AA214: NUMERICAL METHODS FOR COMPRESSIBLE FLOWS

Conservation and Integral Forms and Discontinuities

Outline

- Definition: an equation (or set of equations) is said to be in conservation law form — or more precisely, in divergence form — if it is written as follows

$$\frac{\partial W}{\partial t} + \vec{\nabla} \cdot \vec{\mathcal{F}}(W) = S$$

- If $S = 0$, the equation is said to be in *strong* conservation law form
- For example, many of the equations presented in Chapter 2 are written in strong conservation form

- Recall that the transonic small disturbance equation discussed in Chapter 2 was written as

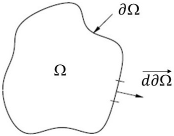
$$\left(1 - M_\infty^2 - (\gamma+1)M_\infty^2 \frac{\frac{\partial \phi}{\partial x}}{\|\vec{v}_\infty\|}\right)\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

- Recall that the transonic small disturbance equation discussed in Chapter 2 was written as

$$\left(1 - M_\infty^2 - (\gamma + 1) M_\infty^2 \frac{\frac{\partial \phi}{\partial x}}{\left\| \vec{v}_\infty \right\|}\right) \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

This equation can be re-written in strong conservation form using

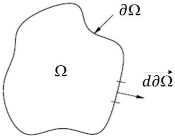
$$\overrightarrow{\mathcal{F}} = \left(\left[(1 - M_\infty^2)\frac{\partial \phi}{\partial x} - (\gamma + 1) M_\infty^2 \frac{\frac{\partial \phi}{\partial x}^2}{2\left\| \vec{v}_\infty \right\|}\right] \quad \frac{\partial \phi}{\partial y} \quad \frac{\partial \phi}{\partial z}\right)^T$$



- The integration over an arbitrary stationary volume Ω enclosed by the surface $\partial\Omega$ of a generic equation written in conservation form can be written as

$$\int_\Omega \frac{\partial W}{\partial t} d\Omega + \int_\Omega \overrightarrow{\nabla} \cdot \overrightarrow{\mathcal{F}}(W)\, d\Omega = \int_\Omega S\, d\Omega$$

Integral Form



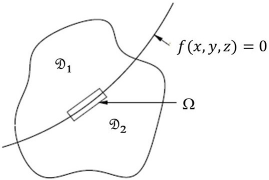
- The integration over an arbitrary stationary volume Ω enclosed by the surface $\partial\Omega$ of a generic equation written in conservation form can be written as

$$\int_{\Omega} \frac{\partial W}{\partial t} d\Omega + \int_{\Omega} \vec{\nabla} \cdot \vec{\mathcal{F}}(W)\, d\Omega = \int_{\Omega} S\, d\Omega$$

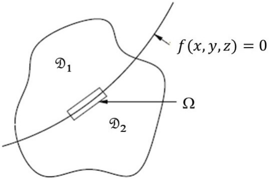
- Dividing by Ω and using the divergence (Gauss, or Ostrogradsky) theorem leads to

$$\begin{aligned} \frac{\partial \overline{W}}{\partial t} + \frac{1}{\Omega} \int_{\partial\Omega} \vec{\mathcal{F}} \cdot \overrightarrow{d\partial\Omega} &= \frac{1}{\Omega} \int_{\Omega} S\, d\Omega \qquad (1) \\ \text{where} \quad \overline{W} &= \frac{1}{\Omega} \int_{\Omega} W\, d\Omega \end{aligned}$$

- The above equation represents the rate of change of the mean value of W over the volume Ω caused by the net flux of $\vec{\mathcal{F}}$ crossing the surface $\partial\Omega$ and the volume source S

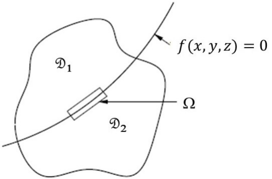


- Let $f(x, y, z) = 0$ represent a surface located at a possible discontinuity within the fluid
- Assume that the flow is continuous within each of the two subdomains shown in the figure above
- Assume also that Ω is placed **symmetrically** about an arbitrary point of the surface and is allowed to shrink to zero



- Now, for the case of a steady flow, Eq. (1) becomes

$$\int_{\partial\Omega} \overrightarrow{\mathcal{F}} \cdot \overrightarrow{d\partial\Omega} = \int_{\Omega} S \, d\Omega$$

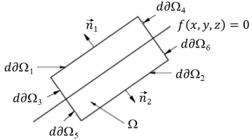


- Now, for the case of a steady flow, Eq. (1) becomes

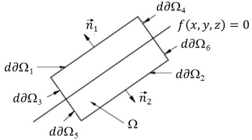
$$\int_{\partial\Omega} \overrightarrow{\mathcal{F}} \cdot \overrightarrow{d\partial\Omega} = \int_{\Omega} S \, d\Omega$$

- As $\Omega \to 0$, the term on the right goes to zero at a faster rate than the surface integration term (h^3 vs h^2, where $h \approx \Omega^{\frac{1}{3}} = \partial\Omega^{\frac{1}{2}}$)
- It follows that for an infinitesimal Ω

$$\int_{\partial\Omega} \overrightarrow{\mathcal{F}} \cdot \overrightarrow{d\partial\Omega} = 0$$



$$0 = \int_{\partial\Omega} \vec{\mathcal{F}} \cdot \overrightarrow{d\partial\Omega} = \sum_{i=1}^{6} \vec{\mathcal{F}}_i \cdot \vec{n}_i \, d\partial\Omega_i, \quad \text{where} \quad \|\vec{n}_i\|_2 = 1, \;\; i = 1, \cdots, 6$$



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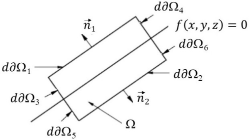
- Since the flow is continuous within each of subdomain $\mathcal{D}_1$ and subdomain $\mathcal{D}_2$, in the limit when $\partial\Omega \to 0$

$$\overrightarrow{\mathcal{F}}_3 \cdot \vec{n}_3 \, d\partial\Omega_3 + \overrightarrow{\mathcal{F}}_4 \cdot \vec{n}_4 \, d\partial\Omega_4 = 0 \quad \text{and} \quad \overrightarrow{\mathcal{F}}_5 \cdot \vec{n}_5 \, d\partial\Omega_5 + \overrightarrow{\mathcal{F}}_6 \cdot \vec{n}_6 d\partial\Omega_6 = 0$$

$$\Longrightarrow \int_{\partial\Omega} \overrightarrow{\mathcal{F}} \cdot \overrightarrow{d\partial\Omega} = \overrightarrow{\mathcal{F}}_1 \cdot \vec{n}_1 \, d\partial\Omega_1 + \overrightarrow{\mathcal{F}}_2 \cdot \vec{n}_2 \, d\partial\Omega_2 = 0$$

$$\Longrightarrow (\overrightarrow{\mathcal{F}}_1 - \overrightarrow{\mathcal{F}}_2) \cdot \vec{n}_1 = 0 \qquad \text{with} \qquad \vec{n}_1 = \frac{\overrightarrow{\nabla} f}{\left\| \overrightarrow{\nabla} f \right\|}$$

$$\Longrightarrow (\overrightarrow{\mathcal{F}}_1 - \overrightarrow{\mathcal{F}}_2) \cdot \overrightarrow{\nabla} f = 0$$



- The jump of $\vec{\mathcal{F}}$ across the surface f is defined as and denoted by

$$\left(\vec{\mathcal{F}}_1 - \vec{\mathcal{F}}_2\right) = \left[\!\left[\vec{\mathcal{F}}\right]\!\right]_1^2$$

$$\Longrightarrow \left[\!\left[\vec{\mathcal{F}}\right]\!\right]_1^2 \cdot \vec{\nabla} f = 0$$

which can also be written as

$$\boxed{[\![\mathcal{F}_x]\!]_1^2 \, \frac{\partial f}{\partial x} + [\![\mathcal{F}_y]\!]_1^2 \, \frac{\partial f}{\partial y} + [\![\mathcal{F}_z]\!]_1^2 \, \frac{\partial f}{\partial z} = 0}$$

- If $\vec{\mathcal{F}}$ is the flux vector of the Euler equations, the above steady jump relations at surface $f(x, y, z) = 0$ represent the **Rankine-Hugoniot** relations across a shock wave

- Consider now the surface $f(x, y, z, t) = 0$ representing a dynamic surface located at a possible moving discontinuity within a volume Ω of a fluid
- Let

$$\vec{\nabla}^{\star} = \left(\frac{\partial}{\partial t} \; \frac{\partial}{\partial x} \; \frac{\partial}{\partial y} \; \frac{\partial}{\partial z} \right)^T$$

and

$$\vec{\mathcal{F}}^{\star}(W) = \left(W^T \; \mathcal{F}_x^T(W) \; \mathcal{F}_y^T(W) \; \mathcal{F}_z^T(W) \right)^T$$

- Then $\dfrac{\partial W}{\partial t} + \vec{\nabla} \cdot \vec{\mathcal{F}}(W) = S$ can be rewritten as $\vec{\nabla}^{\star} \cdot \vec{\mathcal{F}}^{\star}(W) = S$
- Using the above notation, which includes time as a dimension, the previous discussion on stationary discontinuities can be generalized to obtain the following unsteady jump relations for moving discontinuities

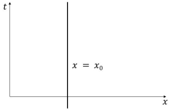
$$[\![\vec{\mathcal{F}}^{\star}]\!]_1^2 \cdot \vec{\nabla}^{\star} f = [\![W]\!]_1^2 \frac{\partial f}{\partial t} + [\![\mathcal{F}_x]\!]_1^2 \frac{\partial f}{\partial x} + [\![\mathcal{F}_y]\!]_1^2 \frac{\partial f}{\partial y} + [\![\mathcal{F}_z]\!]_1^2 \frac{\partial f}{\partial z} = 0$$

Simple Wave Equation

- Consider the model hyperbolic equation with constant wave speed $c \neq 0$ and with scalar variable u

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

  - consider first the case of a stationary discontinuity surface of the form $f(x) = x - x_0 = 0$



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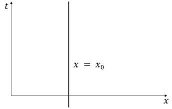


  - in this case, $\overrightarrow{\mathcal{F}}^{\star} = (u \;\, cu)^T$ and $\vec{n}_1 = \frac{\overrightarrow{\nabla}^{\star} f}{\left\|\overrightarrow{\nabla}^{\star} f\right\|} = (0 \;\, 1)^T$, and therefore the jump relation is

$$\left[\!\left[\overrightarrow{\mathcal{F}}^{\star} \right]\!\right]_1^2 \cdot \overrightarrow{\nabla}^{\star} f = [\![cu]\!]_1^2 = c(u_1 - u_2) = 0 \Leftrightarrow u_1 = u_2$$

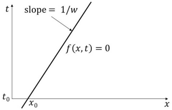
  - this implies that no jump is possible, which is not surprising for a linear equation

Simple Wave Equation

- Consider the model hyperbolic equation with constant wave speed $c \neq 0$ and with scalar variable u (continue)

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

  - consider next the case of a discontinuity surface moving at constant speed w, $f(x, t) = x - x_0 - w(t - t^0) = 0$



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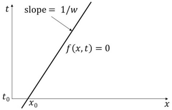
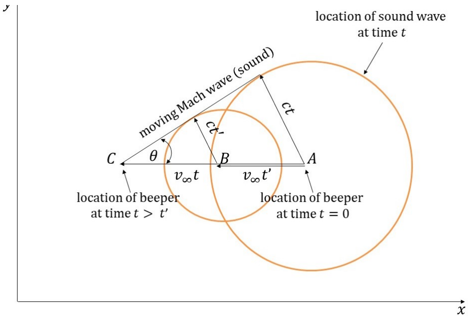


  - in this case, $\vec{n}_1 = \frac{\vec{\nabla}^\star f}{\left\|\vec{\nabla}^\star f\right\|} = \frac{1}{\sqrt{1+w^2}}(-w\ 1)^T$, and therefore the jump relation is

$$\begin{aligned} \left[\!\left[\vec{\mathcal{F}}^\star\right]\!\right]_1^2 \cdot \vec{\nabla}^\star f = -w\,[\![u]\!]_1^2 + [\![cu]\!]_1^2 &= -w(u_1 - u_2) + c(u_1 - u_2) = 0 \\ \Leftrightarrow\quad & (c - w)(u_1 - u_2) = 0 \end{aligned}$$

  - this implies that any jump is possible, as long as it moves at the speed c

Mach Waves



Mach wave: pressure wave traveling with the speed of sound caused by a slight change of pressure added to a compressible flow – these weak waves can combine in **supersonic** flow to become a shock wave if sufficient Mach waves are present at any location

$$\sin\theta = \frac{c}{v_\infty} = \frac{1}{M_\infty} \Rightarrow \tan\theta = \frac{\sin\theta}{\sqrt{1-\sin^2\theta}} = \frac{1}{\sqrt{M_\infty^2 - 1}}$$

Mach Waves: Linearized Small-Perturbation Potential Equation in the Supersonic Regime

- Recall the linearized small-perturbation potential equation modeling a two-dimensional steady flow in either the subsonic or supersonic regime

$$\left(1 - M_\infty^2\right) \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad \Leftrightarrow \quad \vec{\nabla} \cdot \left((1 - M_\infty^2) \frac{\partial \phi}{\partial x} \;\; \frac{\partial \phi}{\partial y} \right)^T = 0$$

- For $M_\infty > 1.2$, this equation is hyperbolic and can describe purely supersonic flows with small perturbations about a supersonic free-stream with velocity $\vec{v}_\infty = \|\vec{v}_\infty\| \, \vec{e}_x$ (recall also that in this case, $\vec{v} = \vec{v}_\infty + \vec{\nabla}\phi = \left(\|\vec{v}_\infty\| + \frac{\partial \phi}{\partial x}\right) \vec{e}_x + \frac{\partial \phi}{\partial y} \vec{e}_y$)
- Consider as a possible stationary discontinuity surface $f(x, y) = a(x - x_0) - b(y - y_0) = 0$, where a and b are constants (stationary w.r.t the object generating it)

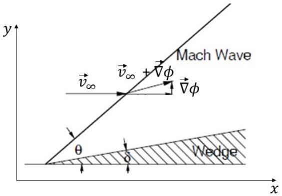
Mach Waves: Linearized Small-Perturbation Potential Equation in the Supersonic Regime

- In this case, $\vec{\mathcal{F}} = \left((1 - M_\infty^2)\dfrac{\partial \phi}{\partial x} \quad \dfrac{\partial \phi}{\partial y}\right)^T$ and $\vec{n}_1 = \dfrac{\vec{\nabla} f}{\left\|\vec{\nabla} f\right\|} = \dfrac{1}{\sqrt{a^2+b^2}}(a \quad -b)^T$, and therefore the jump relation is

$$\begin{aligned}
[\![\vec{\mathcal{F}}]\!]_1^2 \cdot \vec{\nabla} f \quad &= \quad a\left(1 - M_\infty^2\right)\left[\!\left[\frac{\partial \phi}{\partial x}\right]\!\right]_1^2 - b\left[\!\left[\frac{\partial \phi}{\partial y}\right]\!\right]_1^2 = 0 \\
\Leftrightarrow \quad & \quad a\left(1 - M_\infty^2\right)\left(\frac{\partial \phi}{\partial x}\Big|_1 - \frac{\partial \phi}{\partial x}\Big|_2\right) - b\left(\frac{\partial \phi}{\partial y}\Big|_1 - \frac{\partial \phi}{\partial y}\Big|_2\right) = 0
\end{aligned}$$

- if $a = 0$ or $b = 0$, there are no permissible jumps (why?)
- a small perturbation jump can occur across a Mach line $f(x, y)$ with angle θ, in which case the slope of the discontinuity surface is $\dfrac{a}{b} = \tan\theta = \dfrac{1}{\sqrt{M_\infty^2 - 1}}$ (recall that the Mach angle is given by $\sin\theta = \dfrac{1}{M_\infty}$): along this Mach line, the jump relation simplifies to $-\sqrt{M_\infty^2 - 1}\left[\!\left[\dfrac{\partial \phi}{\partial x}\right]\!\right]_1^2 = \left[\!\left[\dfrac{\partial \phi}{\partial y}\right]\!\right]_1^2$

Mach Waves: Linearized Small-Perturbation Potential Equation in the Supersonic Regime



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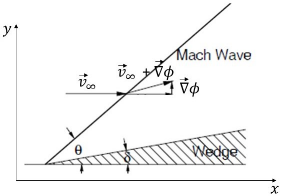
Mach Waves: Linearized Small-Perturbation Potential Equation in the Supersonic Regime



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  - along the Mach line with the slope $\dfrac{a}{b} = \tan\theta = \dfrac{1}{\sqrt{M_\infty^2 - 1}}$, where $-\sqrt{M_\infty^2 - 1}\left[\!\left[\dfrac{\partial \phi}{\partial x}\right]\!\right]_1^2 = \left[\!\left[\dfrac{\partial \phi}{\partial y}\right]\!\right]_1^2$, the flow can turn through an angle δ (small value because small perturbation) from the free-stream direction (see above figure, where $\vec{\nabla}\phi\big|_1 = 0$) such that $\dfrac{\partial \phi}{\partial x}\Big|_2 = \dfrac{-\tan\delta}{\tan\delta + \sqrt{M_\infty^2 - 1}}\,\|\vec{v}_\infty\| \approx \dfrac{-\tan\delta}{\sqrt{M_\infty^2 - 1}}\,\|\vec{v}_\infty\|$ and $\dfrac{\partial \phi}{\partial y}\Big|_2 = \dfrac{\tan\delta\sqrt{M_\infty^2 - 1}}{\tan\delta + \sqrt{M_\infty^2 - 1}}\,\|\vec{v}_\infty\| \approx \tan\delta\,\|\vec{v}_\infty\|$