AA214: NUMERICAL METHODS FOR COMPRESSIBLE FLOWS

Representative Model Problems

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- Combines the convection (or advection) and diffusion equations to describe physical phenomena where physical quantities are transferred inside a physical system due to two processes, namely, convection and diffusion
- Convection is a transport mechanism of a substance or conserved property by a fluid due to the fluid's bulk motion
- Diffusion is the net movement of a substance from a region of high concentration to a region of low concentration
- Also referred to by different communities as the drift-diffusion, Smoluchowski, or *scalar* transport equation

$$\boxed{\frac{\partial c}{\partial t} + \vec{\nabla} \cdot (\vec{a} c) = \vec{\nabla} \cdot (D \vec{\nabla} c) + S}$$

where c is the variable of interest (species concentration for mass transfer, temperature for heat transfer, $\cdots$), D is the diffusivity (or diffusion coefficient), $\vec{a}$ is the average velocity of the quantity that is moving, and S describes sources or sinks of the quantity c

- Common simplifications
  - the diffusion coefficient D is constant, there are no sources or sinks ($S = 0$), and the velocity field describes an *incompressible* flow ($\overrightarrow{\nabla} \cdot \vec{a} = \overrightarrow{\nabla} \cdot \vec{v} = 0$)

$$\boxed{\frac{\partial c}{\partial t} + \vec{a} \cdot \nabla c = D \nabla^2 c}$$

    in this form, the convection-diffusion equation combines both parabolic and hyperbolic partial differential equations
  - stationary convection-diffusion equation

$$\boxed{\overrightarrow{\nabla} \cdot (\vec{a}c) = \overrightarrow{\nabla} \cdot (D \nabla c) + S}$$

- Why is it a good representative model problem (and of what)?
    - for an incompressible flow ($\rho = cst$), the Navier-Stokes equations can be written as

$$\frac{\partial(\rho\vec{v})}{\partial t} + \vec{v} \cdot \vec{\nabla}(\rho\vec{v}) = \nabla^2\left(\frac{\mu}{\rho}(\rho\vec{v})\right) + (\vec{f} - \vec{\nabla}p) \qquad (1)$$

where

$$\begin{aligned}
\nabla^2(\rho\vec{v}) &= \left(\nabla^2(\rho v_x) \;\; \nabla^2(\rho v_y) \;\; \nabla^2(\rho v_z)\right)^T \\
&= \left(\vec{\nabla}\cdot\vec{\nabla}(\rho v_x) \;\; \vec{\nabla}\cdot\vec{\nabla}(\rho v_y) \;\; \vec{\nabla}\cdot\vec{\nabla}(\rho v_z)\right)^T \\
&= \left(\Delta(\rho v_x) \;\; \Delta(\rho v_y) \;\; \Delta(\rho v_z)\right)^T
\end{aligned}$$

and $\vec{f}$ is a body force such as gravity

- Why is it a good representative model problem? (continue)
  - for an incompressible flow ($\rho = cst$), the Navier-Stokes equations can be written as

$$\boxed{\frac{\partial(\rho\vec{v})}{\partial t} + \vec{v}\cdot\vec{\nabla}(\rho\vec{v}) = \nabla^2\left(\frac{\mu}{\rho}(\rho\vec{v})\right) + (\vec{f} - \vec{\nabla}p)}$$

  - compare with the convection-diffusion equation when D is constant and the velocity field describes an incompressible flow ($\vec{\nabla}\cdot\vec{v} = 0$)

$$\boxed{\frac{\partial c}{\partial t} + \vec{a}\cdot\vec{\nabla}c = \nabla^2(Dc) + S}$$

$\Longrightarrow$ the convection-diffusion equation mimics the incompressible Navier-Stokes equations

- Dropping the pressure term from the incompressible Navier-Stokes equations (1) leads to

$$\frac{\partial(\rho\vec{v})}{\partial t} + \vec{v}\cdot\vec{\nabla}(\rho\vec{v}) = \nabla^2\left(\frac{\mu}{\rho}(\rho\vec{v})\right) + \vec{f}$$

- In one-dimension and assuming that μ is constant, the above equation simplifies to Burgers equation (proposed in 1939 by the dutch scientist Johannes Martinus Burgers)

$$\boxed{\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} = \nu\frac{\partial^2 v_x}{\partial x^2} + g_x}$$

where $\nu = \dfrac{\mu}{\rho}$ and $g_x = \dfrac{f_x}{\rho}$

$$\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} = \nu \frac{\partial^2 v_x}{\partial x^2} + g_x$$

- The above equation can be transformed into a linear parabolic equation using the Hopf-Cole transformation ($v_x = -2\nu\phi\dfrac{\partial \phi}{\partial x}$) then solved exactly
- This allows one to compare numerically obtained solutions of this nonlinear equation with the exact one
- For all these reasons, the Burgers equation is often used to investigate the quality of a proposed CFD scheme for viscous (and inviscid, see next) flows

- For $\nu = 0$ and $g_x = 0$, the Burgers equation simplifies to

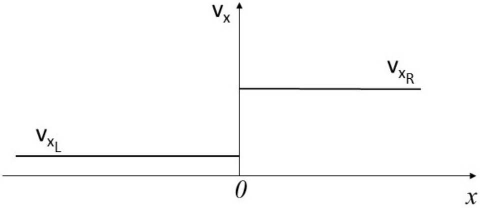
$$\boxed{\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} = 0}$$

  which is known as the inviscid Burgers equation
- It is a prototype for equations whose solution can develop discontinuities (shock waves)
- It can be solved by the method of characteristics
- It can be written in strong conservation law form as follows

$$\boxed{\frac{\partial v_x}{\partial t} + \frac{\partial (\frac{v_x^2}{2})}{\partial x} = 0}$$

- Consider the following inviscid Burgers *problem*

$$\boxed{\begin{aligned} \frac{\partial v_x}{\partial t} + \frac{\partial(\frac{v_x^2}{2})}{\partial x} &= 0 \\ v_x(x,0) &= \begin{cases} v_{x_L} & \text{if} \quad x < 0 \\ v_{x_R} & \text{if} \quad x > 0 \end{cases} \end{aligned} \qquad (2)}$$



- Consider the following inviscid Burgers *problem* (continue)
  - consider now scaling x and t by a constant $\alpha > 0$

$$\bar{x} = \alpha x, \qquad \bar{t} = \alpha t, \qquad \alpha > 0$$

  - since

$$\frac{\partial}{\partial t} = \alpha \frac{\partial}{\partial \bar{t}}, \quad \text{and} \quad \frac{\partial}{\partial x} = \alpha \frac{\partial}{\partial \bar{x}}$$

    the inviscid Burgers equation is not affected by this scaling
  - furthermore, since the initial condition depends only on the sign of x, it is not affected by the above scaling

$\Longrightarrow$ the inviscid Burgers problem defined above is scale invariant

- Scale invariance often implies the risk of multiple solutions
  - if $v_x(x,t)$ is the solution of problem (2), then $u(x,t) = v_x(\alpha x, \alpha t)$ is also a solution of problem (2) for any $\alpha > 0$
  - hence, desiring uniqueness of the solution of the above problem is desiring $u \equiv v_x$ — that is

$$v_x(x,t) = \bar{v}_x\left(\frac{x}{t}\right)$$

  - this implies that the solution $v_x(x,t)$ is constant on the rays (characteristics) $x = ct$, and therefore the solution is said to be *self-similar* [1]
  - in a homework, it will be shown that more precisely, the solution of problem (2) is

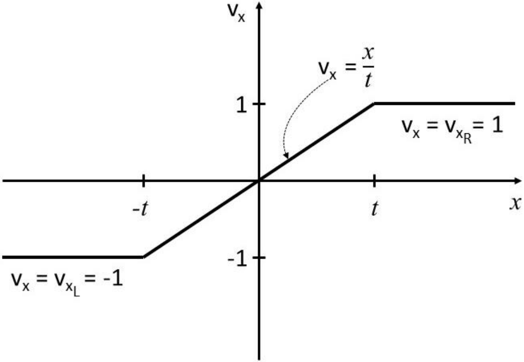
$$v_x(x,t) = \bar{v}_x\left(\frac{x}{t}\right) = \frac{x}{t}$$

  - this solution is called a *rarefaction wave* centered at the origin $(x = t = 0)$

[1] Self-similarity is the property of having a substructure analogous or identical to an overall structure. For example, a part of a line segment is itself a line segment, and thus a line segment exhibits self-similarity.

- Self-similarity in nature (Romanesco broccoli)

- A rarefaction wave can be attached to a constant solution (for a proof, look at the form of Eq. (2))
- It can also join two constants



- In many circumstances, the uniqueness of the solution is enforced by imposing the condition that characteristics must impinge on a discontinuity from both sides, which is known as the *Lax Entropy Condition*
  - consider a shock located along the curve $x = \gamma(t)$ and traveling at the speed $V = \dfrac{dx}{dt} = \dfrac{d\gamma}{dt}$
  - let $v_{x_-}(t)$ and $v_{x_+}(t)$ denote the left and right limits of the solution $v_x(x, t)$ of problem (2) across the shock, respectively
  - the Lax Entropy Condition states that

$$\boxed{v_{x_+}(t) < V < v_{x_-}(t)}$$

    (recall that the flow before a normal shock wave must be supersonic)
  - in particular, the Lax Entropy Condition states that the solution must *jump down*
  - for problem (2), it can be shown that for $\alpha > 0$, the solution jumps up at the discontinuity (see initial condition): Thus, the only admissible solution — that is, the solution in which any shock satisfies the Lax Entropy Condition — is the continuous solution which has no shock

- Scalar conservation laws are simple scalar models of the Euler equations
- They can be written in strong conservation form as

$$\boxed{\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0} \tag{3}$$

- Their integral form in the space-time domain $[x_1,\ x_2] \times [t^1,\ t^2]$ is

$$\int_{x_1}^{x_2} \left[u(x,t^2) - u(x,t^1)\right] dx + \int_{t^1}^{t^2} \left[f\left(u(x_2,t)\right) - f\left(u(x_1,t)\right)\right] dt = 0 \tag{4}$$

- The solutions of the integral form (4) may contain jump discontinuities: In this case, the discontinuous solutions are called *weak solutions* of the differential form (3)
- Jump discontinuities in the differential form (3) must satisfy a jump condition derived from the integral form.
- From (4), it follows that the jump condition for a jump discontinuity traveling at a speed V is

$$\boxed{f(u_+) - f(u_-) = V(u_+ - u_-) \Leftrightarrow [\![f(u)]\!]_+^- = V[\![u]\!]_+^-} \qquad (5)$$

and therefore is analogous to the Rankine-Hugoniot relations $\Big($recall $[\![\overrightarrow{\mathcal{F}}^\star]\!]_1^2 \cdot \overrightarrow{\nabla}^\star g = 0$, here with $\overrightarrow{\mathcal{F}}^\star(u) = (u \;\, f(u))^T$ and $g(x,t) = x - x_o - V(t - t^0)\Big)$

- Using chain rule, the non conservation form (or wave speed form) of a scalar conservation law is

$$\boxed{\begin{aligned} \frac{\partial u}{\partial t} + a(u)\frac{\partial u}{\partial x} &= 0 \\ &\text{where} \\ a(u) &= \frac{df}{du} \end{aligned}}$$

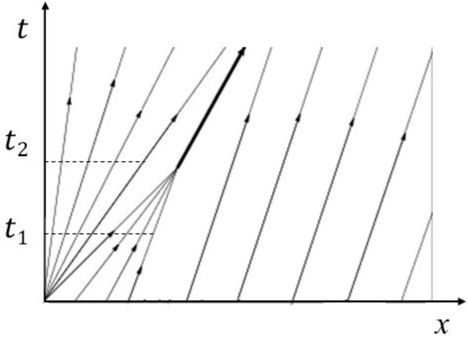
- $a(u)$ is called the wave speed

- Examples
  - $f(u) = \dfrac{u^2}{2} \Rightarrow$ Burgers equation
  - $f(u) = au \Rightarrow$ linear advection
  - $f(u) = \dfrac{u^2}{u^2 + c(1-u)^2}$, where c is a constant $\Rightarrow$ Bucky-Leverett equation which is a simple model of two-phase flow in a porous medium

- Scalar conservation laws support features analogous to *simple* expansion waves
- For scalar conservation laws, an *expansion wave* (or a *rarefaction wave*) is any region in which the wave speed $a(u)$ increases from left to right

$$\boxed{a\left(u(x,t)\right) \leq a\left(u(y,t)\right), \qquad b_1(t) \leq x \leq y \leq b_2(t)}$$

- A *centered expansion fan* is an expansion wave where all characteristics originate at a single point in the $x-t$ plane
- **Centered expansion fans must originate in the initial conditions or at intersections between shocks or contacts** (see definitions next)

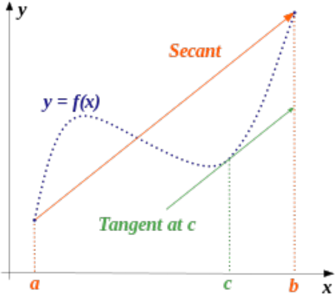

t
t2
t1
x

- Scalar conservation laws support features analogous to *simple* compression and shock waves
- For scalar conservation laws, a *compression wave* is any region in which the wave speed $a(u)$ decreases from left to right

$$\boxed{a\left(u(x,t)\right) \geq a\left(u(y,t)\right), \qquad b_1(t) \leq x \leq y \leq b_2(t)}$$

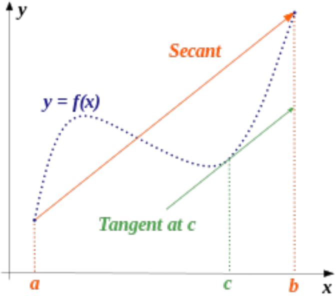
- A *centered compression fan* is a compression wave where all characteristics converge on a single point in the $x - t$ plane
- **The converging characteristics in a compression wave must eventually intersect, creating a *shock wave***

- Mean value theorem: Let $f : [a, b] \to \mathbb{R}$ be a continuous function on the closed interval $[a, b]$, and differentiable on the open interval $]a, b[$, where $a < b$. Then, there exists some $c \in]a, b[$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a} \iff f(b) - f(a) = f'(c)(b - a)$ [2]



[2]Note that $f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{1}{b-a} \int_a^b f'(x)dx =$ average tangent

- Mean value theorem: Let $f : [a, b] \to \mathbb{R}$ be a continuous function on the closed interval $[a, b]$, and differentiable on the open interval $]a, b[$, where $a < b$. Then, there exists some $c \in]a, b[$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a} \iff f(b) - f(a) = f'(c)(b - a)$ [2]



- A shock wave is a jump discontinuity governed by the jump condition $f(u_+) - f(u_-) = V(u_+ - u_-)$ (see Eq. (5)): From the mean value theorem, it follows that

$$V = \frac{df}{du}(\xi) = a(\xi), \qquad u_- \leq \xi \leq u_+$$

[2]Note that $f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{1}{b-a} \int_a^b f'(x)dx$ = average tangent

- A shock wave may originate in a jump discontinuity in the initial conditions or it may form spontaneously from a smooth compression wave
- In addition to the jump condition (5), shock waves must satisfy (think of the Lax Entropy Condition)

$$\boxed{a(u_-) \geq V \geq a(u_+)}$$

- If wave speeds are interpreted as slopes in the $x - t$ plane, then the above equation implies that **waves (characteristics) terminate on shocks and never originate in shocks** (shocks only absorb waves — they never emit waves)

- Scalar conservation laws support features analogous to contact discontinuities – that is, surfaces that separate flow zones of different density and temperature, but same pressure and velocity
- For scalar conservation laws, a *contact discontinuity* is a jump discontinuity from u_- to u_+ such that

$$\boxed{a(u_-) = a(u_+)}$$

- Like contacts in the Euler equations, **contacts in scalar conservation laws must originate in the initial conditions or at the intersections of shocks**

- In the theory of hyperbolic equations, a Riemann problem (named after Bernhard Riemann) consists of a *conservation law* equipped with uniform initial conditions on an infinite spatial domain, except for a single jump discontinuity
- In one-dimension (1D), for a hyperbolic problem governing the field u, the Riemann problem centered on $x = x_0$ and $t = t^0$ has the following initial conditions

$$u(x, t^0) = \begin{cases} u_L & \text{if} \quad x < x_0 \\ u_R & \text{if} \quad x > x_0 \end{cases}$$

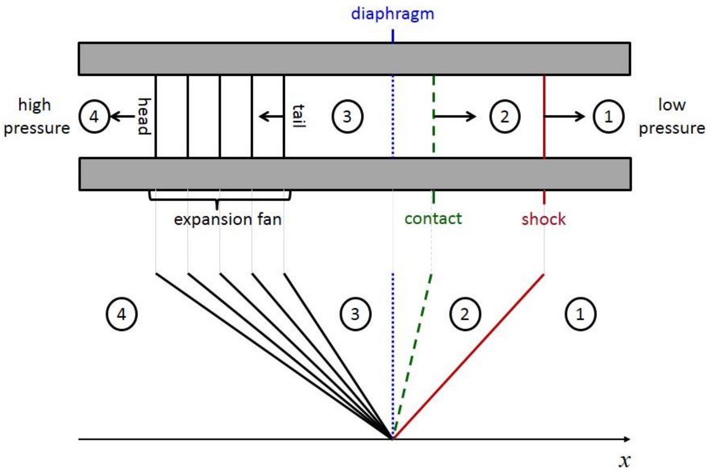
- For example, problem (2) is a Riemann problem
- For convenience, the remainder of this chapter uses $x_0 = 0$ and $t^0 = 0$

- In 1D, the Riemann problem has an exact analytical solution for the: 1) Euler equations; 2) scalar conservation laws; and 3) any linear system of equations
- Furthermore, the solution is self-similar (or self-preserving): It stretches uniformly in space as time increases but otherwise retains its shape, so that $u(x, t^1)$ and $u(x, t^2)$ are "similar" to each other for any two times t^1 and t^2 — in other words, the solution depends on the single variable $\frac{x}{t}$ rather than on x and t separately
- The Riemann problem is very useful for the understanding of the Euler equations because **shocks and rarefaction waves may appear as characteristics in the solution**
- Riemann problems appear in a natural way in finite volume methods for the solution of equations of conservation laws due to the discreteness of the grid: They give rise to the *Riemann solvers* which are very popular in CFD

Shock Tube

- Consider a 1D tube containing two regions of stagnant fluid at different pressures
- Suppose that the two regions are initially separated by a rigid diaphragm
- Suppose that this diaphragm is instantly removed (for example, by a small explosion)
  - pressure imbalance $\Rightarrow$ 1D unsteady flow containing a steadily moving shock, a *steadily* moving *simple* centered expansion fan, and a steadily moving contact discontinuity separating the shock and expansion
  - the shock, expansion, and contact separate regions of uniform flow

Shock Tube


diaphragm
high pressure
4
head
tail
3
2
1
low pressure
expansion fan
contact
shock
4
3
2
1
x

Shock Tube

- The flow in a shock tube has always zero initial velocity
- Removing this restriction, the shock tube problem becomes a Riemann problem and thus is a special case of the Riemann problem
- Major result:
  - like the shock tube problem, the Riemann problem may give rise to a steadily moving shock, a steadily moving *simple* centered expansion fan, and a steadily moving contact separating the shock and expansion; and the shock, expansion, and contact separate regions of uniform flow
  - unlike the shock tube however, one or two of these waves may be absent

Governing Equations

$$
\begin{aligned}
\frac{\partial W}{\partial t} + \frac{\partial \mathcal{F}_x}{\partial x} &= 0 \\
W = (\rho \;\; \rho v_x \;\; E)^T, \quad \mathcal{F}_x &= \left(\rho v_x \;\; \rho v_x^2 + p \;\; (E+p)v_x\right)^T \\
W(x,0) &= \begin{cases} W_L = (\rho_L \;\; \rho_L v_{x_L} \;\; E_L)^T & \text{if} \quad x < 0 \\ W_R = (\rho_R \;\; \rho_R v_{x_R} \;\; E_R)^T & \text{if} \quad x > 0 \end{cases}
\end{aligned} \tag{6}
$$

with $p = (\gamma - 1)\left(E - \rho \frac{v_x^2}{2}\right)$, and the speed of sound c given by $c^2 = \frac{\gamma p}{\rho}$

Exact Solution

- First, consider the shock
  - **in a frame moving with the (steadily moving) shock**, the Rankine-Hugoniot conditions[3] can be written as

$$
\begin{aligned}
\rho_2(v_{x_2} - V) &= \rho_1(v_{x_1} - V) &\quad (7)\\
\rho_2(v_{x_2} - V)^2 + p_2 &= \rho_1(v_{x_1} - V)^2 + p_1 &\quad (8)\\
(E_2 + p_2)(v_{x_2} - V) &= (E_1 + p_1)(v_{x_1} - V) &\quad (9)
\end{aligned}
$$

    where V is the speed of the shock
  - recall the expression of the speed of sound

$$c^2 = \gamma \frac{p}{\rho}$$

[3] In this case, the Rankine-Hugoniot conditions are given by $[\![\vec{\mathcal{F}}_x]\!]_1^2 \cdot \vec{e}_x = [\![\mathcal{F}_x]\!]_1^2 = 0$

Exact Solution

- First, consider the shock (continue)
  - from the Rankine-Hugoniot conditions applied in a frame moving with the (steadily moving) shock it follows that

$$\frac{c_2^2}{c_1^2} = \left(\frac{p_2}{p_1}\right) \frac{\frac{\gamma+1}{\gamma-1} + \left(\frac{p_2}{p_1}\right)}{1 + \frac{\gamma+1}{\gamma-1}\left(\frac{p_2}{p_1}\right)} \quad (10)$$

$$v_{x_2} = v_{x_1} + \frac{c_1}{\gamma} \frac{\left(\frac{p_2}{p_1}\right) - 1}{\sqrt{\frac{\gamma+1}{2\gamma}\left(\left(\frac{p_2}{p_1}\right) - 1\right) + 1}} \quad (11)$$

$$V = v_{x_1} + c_1 \sqrt{\frac{\gamma+1}{2\gamma}\left(\left(\frac{p_2}{p_1}\right) - 1\right) + 1} \quad (12)$$

$\Longrightarrow$ functions of the ratio $\frac{p_2}{p_1}$

Exact Solution

- Next, consider the contact discontinuity
  - by definition

$$v_{x_3} = v_{x_2} \qquad (13)$$

$$p_3 = p_2 \qquad (14)$$

- Finally, consider the *simple centered expansion fan*
  - recall that for the 1D Euler equations, an expansion wave is a wave where the wave speed (v_x, or $v_x \pm c$) increases monotonically from left to right
  - recall that a simple wave is a wave where all states lie on the same integral curve of one of the characteristic families $\Rightarrow$ a simple wave in the entropy characteristic family ξ_0 is a wave where $d\xi_+ = d\xi_- = 0$ and therefore $v_x = cst$ and $p = cst$ $\Rightarrow$ entropy waves cannot create expansions
  - it follows that the simple centered expansion fan here is a simple centered acoustic fan associated with the characteristic curve $dx = (v_x - c)dt$

- Finally, consider the *simple centered expansion fan* (continue)
  - along the integral curve of a simple centered expansion fan associated with the characteristic curve $dx = (v_x - c)dt$, the two Riemann invariants ξ_0 (entropy) and ξ_+ are constant

$$d\xi_0 = d\rho - \frac{dp}{c^2} = 0 \Rightarrow p = cst\rho^\gamma \text{ and } c = \sqrt{cst\gamma}\rho^{\frac{\gamma-1}{2}} \Rightarrow \int \frac{dp}{\rho c} = \frac{2c}{\gamma - 1}$$

$$d\xi_+ = dv_x + \frac{dp}{\rho c} = 0 \Rightarrow \xi_+ = v_x + \frac{2c}{\gamma - 1} = cst \text{ for } dx = (v_x + c)dt$$

$$(\text{and } \xi_- = v_x - \frac{2c}{\gamma - 1} \text{ for } dx = (v_x - c)dt)$$

  - hence, along the integral curve of a simple centered expansion fan associated with the characteristic curve $dx = (v_x - c)dt$ and on this characteristic curve

$$s = cst, \quad v_x + \frac{2c}{\gamma - 1} = cst, \quad \text{and} \quad v_x - \frac{2c}{\gamma - 1} = cst$$

therefore in this flow region, all flow properties are constant and $dx = (v_x - c)dt$ becomes the straight line $x = (v_x - c)t + cst$

- Finally, consider the *simple centered expansion fan* (continue)
  - now, along the integral curve of a simple centered expansion fan associated with the characteristic curve $x = (v_x - c)t$ ($cst = 0$ as fan centered at 0)
$$v_x + \frac{2c}{\gamma - 1} = v_{x_4} + \frac{2c_4}{\gamma - 1}$$
  - hence *along this integral curve and on the characteristic curve* $x = (v_x - c)t \Leftrightarrow c = v_x - \dfrac{x}{t}$ the following holds
$$v_x + \frac{2}{\gamma - 1}\left(v_x - \frac{x}{t}\right) = v_{x_4} + \frac{2c_4}{\gamma - 1}$$

$$\boxed{\Longrightarrow \begin{cases} v_x(x,t) &= \dfrac{2}{\gamma+1}\left(\dfrac{x}{t} + \dfrac{\gamma-1}{2}v_{x_4} + c_4\right) \\ c(x,t) &= \dfrac{2}{\gamma+1}\left(\dfrac{x}{t} + \dfrac{\gamma-1}{2}v_{x_4} + c_4\right) - \dfrac{x}{t} \\ p &= p_4\left(\dfrac{c}{c_4}\right)^{\frac{2\gamma}{\gamma-1}} \text{ (from the isentropic relations)} \end{cases} \qquad (15)}$$

- Combine now the shock, contact, and expansion results to determine $\left(\frac{p_2}{p_1}\right)$ across the shock in terms of the known ratio $\frac{p_4}{p_1} = \frac{p_L}{p_R}$
  - simple wave condition $v_x + \frac{2c}{\gamma - 1} = cst$ implies

$$v_{x_3} + \frac{2c_3}{\gamma - 1} = v_{x_4} + \frac{2c_4}{\gamma - 1} \tag{16}$$

  - from the third of equations (15) and (16) it follows that

$$v_{x_3} = v_{x_4} + \frac{2c_4}{\gamma - 1}\left[1 - \left(\frac{p_3}{p_4}\right)^{\frac{\gamma-1}{2\gamma}}\right] \tag{17}$$

  - from (13), (14) and (16) it follows that

$$v_{x_2} = v_{x_4} + \frac{2c_4}{\gamma - 1}\left[1 - \left(\frac{p_2}{p_4}\right)^{\frac{\gamma-1}{2\gamma}}\right] = v_{x_4} + \frac{2c_4}{\gamma - 1}\left[1 - \left(\frac{p_1}{p_4}\left(\frac{p_2}{p_1}\right)\right)^{\frac{\gamma-1}{2\gamma}}\right] \tag{18}$$

- Solving equation (18) for $\dfrac{p_4}{p_1}$ gives

$$\frac{p_4}{p_1} = \left(\frac{p_2}{p_1}\right)\left[1 + \frac{\gamma - 1}{2c_4}(v_{x_4} - v_{x_2})\right]^{-\frac{2\gamma}{\gamma-1}} \tag{19}$$

- Finally, combining (11) and (19) delivers the nonlinear equation in $\left(\dfrac{p_2}{p_1}\right)$

$$\boxed{\frac{p_4}{p_1} = \left(\frac{p_2}{p_1}\right)\left\{1 + \frac{\gamma - 1}{2c_4}\left[v_{x_4} - v_{x_1} - \frac{c_1}{\gamma}\frac{\left(\frac{p_2}{p_1}\right) - 1}{\sqrt{\frac{\gamma+1}{2\gamma}\left(\left(\frac{p_2}{p_1}\right) - 1\right) + 1}}\right]\right\}^{-\frac{2\gamma}{\gamma-1}} \qquad (20)}$$

which can be solved by a preferred numerical method to obtain $\left(\dfrac{p_2}{p_1}\right)$ and therefore p_2

- Once p_2 is found, equation (11) gives v_{x_2}, equation (10) gives c_2, and equation (12) gives the speed of the shock V, which completely determines the state 2
- Then, equations (13) and (14) give v_{x_3} and p_3 and equation (16) gives c_3, which completely determines state 3
- Finally, the first, second, and third of equations (15) deliver v_x, c, and p inside the expansion, respectively
- In some cases (depending on the values of W_L and W_R), the Riemann problem may yield only one or two waves, instead of three: To a large extent, the solution procedure described above handles such cases automatically

- The exact solution of the Riemann problem (6) is (relatively) expensive because finding p_2 requires solving the nonlinear equation (20)
- To this effect, approximate Riemann problems are often constructed as *surrogate* Riemann problems for the Euler equations
- Here, the family of approximate Riemann problems based on *a* linearization of problem (6) is considered in the general case of m dimensions

- Consider the linear Riemann problem

$$\boxed{\begin{array}{rcl} \dfrac{\partial W}{\partial t} + A\dfrac{\partial W}{\partial x} & = & 0 \\ W(x,0) & = & \left\{\begin{array}{lll} W_L & \text{if} & x < 0 \\ W_R & \text{if} & x > 0 \end{array}\right. \end{array} \qquad (21)}$$

where A is an $m \times m$ **constant** matrix whose construction is discussed in the next section

- Assume that A is diagonalizable

$$A = Q^{-1}\Lambda Q, \qquad \Lambda = \text{diag}\,(\lambda_1,\ \cdots\ ,\ \lambda_m)$$

where Q and Λ are constant matrices, and that r_i and l_i, $i = 1, \ldots,\ m$ are its right and left eigenvectors, respectively

$$Ar_i = \lambda_i r_i, \qquad A^T l_i = \lambda_i l_i \ (\text{or } l_i^T A = \lambda_i l_i^T\ )$$

- In the linear case, the change to characteristic variables $d\xi = QdW$ simplifies to

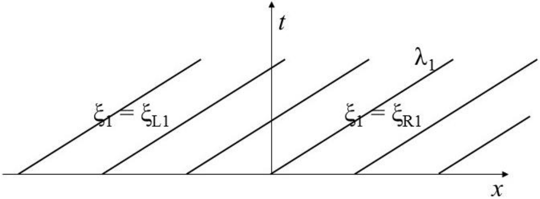
$$\boxed{\xi = QW}$$

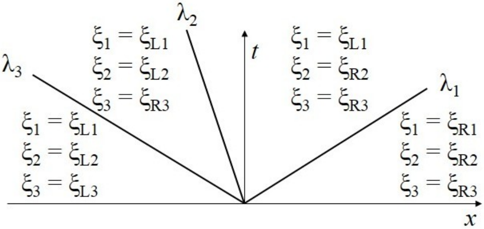
and leads to the following characteristic form of problem (21)

$$\begin{aligned} \frac{\partial \xi}{\partial t} + \Lambda \frac{\partial \xi}{\partial x} &= 0 \\ \xi(x, 0) &= \begin{cases} \xi_L = QW_L & \text{if} \quad x < 0 \\ \xi_R = QW_R & \text{if} \quad x > 0 \end{cases} \end{aligned}$$

- The individual form of the above problem is

$$\begin{aligned} \frac{\partial \xi_i}{\partial t} + \lambda_i \frac{\partial \xi_i}{\partial x} &= 0, \qquad i = 1, \ \ldots, \ m \\ \xi_i(x, 0) &= \begin{cases} \xi_{L_i} = l_i^T W_L & \text{if} \quad x < 0 \\ \xi_{R_i} = l_i^T W_R & \text{if} \quad x > 0 \end{cases} \end{aligned} \tag{22}$$


t
λ_1
$\xi_1 = \xi_{L1}$
$\xi_1 = \xi_{R1}$
x


λ_2
λ_3
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$\xi_1 = \xi_{L1}$
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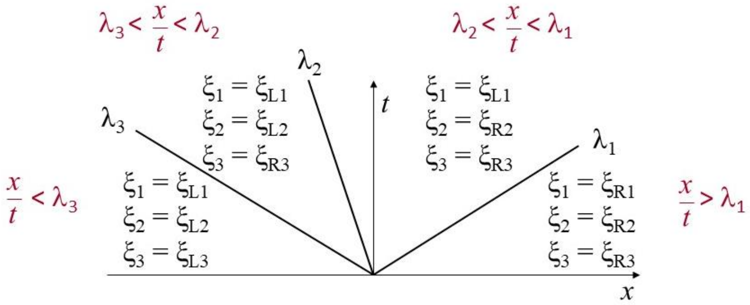
$\xi_3 = \xi_{R3}$
x


$\lambda_3 < \frac{x}{t} < \lambda_2$
$\lambda_2 < \frac{x}{t} < \lambda_1$
λ_2
λ_3
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t
x
$\xi_1 = \xi_{L1}$
$\xi_2 = \xi_{L2}$
$\xi_3 = \xi_{R3}$
$\xi_1 = \xi_{L1}$
$\xi_2 = \xi_{R2}$
$\xi_3 = \xi_{R3}$
$\frac{x}{t} < \lambda_3$
$\xi_1 = \xi_{L1}$
$\xi_2 = \xi_{L2}$
$\xi_3 = \xi_{L3}$
$\xi_1 = \xi_{R1}$
$\xi_2 = \xi_{R2}$
$\xi_3 = \xi_{R3}$
$\frac{x}{t} > \lambda_1$

- Since λ_i is constant, the solution of problem (22) is trivial: For $m = 3$, it can be written as $(\lambda_1 > \lambda_2 > \lambda_3)$

$$\xi(x,t) = \bar{\xi}\left(\frac{x}{t}\right) = \begin{cases} (\xi_{L_1}\ \xi_{L_2}\ \xi_{L_3})^T & \text{if} \quad \dfrac{x}{t} < \lambda_3 \\ (\xi_{L_1}\ \xi_{L_2}\ \xi_{R_3})^T & \text{if} \quad \lambda_3 < \dfrac{x}{t} < \lambda_2 \\ (\xi_{L_1}\ \xi_{R_2}\ \xi_{R_3})^T & \text{if} \quad \lambda_2 < \dfrac{x}{t} < \lambda_1 \\ (\xi_{R_1}\ \xi_{R_2}\ \xi_{R_3})^T & \text{if} \quad \dfrac{x}{t} > \lambda_1 \end{cases} \tag{23}$$

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- If $\Delta W = W_R - W_L$, then $\Delta\xi = Q\Delta W$, and $\Delta\xi_i = l_i^T \Delta W$ is often referred to as the *strength* or *amplitude* of the i-th wave

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- Let

$$\Delta\xi^1 = (\Delta\xi_1\ 0\ 0)^T,\ \Delta\xi^2 = (0\ \Delta\xi_2\ 0)^T,\ \Delta\xi^3 = (0\ 0\ \Delta\xi_3)^T$$

Note that the superscripts used above are NOT powers: They are used only to distinguish each of the above vector quantities from the scalar jump $\Delta\xi_i$ in the i-th characteristic

- Noting that $\Delta\xi^1 + \Delta\xi^2 + \Delta\xi^3 = \Delta\xi = \xi_R - \xi_L$, the solution (23) can be rewritten as

$$\xi(x,t) = \bar{\xi}\left(\frac{x}{t}\right) = \begin{cases} \xi_L = \xi_R - \Delta\xi^3 - \Delta\xi^2 - \Delta\xi^1 & \text{if} \quad \dfrac{x}{t} < \lambda_3 < \lambda_2 < \lambda_1 \\ & \\ & \\ & \end{cases}$$

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$$\boxed{W\left(\frac{x}{t}\right) = \begin{cases} W_L = W_R - \Delta\xi_3 r_3 - \Delta\xi_2 r_2 - \Delta\xi_1 r_1 & \text{if} \quad \frac{x}{t} < \lambda_3 < \lambda_2 < \lambda_1 \\ W_L + \Delta\xi_3 r_3 = W_R - \Delta\xi_2 r_2 - \Delta\xi_1 r_1 & \text{if} \quad \lambda_3 < \frac{x}{t} < \lambda_2 < \lambda_1 \\ W_L + \Delta\xi_3 r_3 + \Delta\xi_2 r_2 = W_R - \Delta\xi_1 r_1 & \text{if} \quad \lambda_3 < \lambda_2 < \frac{x}{t} < \lambda_1 \\ W_L + \Delta\xi_3 r_3 + \Delta\xi_2 r_2 + \Delta\xi_1 r_1 = W_R & \text{if} \quad \lambda_3 < \lambda_2 < \lambda_1 < \frac{x}{t} \end{cases}} \quad (25)$$

- Many CFD methods do not use the solution of a Riemann problem directly, whether expressed in terms of ξ or W, but use instead only the flux at $x = 0$
- Here (**linear** Riemann problem), the flux function at $x = 0$ is $AW(0)$
- From (25), it follows that

$$AW(0) = \begin{cases} AW_L = AW_R - \Delta\xi_3\lambda_3 r_3 - \Delta\xi_2\lambda_2 r_2 - \Delta\xi_1\lambda_1 r_1 & \text{if} \quad 0 < \lambda_3 < \lambda_2 < \lambda_1 \\ AW_L + \Delta\xi_3\lambda_3 r_3 = AW_R - \Delta\xi_2\lambda_2 r_2 - \Delta\xi_1\lambda_1 r_1 & \text{if} \quad \lambda_3 < 0 < \lambda_2 < \lambda_1 \\ AW_L + \Delta\xi_3\lambda_3 r_3 + \Delta\xi_2\lambda_2 r_2 = AW_R - \Delta\xi_1\lambda_1 r_1 & \text{if} \quad \lambda_3 < \lambda_2 < 0 < \lambda_1 \\ AW_L + \Delta\xi_3\lambda_3 r_3 + \Delta\xi_2\lambda_2 r_2 + \Delta\xi_1\lambda_1 r_1 = AW_R & \text{if} \quad \lambda_3 < \lambda_2 < \lambda_1 < 0 \end{cases}$$

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- Let $\lambda_i^- = \min(0, \lambda_i)$ and $\lambda_i^+ = \max(0, \lambda_i) \Rightarrow \lambda_i^+ - \lambda_i^- = |\lambda_i|$
- Then, the flux function at $x = 0$ can be rewritten as

$$\begin{aligned} AW(0) &= AW_L + \sum_{i=1}^{3} \lambda_i^- \Delta\xi_i r_i = AW_R - \sum_{i=1}^{3} \lambda_i^+ \Delta\xi_i r_i \\ &= \frac{1}{2} A(W_R + W_L) - \frac{1}{2} \sum_{i=1}^{3} |\lambda_i| \Delta\xi_i r_i \qquad (26) \end{aligned}$$

- Note that

$$\lambda_i^+ - \lambda_i^- = |\lambda_i| \Rightarrow \Lambda^+ - \Lambda^- = |\Lambda|$$

$$\lambda_i^+ + \lambda_i^- = \lambda_i \Rightarrow \Lambda^+ + \Lambda^- = \Lambda$$

(Definitions) $$A^+ = Q^{-1}\Lambda^+ Q, \quad A^- = Q^{-1}\Lambda^- Q, \quad |A| = Q^{-1}|\Lambda| Q$$

$$A^+ + A^- = A, \qquad A^+ - A^- = |A|$$

- Note that

$$\lambda_i^+ - \lambda_i^- = |\lambda_i| \Rightarrow \Lambda^+ - \Lambda^- = |\Lambda|$$
$$\lambda_i^+ + \lambda_i^- = \lambda_i \Rightarrow \Lambda^+ + \Lambda^- = \Lambda$$

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$$A^+ + A^- = A, \qquad A^+ - A^- = |A|$$

- It follows that

$$\sum_{i=1}^{3} |\lambda_i| \underbrace{\Delta\xi_i r_i}_{Q^{-1}\Delta\xi^i} = Q^{-1} \sum_{i=1}^{3} |\lambda_i| \Delta\xi^i = Q^{-1}|\Lambda| \underbrace{\Delta\xi}_{Q\Delta W} = |A|(W_R - W_L)$$

- Note that

$$\lambda_i^+ - \lambda_i^- = |\lambda_i| \Rightarrow \Lambda^+ - \Lambda^- = |\Lambda|$$
$$\lambda_i^+ + \lambda_i^- = \lambda_i \Rightarrow \Lambda^+ + \Lambda^- = \Lambda$$

(Definitions) $\quad A^+ = Q^{-1}\Lambda^+ Q, \quad A^- = Q^{-1}\Lambda^- Q, \quad |A| = Q^{-1}|\Lambda| Q$

$$A^+ + A^- = A, \qquad A^+ - A^- = |A|$$

- It follows that

$$\sum_{i=1}^{3} |\lambda_i| \underbrace{\Delta\xi_i r_i}_{Q^{-1}\Delta\xi^i} = Q^{-1} \sum_{i=1}^{3} |\lambda_i| \Delta\xi^i = Q^{-1}|\Lambda| \underbrace{\Delta\xi}_{Q\Delta W} = |A|(W_R - W_L)$$

- Hence, the solution (26) can be written as

$$AW(0) = AW_L + A^-(W_R - W_L) = AW_R - A^+(W_R - W_L)$$

$$\Longrightarrow \boxed{AW(0) = \frac{1}{2}A(W_R + W_L) - \frac{1}{2}|A|(W_R - W_L)} \qquad (27)$$

- Consider first any nonlinear *scalar* function $f(w)$, where w is also a scalar variable, and let

$$a(w) = \frac{df(w)}{dw}$$

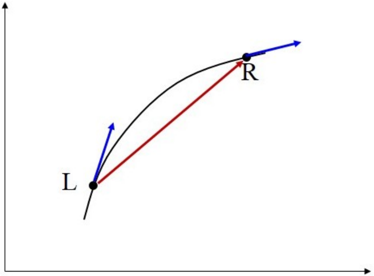
- Two linear approximations of the function $f(w)$ are
  - the *tangent* line approximation(s)
  - the *secant* line approximation

- Tangent line approximations

$$\text{about } w_R: \quad f(w) \approx f(w_R) + a(w_R)\,(w - w_R)$$

$$\text{about } w_L: \quad f(w) \approx f(w_L) + a(w_L)\,(w - w_L)$$

These two approximations are more accurate near w_R and w_L, respectively



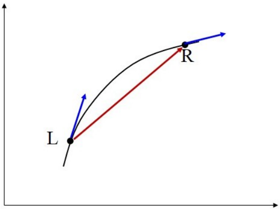
- Secant line approximation

$$f(w) \approx f(w_R) + a_{RL}(w - w_R) \Leftrightarrow f(w) \approx f(w_L) + a_{RL}(w - w_L)$$

where

$$a_{RL} = \frac{f(w_R) - f(w_L)}{(w_R - w_L)}$$

It is more accurate on average over the entire region between w_L and w_R



- The mean value theorem connects tangent line and secant line approximations as follows

$$a_{RL} = a(\eta) \quad \text{for } \eta \text{ between } w_L \text{ and } w_R$$

which essentially states that secant line slopes are average tangent line slopes

- Consider next any nonlinear *vector* function $f(W)$, where W is also a vector
- The *tangent plane* approximation about W_L is defined as

$$\boxed{f(W) \approx f(W_L) + A(W_L)(W - W_L)}$$

where $A = \dfrac{df}{dW}$ is the Jacobian matrix

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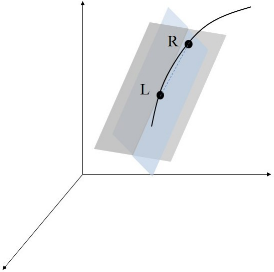
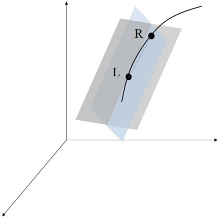
- A *secant plane* is any plane containing the line connecting W_L and W_R: There are an infinite number of such planes



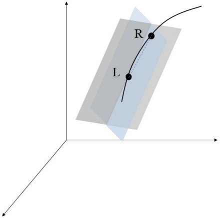


- *Secant plane* approximations are defined as follows

$$\boxed{f(W) \approx f(W_L) + A_{RL}(W - W_L) = f(W_R) + A_{RL}(W - W_R)}$$

where A_{RL} is *any* matrix such that

$$\boxed{f(W_R) - f(W_L) = A_{RL}(W_R - W_L)} \qquad (28)$$



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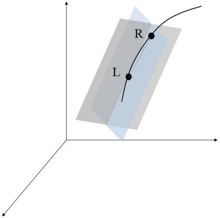
- Note that if each of W and $f(W)$ is a vector with m components, A_{RL} is a matrix with m^2 elements: Hence, equation (28) consists of m equations with m^2 unknowns

- Example 1

$$\begin{pmatrix} \dfrac{f_1(W_R) - f_1(W_L)}{W_{R_1} - W_{L_1}} & 0 & 0 \\ 0 & \dfrac{f_2(W_R) - f_2(W_L)}{W_{R_2} - W_{L_2}} & \\ 0 & 0 & \dfrac{f_3(W_R) - f_3(W_L)}{W_{R_3} - W_{L_3}} \end{pmatrix}$$

- Example 2

$$\frac{1}{3}\begin{pmatrix} \dfrac{f_1(W_R) - f_1(W_L)}{W_{R_1} - W_{L_1}} & \dfrac{f_1(W_R) - f_1(W_L)}{W_{R_2} - W_{L_2}} & \dfrac{f_1(W_R) - f_1(W_L)}{W_{R_3} - W_{L_3}} \\ \dfrac{f_2(W_R) - f_2(W_L)}{W_{R_1} - W_{L_1}} & \dfrac{f_2(W_R) - f_2(W_L)}{W_{R_2} - W_{L_2}} & \dfrac{f_2(W_R) - f_2(W_L)}{W_{R_3} - W_{L_3}} \\ \dfrac{f_3(W_R) - f_3(W_L)}{W_{R_1} - W_{L_1}} & \dfrac{f_3(W_R) - f_3(W_L)}{W_{R_2} - W_{L_2}} & \dfrac{f_3(W_R) - f_3(W_L)}{W_{R_3} - W_{L_3}} \end{pmatrix}$$



$$f(W_R) - f(W_L) = A_{RL}(W_R - W_L)$$

- By analogy with the scalar case, suppose that one requires that in the vector case, secant planes be average tangent planes: In this case,

$$\boxed{A_{RL} = A(W_{RL})}$$

where W_{RL} is an average between W_R and W_L, and there are only m unknowns — the components of W_{RL} — that can be determined by solving equation (28)

- Consider now the one-dimensional Euler equations: For these equations, the conservative state vector W, flux vector $\mathcal{F}_x$, and Jacobian matrix $A = \dfrac{\partial \mathcal{F}_x}{\partial W}$ can be written as

$$
\begin{aligned}
W &= \begin{pmatrix}\rho & \rho v_x & E\end{pmatrix}^T = \begin{pmatrix}\rho & \rho v_x & \frac{1}{\gamma}\rho h + \frac{1}{2\gamma}(\gamma-1)\rho v_x^2\end{pmatrix}^T \\
\mathcal{F}_x &= \begin{pmatrix}\rho v_x & \rho v_x^2 + p & (E+p)v_x\end{pmatrix}^T = \begin{pmatrix}\rho v_x & \frac{\gamma-1}{\gamma}\rho h + \frac{\gamma+1}{2\gamma}\rho v_x^2 & \rho h v_x\end{pmatrix}^T \\
A &= \begin{pmatrix} 0 & 1 & 0 \\ \frac{\gamma-3}{2}v_x^2 & (3-\gamma)v_x & \gamma-1 \\ -v_x h + \frac{1}{2}(\gamma-1)v_x^3 & h-(\gamma-1)v_x^2 & \gamma v_x \end{pmatrix}
\end{aligned} \tag{29}
$$

where $h = \dfrac{H}{\rho}$ is the specific enthalpy and $H = E + p$ is the total enthalpy per unit volume

- Choose $A_{RL} = A(W_{RL})$: In this case, equation (29) leads to the *Roe-average Jacobian* matrix

$$A_{RL} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{\gamma-3}{2} v_{x_{RL}}^2 & (3-\gamma) v_{x_{RL}} & \gamma - 1 \\ -v_{x_{RL}} h_{RL} + \frac{1}{2}(\gamma-1) v_{x_{RL}}^3 & h_{RL} - (\gamma-1) v_{x_{RL}}^2 & \gamma v_{x_{RL}} \end{pmatrix} \tag{30}$$

- Solving equation (28) using the above Roe-average Jacobian matrix leads after several algebraic manipulations to

$$v_{x_{RL}} = \frac{\sqrt{\rho_R} v_{x_R} + \sqrt{\rho_L} v_{x_L}}{\sqrt{\rho_R} + \sqrt{\rho_L}}$$

$$h_{RL} = \frac{\frac{H_R}{\sqrt{\rho_R}} + \frac{H_L}{\sqrt{\rho_L}}}{\sqrt{\rho_R} + \sqrt{\rho_L}} = \frac{\sqrt{\rho_R} h_R + \sqrt{\rho_L} h_L}{\sqrt{\rho_R} + \sqrt{\rho_L}}$$

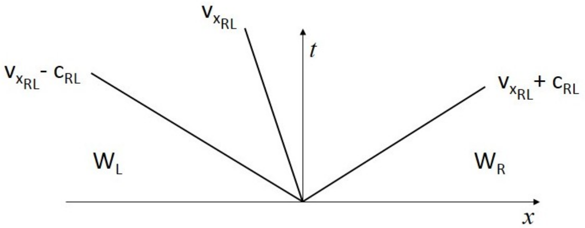
- The usual perfect gas relationships hold between the Roe-averaged quantities: for example

$$h_{RL} = \frac{1}{2} v_{x_{RL}}^2 + \frac{c_{RL}^2}{\gamma - 1} \Longrightarrow c_{RL} = \sqrt{(\gamma-1)\left(h_{RL} - \frac{1}{2} v_{x_{RL}}^2\right)}$$

- Finally, define $\rho_{RL} = \sqrt{\rho_R \rho_L}$

$$\Longrightarrow p_{RL} = \frac{\rho_{RL} c_{RL}^2}{\gamma}$$

- Roe's approximate Riemann solver



- Roe's approximate Riemann solver for the Euler equations (vector function $f = \mathcal{F}_x$) is based on two ideas: (1) the linear (secant) approximation of the flux vector

$$\mathcal{F}_x(W) \approx \widehat{\mathcal{F}}_x(W) = \mathcal{F}_x(W_L) + A_{RL}(W - W_L) = \mathcal{F}_x(W_R) + A_{RL}(W - W_R) \qquad (31)$$

  where A_{RL} is the Roe-average Jacobian given in (30); and (2) the exact solution of the linear Riemann problem (21) with $A = A_{RL}$ (see also (25) for $A = A_{RL}$)
- Indeed, substituting (31) into the Euler equations (6) gives

$$\frac{\partial W}{\partial t} + \frac{\partial}{\partial x}\left\{\mathcal{F}_x(W_L) + A_{RL}(W - W_L)\right\} = \frac{\partial W}{\partial t} + A_{RL}\frac{\partial W}{\partial x} = 0$$

- From (31) and (27), it follows that Roe's approximate Riemann solver computes the fluxes at $x = 0$ as

$$\begin{aligned}
\widehat{\mathcal{F}}_x\left(W(0)\right) &= \mathcal{F}_x(W_L) + A_{RL}\left(W(0) - W_L\right) = \mathcal{F}_x(W_R) + A_{RL}\left(W(0) - W_R\right) \\
&= \frac{1}{2}\left(\mathcal{F}_x(W_R) + \mathcal{F}_x(W_L)\right) + A_{RL}W(0) - \frac{1}{2}A_{RL}(W_R + W_L) \\
&= \frac{1}{2}\left(\mathcal{F}_x(W_R) + \mathcal{F}_x(W_L)\right) + \frac{1}{2}A_{RL}(W_R + W_L) - \frac{1}{2}|A_{RL}|(W_R - W_L) \\
&\qquad - \frac{1}{2}A_{RL}(W_R + W_L)
\end{aligned}$$

$$\Longrightarrow \boxed{\widehat{\mathcal{F}}_x\left(W(0)\right) = \frac{1}{2}\left(\mathcal{F}_x(W_R) + \mathcal{F}_x(W_L)\right) - \frac{1}{2}|A_{RL}|(W_R - W_L)}$$

- Like the true (exact) Riemann solver, Roe's approximate Riemann solver yields three equally-spaced waves (see previous Figure)
- Unlike in the true Riemann solver however, all three waves in Roe's approximate Riemann solver have zero spread (hence, Roe's approximate Riemann solver cannot capture the finite spread of the expansion fan)
- Roe's approximate Riemann solver for the Euler equations is roughly 2.5 times faster than the exact Riemann solver
- What about its accuracy?

- Suppose that the exact Riemann problem yields a single shock or a single contact with speed V (recall that unlike in the shock tube problem, one or two of the shock, contact, and expansion waves may be absent in the solution of the exact Riemann problem)
- The shock or contact must satisfy the Rankine-Hugoniot conditions

$$\mathcal{F}_x(W_R) - \mathcal{F}_x(W_L) = V(W_R - W_L) \qquad (\Rightarrow V = \text{cst})$$

- For Roe's approximate Riemann solver, A_{RL} must satisfy the secant plane condition

$$\mathcal{F}_x(W_R) - \mathcal{F}_x(W_L) = A_{RL}(W_R - W_L)$$

  - it follows that

$$A_{RL}(W_R - W_L) = V(W_R - W_L)$$

    which implies that V is a characteristic value (eigenvalue) of A_{RL} and $W_R - W_L$ is a right characteristic vector (eigenvector) of A_{RL}

- Let $V = \lambda_j$ and $W_R - W_L = r_j$: then, the strength of the i-th wave is given by

$$\Delta\xi_i = l_i^T (W_R - W_L) = l_i^T r_j = \delta_{ij} \, (QQ^{-1} = I) = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$$

$\Rightarrow$ two of the three waves have zero strength

- Since $\Delta W = Q^{-1} \Delta\xi = \sum_{i=1}^{3} r_i \Delta\xi_i$, it follows that the single non trivial wave makes the full transition between W_L and W_R at the speed V

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- It follows that for a single shock or a single contact, Roe's approximate Riemann solver yields the exact solution!
- Except in the above case however, Roe's approximate Riemann solver deviates substantially from the true Riemann solver: more specifically, unlike the true nonlinear flux function, Roe's linear flux function allows for *expansion shocks* — that is, jump discontinuities that satisfy the Rankine-Hugoniot relations but expand rather than compress the flow and therefore violate the second law of thermodynamics