

We constructed numerical examples of a fertility transition using the Coale-Demeny model life tables for Western female stable populations (Coale and

Demeny, 1983). From this family of models we selected an initial maternity function $\phi_0(x) = \phi(x, 0)$ corresponding to an expectation of life at birth of $e_0 = 65$, and a NRR of 3. The initial population was constructed as a stable population with these characteristics.

For the transition in vital rates, we assumed that mortality rates were constant in time, but fertility declined to a final level at which the net reproductive rate is unity. To allow the age-pattern of fertility to change over the transition, we described the changing component of the maternity function in equation (13) by

$$\phi_1(x) = A \phi_0(x) e^{-Kx}. \quad (22)$$

In one set of calculations we set $K = 0$, corresponding to a fertility decline in which the age-pattern of fertility was always the same as in the initial $\phi_0(x)$. In a second set of calculations, to represent the typical situation in which fertility declines are larger at later ages, we set $K = 0.1$. Given that $\phi_0 = \phi_F + \phi_1$, that the initial NRR is 3 and the final NRR is 1, and a value of the shape parameter K , we determined the constant A in equation (22). Figure 1 displays the initial and final maternity functions for the cases $K = 0$ and $K = 0.1$. Notice that in the latter case there is a substantial change in the shape of the function as the transition proceeds from ϕ_0 to ϕ_F . The mean and variance of the age at reproduction as determined by the initial maternity function ϕ_0 are about 27 years and 44, respectively. When $K = 0$, these moments are the same for ϕ_F as for ϕ_0 . On the other hand, when $K = 0.1$, the mean and variance of ϕ_F are substantially smaller at about 23 years and 30, respectively.

The trajectory of changing is described by the function $f(t)$. In one set of calculations we took $f(t)$ to decline linearly,

$$f(t) = 1 - \left(\frac{t}{\gamma}\right).$$

In a second set of calculations we took $f(t)$ to be a logistic function

$$f(t) = 1.03 - \frac{1.5}{(1.4 + 48.6e^{-St})};$$

here the value of S is chosen to yield $f(\gamma) = 0$ for any desired γ . Figure 2 shows the differences between the linear and logistic declines for transitions that take $\gamma = 25, 50, 75$ and 100 years.

Exact values of the population momentum were computed using a direct numerical iteration of the renewal equation. Figure 3 displays these numerical values, and compares four cases: linear decline with no change (i.e., $K = 0$) in the age-pattern of fertility (labeled L1); linear decline with changing age-pattern (i.e., $K = 0.1$) of fertility (labeled L2); logistic decline with no change (i.e., $K = 0$) in the age-pattern of fertility (labeled L3); logistic decline with changing age-pattern (i.e., $K = 0.1$) of fertility (labeled L4). For a given γ , the logistic function initially decreases slower than the linear function and eventually declines faster than the linear. Hence we expect, and Figure 3 confirms, that the difference in momentum between the two functions should be small. Keyfitz (1971) pointed out that changes in the level of fertility are more important than changes in the shape of fertility in affecting the number of births and hence the momentum. In agreement with his assessment, we see in Figure 3 that there are only small differences in momentum, for a given γ , with and without change in the age-structure of fertility. The main lesson of Figure 3 is that the most important factor affecting momentum is the speed of fertility level decline, measured by γ ; we see that momentum increases significantly with γ in all cases.

How well do our analytical expressions perform? We show the values of M_1 from (19), M_2 from (23) and the numerically computed M at different γ for our four cases: constant fertility structure and linear decline in Figure 4; changing fertility structure and linear decline in Figure 5; constant fertility structure and logistic decline in Figure 6; and changing fertility structure with logistic level decline in Figure 7. We know that M_1 is exact for γ less than 15 years, while M_2 is exact for γ less than 30 years. But Figures 4 through 7 show that M_1 is an accurate approximation for γ as large as 40 years, while M_2 is an accurate approximation to momentum even for values of γ as large as 100 years.

DISCUSSION

Fertility transitions in this century have ranged from rapid ones, such as China's, in which replacement fertility was achieved in 15 years or less, to relatively slow ones, such as India's, with a time to completion of perhaps 40 years. For transitions in this range, our simple formula M_1 is quite adequate. Furthermore, differences between linear and logistic declines are small, as noted in the preceding section, so that a sound qualitative picture of momentum can be built up by examining M_1 for a linear decline in fertility.

For a linear decline completed in γ years, the slope of $f(t)$ is just $-(1/\gamma)$, and equation (19) yields

$$M_1 = M_M \left(\frac{e^{r\gamma} - 1}{r\gamma} \right).$$

If the transition time and the initial growth rate satisfy $r\gamma < 1$, we can approximate this by

$$M_1 \simeq M_M \left(1 + \frac{r\gamma}{2} + \frac{r^2\gamma^2}{6} \right).$$

Clearly the momentum M_1 increases approximately linearly with the transition period when γ is small, and then increases faster than linearly.

The actual momentum for a transition that takes a finite time γ can be much larger than Keyfitz's momentum. For the initial fertility schedule that we used in the preceding section, starting with $R_0 = 3$ and $\mu_0 = 27$ years, we have $r = 0.042$. Using our equation for a linear fertility decline, the actual momentum M_1 will be 40 percent higher than the Keyfitz momentum if the transition takes 15 years, 57 percent higher if the transition takes 20 years, and nearly double the Keyfitz momentum if the transition takes 30 years. In general, M_1 will be nearly double M_M for transitions that take about $\gamma_d = (1.3/r)$ years. These differences are sizeable if we consider that they multiply rather large initial populations.

Another perspective on the effect of the speed of fertility decline may be obtained by comparing two cases. In the first case, suppose that fertility decline is instantaneous but is delayed to the time $t = \gamma$ instead of occurring at time $t = 0$. If the decline had occurred at $t = 0$ the momentum would be M_K , but the delay causes the population momentum to increase to the value $M_K e^{r\gamma}$. In the second case, suppose that fertility declines linearly starting at $t = 0$ and ending at replacement level at time $t = \gamma$. It is clear that the momentum M_1 in the second case will be less than the momentum in the first case; but by how much? The answer is that the momentum in the second case can exceed half the momentum in the first case, when transitions are rapid. To illustrate, with $r = 0.042$, and $\gamma = \alpha = 15$ years, momentum with a linear decline is 74 percent of the momentum with a 15-year delay; for $\gamma = 30$ years the ratio drops to 57 percent.

What questions about momentum remain to be answered? Preston (1986) and Kim and Schoen (1991) have examined the aging of a population's age-structure in response to a sudden drop in fertility level; our results can be used

to see how their results change when the transition takes a finite time. Our general result equation (12) covers changes in fertility or mortality or both, but the explicit expressions M_1 and M_2 describe only fertility transitions. Given the recent and remarkable declines in human mortality rates, it would be apropos to study mortality transitions, and our general equation (12) is the right point of departure.

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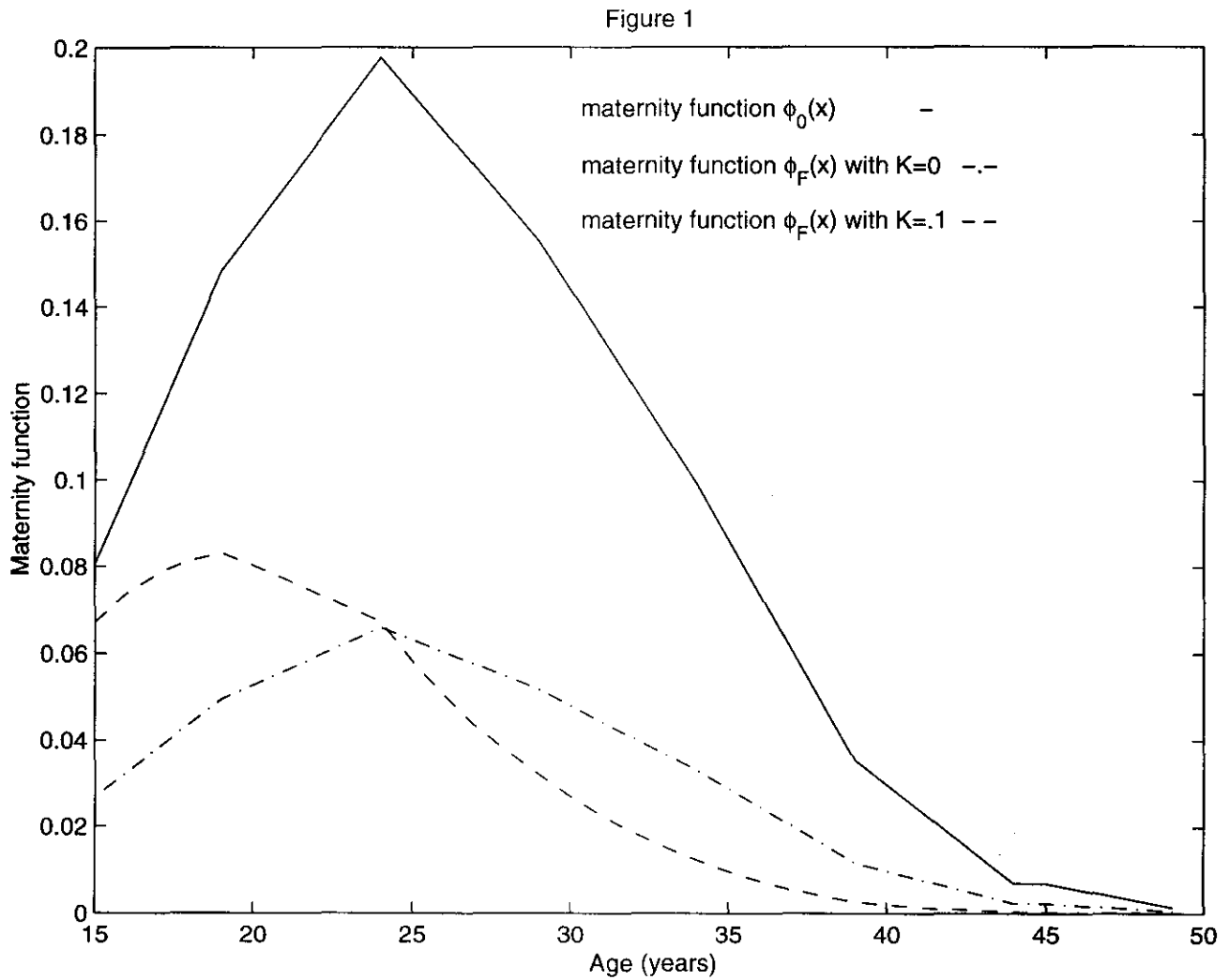


Figure 1. Initial and final maternity functions $\phi_0(x)$ and $\phi_F(x)$. With a constant age-pattern of fertility ($K = 0$), $\phi_F(x)$ has the same mean and variance of age at reproduction as $\phi(x, 0)$, about 27 years and 44. Typically, fertility declines faster at later ages; we describe such a case by setting $K = 0.1$ corresponding to a $\phi_F(x)$ with the mean of 23 years and a variance of 30. In both cases the NRR declines from 3 to the replacement value 1.

Figure 2

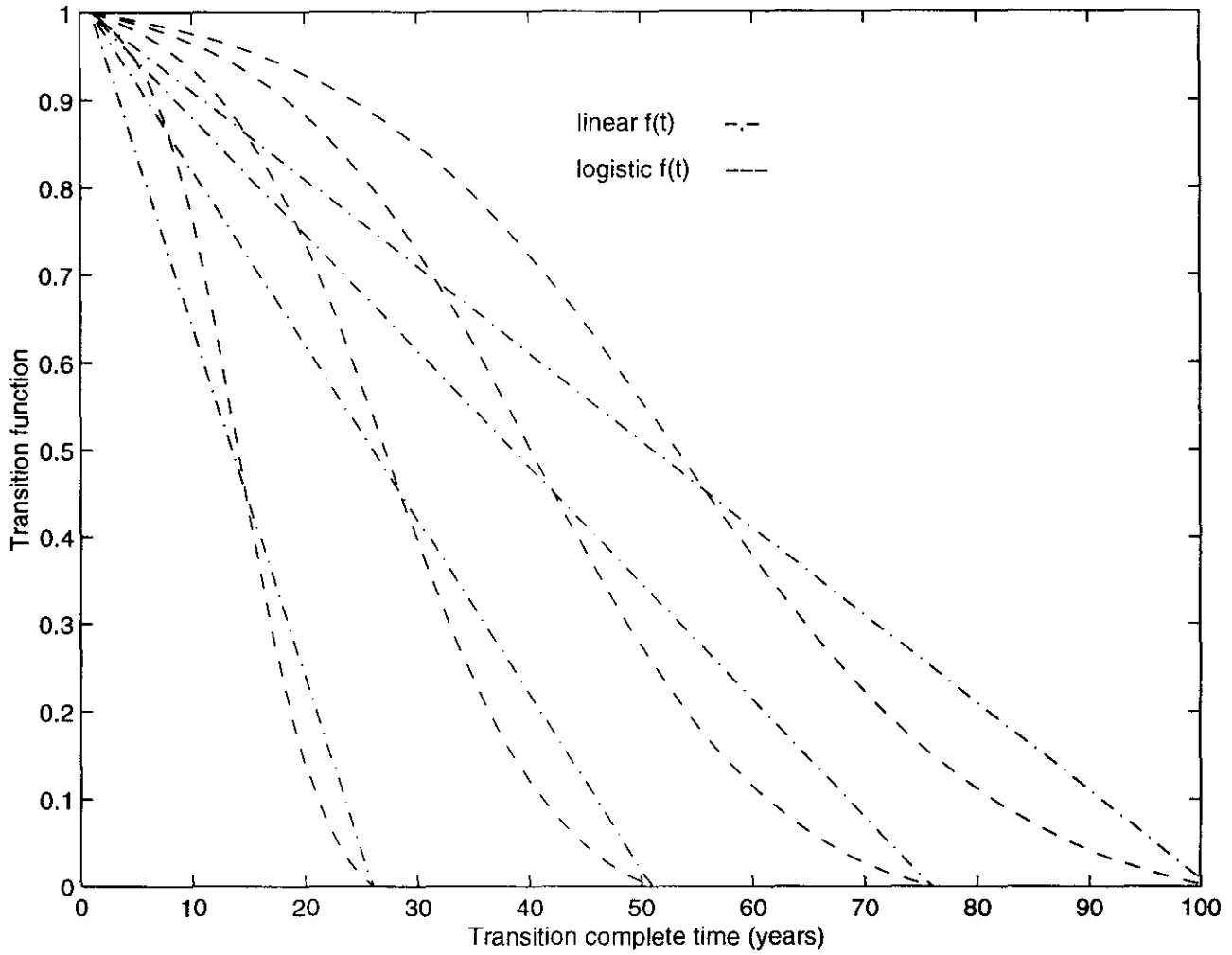


Figure 2. The linear and logistic transition functions $f(t)$ stop changing at the same time γ . For a given γ , the logistic function initially decreases slower than the linear function and eventually declines faster than the linear.

Figure 3

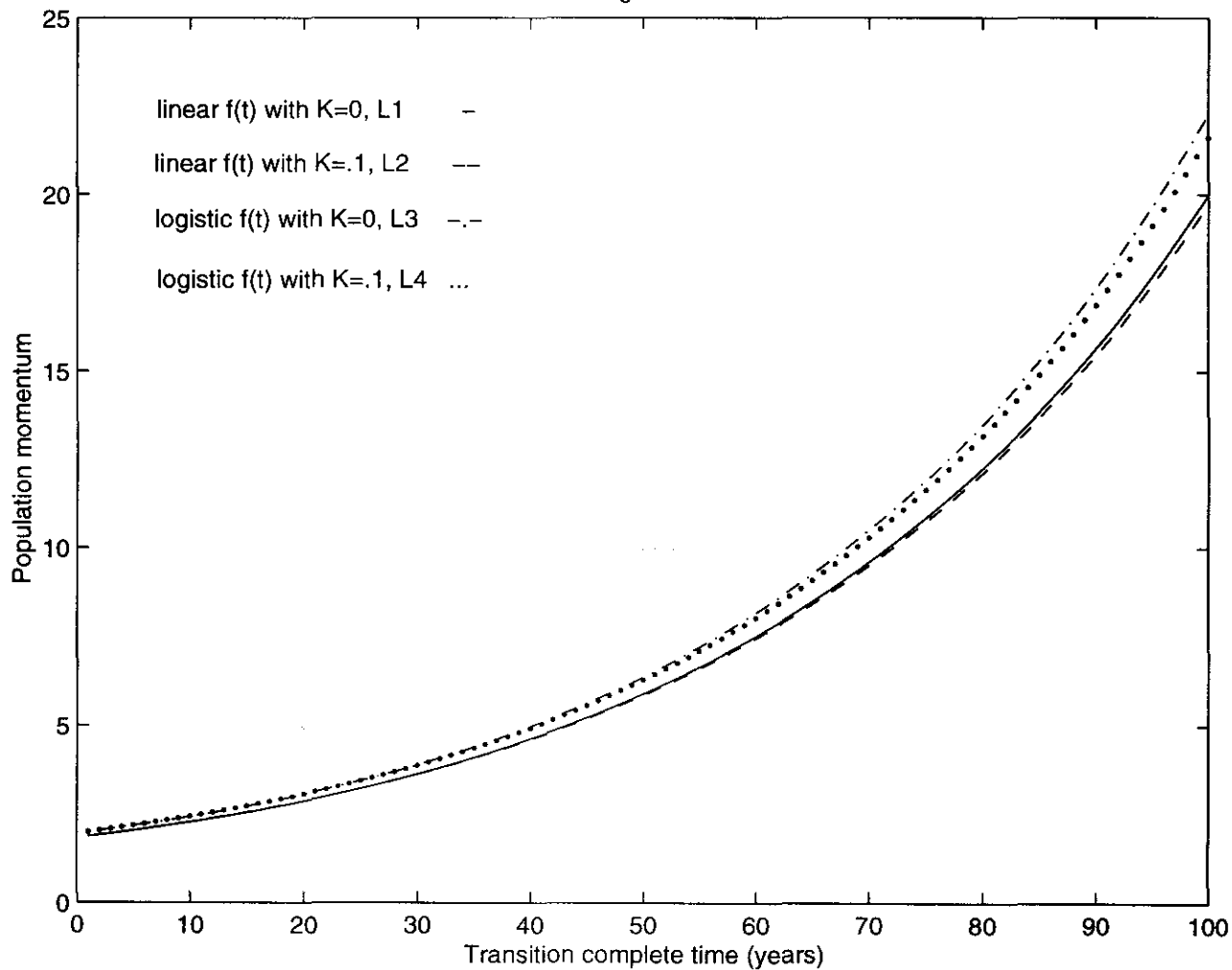


Figure 3. The values of population momentum computed using direct numerical iteration. For any γ , the differences between four cases -- linear and logistic decline with and without a change in the age-pattern of fertility -- are small. Note that in all cases momentum increases significantly with γ .

Figure 4

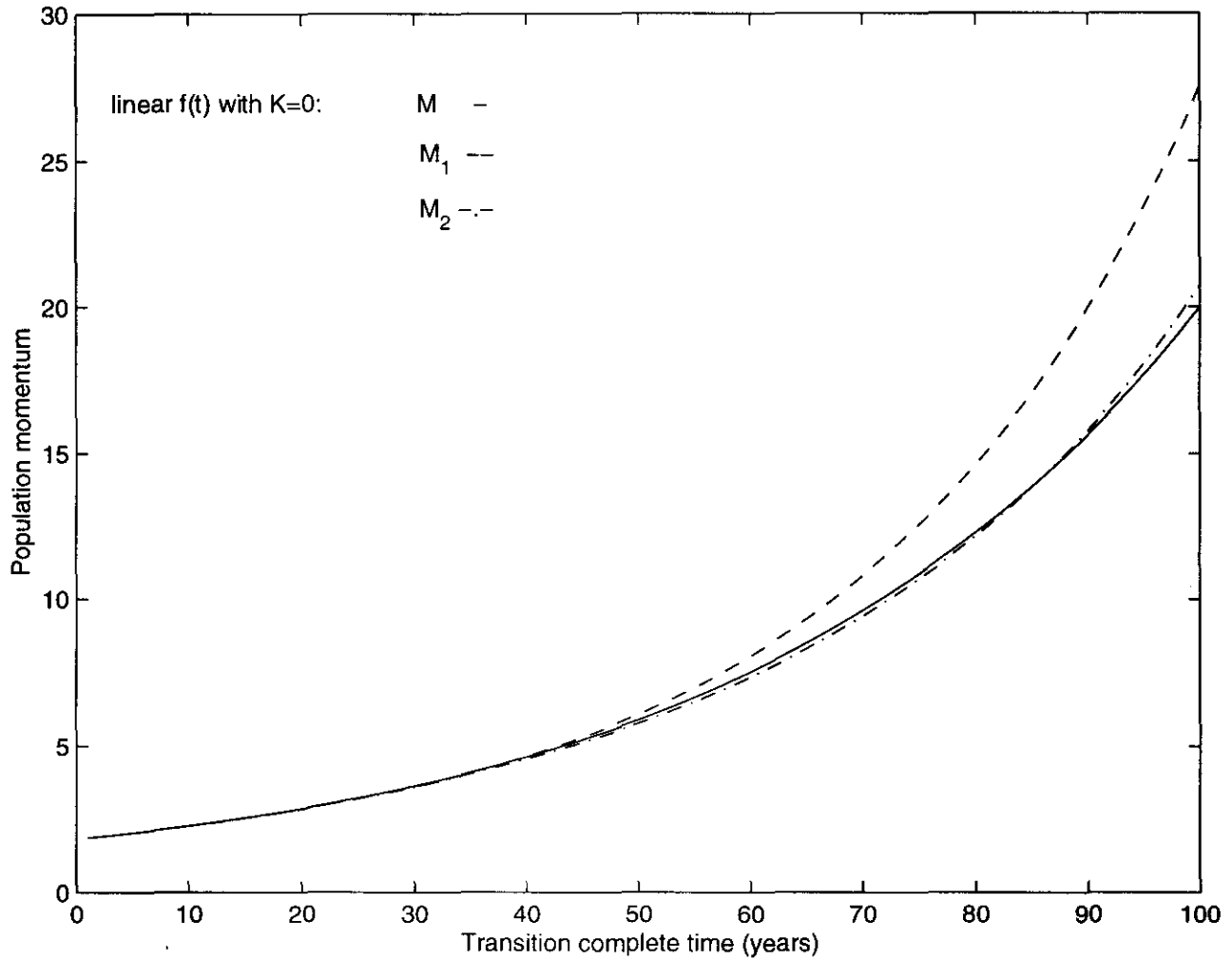


Figure 4. Values of momentum M obtained by numerical computation, M_1 from equation (19), and M_2 from equation (23), for a linear decline with constant age-pattern of fertility. M_1 is an accurate approximation for γ as large as 40 years, while M_2 is an accurate approximation to momentum even for values of γ as large as 100 years.

Figure 5

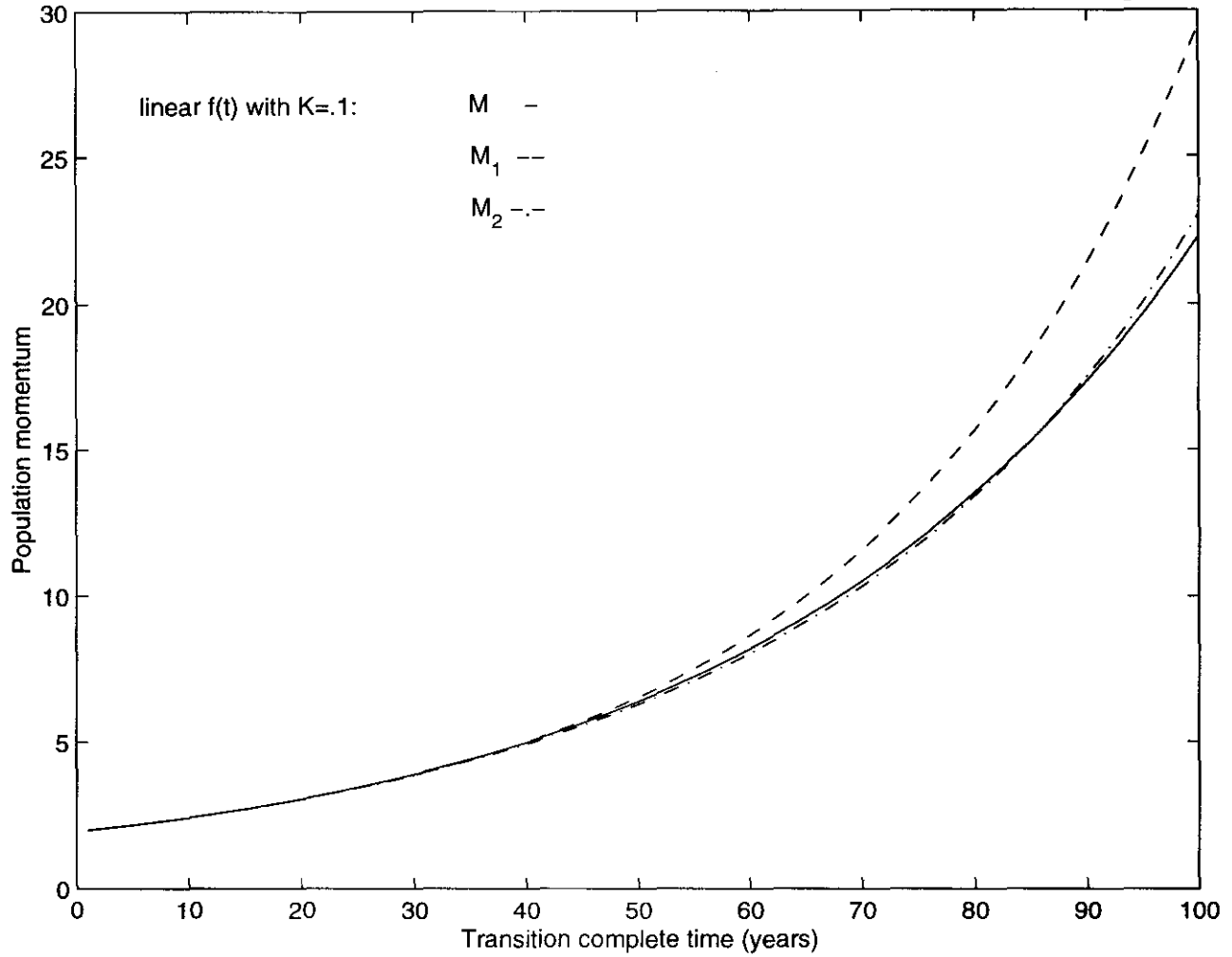


Figure 5. For a linear decline with changing age-pattern of fertility, variables as in Figure 4. Again, M_1 is an accurate approximation for γ as large as 40 years, while M_2 is an accurate approximation to momentum even for values of γ as large as 100 years.

Figure 6

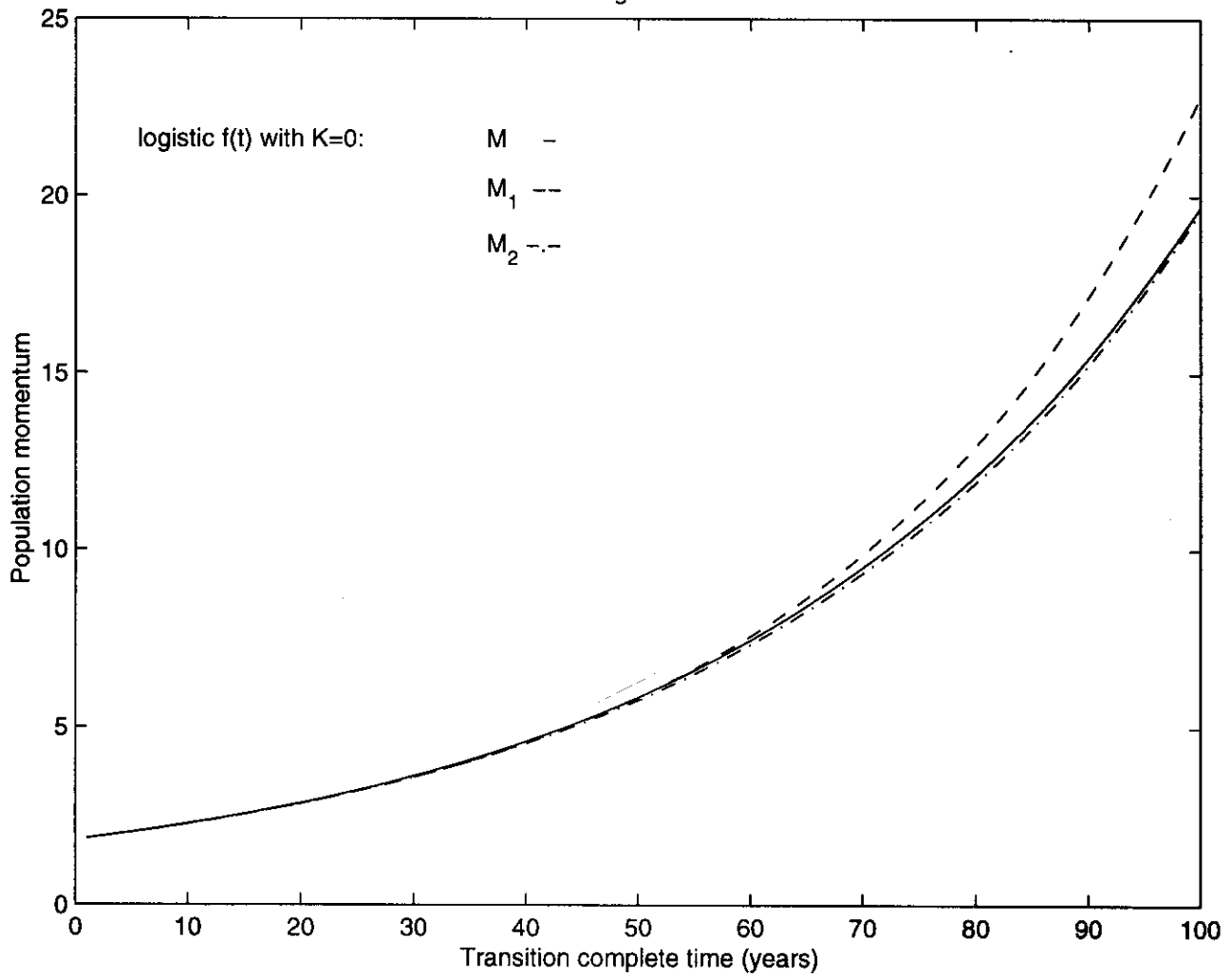


Figure 6. For a logistic decline with constant age-pattern of fertility, variables as in Figure 4. Again, M_1 is an accurate approximation for γ as large as 40 years, while M_2 is an accurate approximation to momentum even for values of γ as large as 100 years.

Figure 7

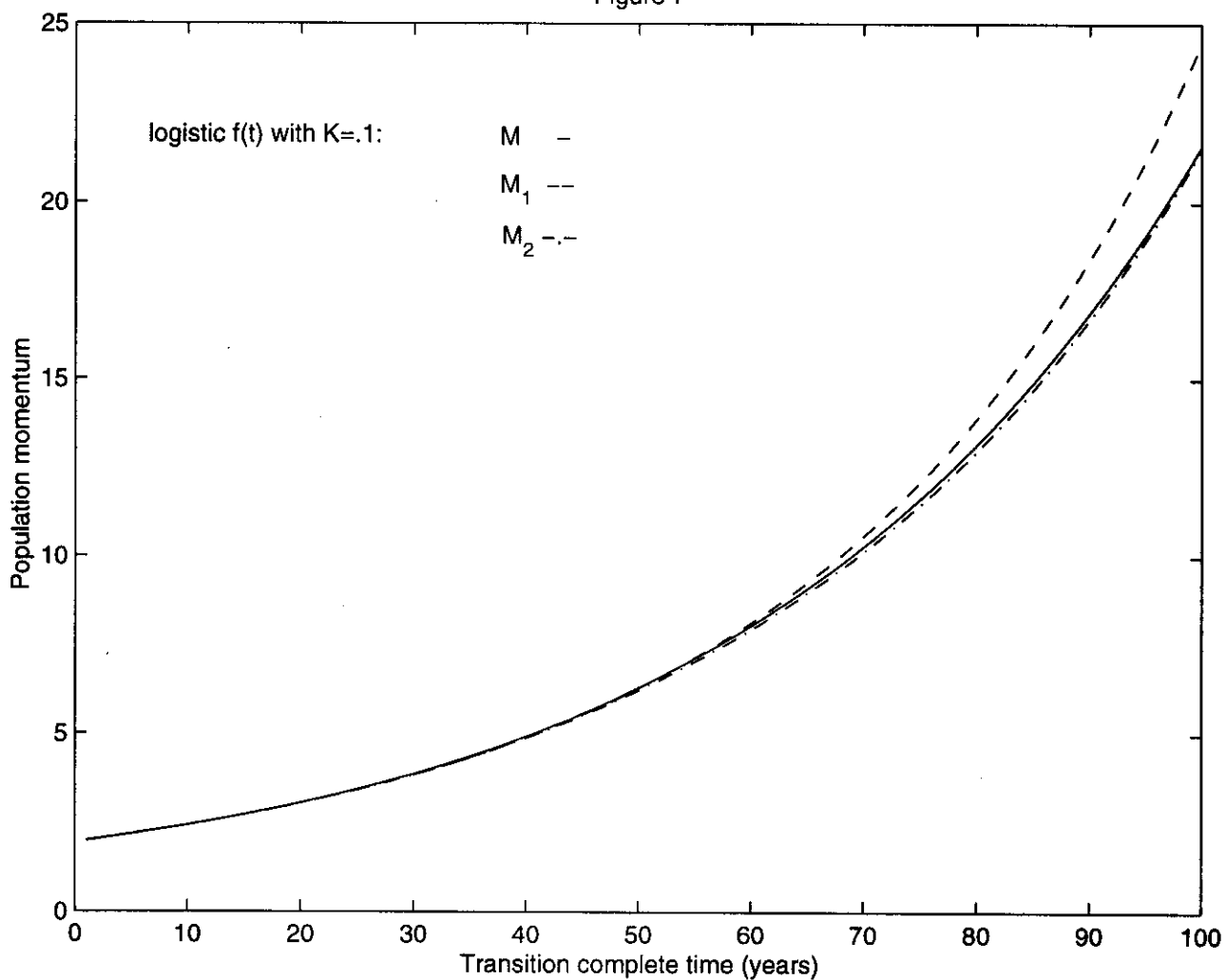


Figure 7. For a logistic decline with a changing age-pattern of fertility, variables as in Figure 4. Here too, M_1 is an accurate approximation for γ as large as 40 years, while M_2 is an accurate approximation to momentum even for values of γ as large as 100 years.