AA214: NUMERICAL METHODS FOR COMPRESSIBLE FLOWS

Hierarchy of Mathematical Models

Outline

- Throughout this chapter — and as a matter of fact, this entire course — the flow is assumed to be compressible and the fluid is assumed to be a *perfect gas* (thermally and calorically)
- The Equation Of State (EOS) of a thermally perfect gas is

$$p = \rho R T \Rightarrow p = p(\rho, T) \text{ or } T = T(p, \rho)$$

where p denotes the gas pressure, ρ its density, T its temperature, and R is the specific gas constant (in SI units, R $=$ 287.058 m^2/s^2/K)
- The internal energy per unit mass e of a calorically perfect gas is given by

$$e = C_v T = \frac{R}{\gamma - 1} T \Rightarrow e = e(T) \text{ or } T = T(e)$$

where C_v denotes the heat capacity at constant volume of the gas and γ denotes the ratio of its heat capacities (C_p/C_v, where C_p denotes the heat capacity at constant pressure)

Nomenclature

γ	heat capacity ratio
ρ	density
p	pressure
T	temperature
superscript T	transpose
$\vec{v}$	velocity vector
e	internal energy per unit mass
$E = \rho e + \frac{1}{2}\rho\|\vec{v}\|^2$	total energy per unit volume
$H = E + p$	total enthalpy per unit volume
τ	(deviatoric) viscous stress tensor/matrix
$\mu \quad (\tau = \mu dv/dy)$	(laminar) dynamic (absolute) molecular viscosity
	$\rightarrow$ measure of force
ν	(laminar) kinematic molecular viscosity, μ/ρ
	$\rightarrow$ measure of velocity
κ	thermal conductivity
$\mathbb{I}$	identity tensor/matrix
M	Mach number
R_e	Reynolds number, $\rho\|\vec{v}\|L_c/\mu = \|\vec{v}\|L_c/\nu$
L_c	characteristic length
t	time
$\vec{t}$	unitary axis for the time dimension
subscript t	turbulence eddy quantity
subscripts x, y, z (or occasionally i, j)	components in the x, y, and z directions
$\vec{e}_x$ ($\vec{e}_y$, or $\vec{e}_z$)	unitary axis in the x (y, or z) direction
subscript ∞	free-stream quantity

- Navier-Stokes equations
  - Reynolds-averaged Navier-Stokes equations (RANS)
  - large eddy simulation (LES)
- Euler equations
- Full potential equation
- Linearized Small-Perturbation Potential Equation
  - subsonic and supersonic regimes
  - transonic regime

- The fluid of interest is a continuum
- The fluid of interest is not moving at relativistic velocities
- The fluid stress is the sum of a pressure term and a diffusing viscous term proportional to the gradient of the velocity

$$\boxed{\sigma = -p\mathbb{I} + \tau = -p\mathbb{I} + \underbrace{2\mu\left[\frac{1}{2}(\nabla + \nabla^T)\vec{v} - \frac{1}{3}(\vec{\nabla}\cdot\vec{v})\mathbb{I}\right]}_{\tau}} \quad (1)$$

where

$$\vec{v} = (v_x \; v_y \; v_z)^T, \quad \nabla\vec{v} = \begin{pmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_y}{\partial x} & \frac{\partial v_z}{\partial x} \\ \frac{\partial v_x}{\partial y} & \frac{\partial v_y}{\partial y} & \frac{\partial v_z}{\partial y} \\ \frac{\partial v_x}{\partial z} & \frac{\partial v_y}{\partial z} & \frac{\partial v_z}{\partial z} \end{pmatrix}, \quad \nabla^T\vec{v} = \begin{pmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial y} & \frac{\partial v_x}{\partial z} \\ \frac{\partial v_y}{\partial x} & \frac{\partial v_y}{\partial y} & \frac{\partial v_y}{\partial z} \\ \frac{\partial v_z}{\partial x} & \frac{\partial v_z}{\partial y} & \frac{\partial v_z}{\partial z} \end{pmatrix}$$

$$\vec{\nabla} = \left(\frac{\partial}{\partial x} \; \frac{\partial}{\partial y} \; \frac{\partial}{\partial z}\right)^T \Rightarrow \vec{\nabla}\cdot\vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

- Eulerian setting
- Dimensional form

$$\frac{\partial W}{\partial t} + \vec{\nabla} \cdot \vec{\mathcal{F}}(W) = \vec{\nabla} \cdot \vec{\mathcal{R}}(W)$$

$$W = \left(\rho \;\; \rho\vec{v}^{\,T} \;\; E\right)^T$$

$$\vec{\mathcal{F}}(W) = \left(\mathcal{F}_x^T(W) \;\; \mathcal{F}_y^T(W) \;\; \mathcal{F}_z^T(W)\right)^T$$

$$\vec{\mathcal{R}}(W) = \left(\mathcal{R}_x^T(W) \;\; \mathcal{R}_y^T(W) \;\; \mathcal{R}_z^T(W)\right)^T$$

- One continuity equation, three momentum equations and one energy equation $\Rightarrow$ five equations
- Closed system $\left(\rho,\ \vec{v},\ e,\ T = T(e),\ p = p(\rho, T)\right)$

$$
\left(\mathcal{F}_x(W)\ \mathcal{F}_y(W)\ \mathcal{F}_z(W)\right) = \begin{pmatrix} \rho\vec{v}^{\,T} \\ \rho v_x\vec{v}^{\,T} + p\vec{e}_x^{\,T} \\ \rho v_y\vec{v}^{\,T} + p\vec{e}_y^{\,T} \\ \rho v_z\vec{v}^{\,T} + p\vec{e}_z^{\,T} \\ (E+p)\vec{v}^{\,T} \end{pmatrix}
$$

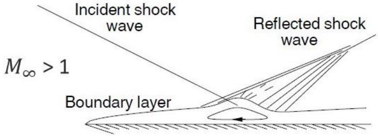
$$
\left(\mathcal{R}_x(W)\ \mathcal{R}_y(W)\ \mathcal{R}_z(W)\right) = \begin{pmatrix} \vec{0}^{\,T} \\ \left(\tau\cdot\vec{e}_x\right)^{T} \\ \left(\tau\cdot\vec{e}_y\right)^{T} \\ \left(\tau\cdot\vec{e}_z\right)^{T} \\ \left(\tau\cdot\vec{v} + \kappa\nabla T\right)^{T} \end{pmatrix}
$$

$$
\vec{e}_x^{\,T} \;=\; (1\ 0\ 0), \quad \vec{e}_y^{\,T} = (0\ 1\ 0), \quad \vec{e}_z^{\,T} = (0\ 0\ 1), \quad \vec{0}^{\,T} = (0\ 0\ 0)
$$

- The Navier-Stokes equations are named after Claude-Louis Navier (French engineer) and George Gabriel Stokes (Irish mathematician and physicist)
- They are generally accepted as an adequate description for aerodynamic flows at standard temperatures and pressures
- Because of mesh resolution requirements however, they are practically useful "as is" only for laminar viscous flows, and low Reynolds number turbulent viscous flows
- Today, mathematicians have not yet proven that in three dimensions solutions always exist, or that if they do exist, then they are smooth
- The above problem is considered one of the seven most important open problems in mathematics: the Clay Mathematics Institute offers $ 1,000,000 prize for a solution or a counter-example

Motivations

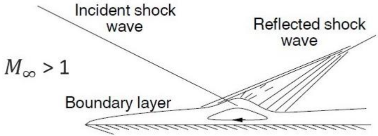
- Consider the flow graphically depicted in the figure below



  - an oblique shock wave impinges on a boundary layer
  - the adverse pressure gradient ($dP/ds > 0$) produced by the shock can propagate upstream through the subsonic part of the boundary layer and, if sufficiently strong, can separate the flow forming a circulation within a separation bubble
  - the boundary layer thickens near the incident shock wave and then necks down where the flow reattaches to the wall, generating two sets of compression waves bounding a rarefaction fan, which eventually form the reflected shockwave

Motivations

- Consider the flow graphically depicted in the figure below (continue)



  - the Navier-Stokes equations describe well this problem
  - but at Reynolds numbers of interest to aerodynamics (high R_e), their practical discretization cannot capture adequately the inviscid-viscous interactions described above
  - today, this problem and most turbulent viscous flow problems of interest to aerodynamics require turbulence modeling to represent scales of the flow that are not resolved by practical grids
  - the Reynolds-Averaged Navier–Stokes (RANS) equations are one approach for modeling a class of turbulent flows

Approach

- The RANS equations are time-averaged equations of motion for fluid flow

$$W \rightarrow \overline{W} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{t^0}^{t^0+T} W \, dt$$

- The main idea is to decompose an instantaneous quantity into time-averaged and fluctuating components

$$W = \underbrace{\overline{W}}_{time-averaged} + \underbrace{W'}_{fluctuation}$$

- The substitution of this decomposition (first proposed by the Irish engineer Osborne Reynolds) into the Navier-Stokes equations, the averaging of the resulting equations and the injection in them of various approximations based on knowledge of the properties of flow turbulence lead to a *closure* problem induced by the arising non-linear Reynolds stress term

$$R_{ij} = -\overline{v_i' v_j'} \qquad \left(-\frac{\bar{p}}{\rho} + \nu \left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i} \right) - \overline{v_i' v_j'} \right)$$

- Additional modeling of R_{ij} is therefore required to close the RANS equations, which has led to many different *turbulence models*

Approach

- Many of these turbulence models are based on
  - the Boussinesq assumption $R_{ij} = R_{ij}(\nu_t)$ — that is, on assuming that the additional turbulence stresses are given by augmenting the laminar molecular viscosity μ with a (turbulence) *eddy* viscosity μ_t (which leads to augmenting the laminar kinematic molecular viscosity ν with a (turbulence) kinematic eddy viscosity ν_t) (see Eq. (1))
  - a parameterization $\nu_t = \nu_t(\chi_1, \cdots, \chi_m)$
  - additional transport equations similar to the Navier-Stokes equations for modeling the dynamics of the parameters $\chi_1, \cdots, \chi_m$

Equations

- In any case, whatever RANS turbulence model is chosen, W is augmented by the m parameters of the chosen turbulence model (usually, $m = 1$ or 2)

$$W_{aug} \leftarrow \left(\rho \ \rho\vec{v}^{\,T} \ E \ \chi_1 \ \cdots \ \chi_m\right)^T$$

and the standard Navier-Stokes equations are transformed into the RANS equations which have the same form but are written in terms of $\overline{W}$ and feature a source term S that is turbulence model dependent

$$\frac{\partial \overline{W}}{\partial t} + \vec{\nabla} \cdot \vec{\mathcal{F}}(\overline{W}) = \vec{\nabla} \cdot \vec{\mathcal{R}}(\overline{W}) + S(\overline{W}, \ \chi_1, \ \cdots, \ \chi_m)$$

- The fluid of interest is inviscid (or the viscous effects are negligible)
- There are no thermal conduction effects (or they are negligible)

$$\Longrightarrow \left\{ \begin{array}{l} \mu = 0 \Rightarrow \tau = 0 \\ \kappa = 0 \end{array} \right\} \Rightarrow \vec{\mathcal{R}}(W) = 0$$

- Eulerian setting
- Dimensional form

$$\frac{\partial W}{\partial t} + \vec{\nabla} \cdot \vec{\mathcal{F}}(W) = (0\ \vec{0}\ 0)^T$$

- One continuity equation, three momentum equations and one energy equation

- The Euler equations are named after Leonhard Euler (Swiss mathematician and physicist)
- Historically, only the continuity and momentum equations have been derived by Euler around 1757, and the resulting system of equations was underdetermined except in the case of an incompressible fluid
- The energy equation was contributed by Pierre-Simon Laplace (French mathematician and astronomer) in 1816 who referred to it as the adiabatic condition
- The Euler equations are *nonlinear hyperbolic* equations and their general solutions are waves
- Waves described by the Euler equations can break and give rise to shock waves

- Mathematically, this is a nonlinear effect and represents the solution becoming multi-valued
- Physically, this represents a breakdown of the assumptions that led to the formulation of the differential equations
- Weak solutions are then formulated by working with jumps of flow quantities (density, velocity, pressure, entropy) using the Rankine-Hugoniot shock conditions
- In real flows, these discontinuities are smoothed out by viscosity
- Shock waves with Mach numbers just ahead of the shock greater than 1.3 are usually strong enough to cause boundary layer separation and therefore require using the Navier-Stokes equations
- Shock waves described by the Navier-Stokes equations would represent a jump as a smooth transition — of length equal to a few mean free paths [1] — between the same values given by the Euler equations

[1] The mean free path is the average distance over which a moving particle (such as an atom, a molecule, or a photon) travels before substantially changing its direction or energy (or, in a specific context, other properties), typically as a result of one or more successive collisions with other particles.

- Flow is isentropic
  $\Longrightarrow$ flow contains weak (or no) shocks and with peak Mach numbers below 1.3

- **And** flow is irrotational – that is, $\vec{\nabla} \times \vec{v} = 0$
  $\Longrightarrow \vec{v} = \vec{\nabla}\Phi$, where Φ is referred to as the velocity potential
  $$\Longrightarrow \left\{ \begin{array}{c} \vec{\nabla} \times \vec{\nabla}\Phi = 0 \\ \vec{\nabla} \times \vec{v} = 0 \end{array} \right.$$
  $\Longrightarrow$ not suitable in flow regions where vorticity is known to be important (for example, wakes and boundary layers)

- Steady flow (but the potential flow approach equally applies to unsteady flows)
  - from the isentropic flow conditions ($p/\rho^\gamma = \text{cst}$) and $\vec{v} = \vec{\nabla}\Phi$, it follows that

$$
\begin{aligned}
T &= T_\infty \left[1 - \frac{\gamma-1}{2} M_\infty^2 \left(\frac{\frac{\partial\Phi}{\partial x}^2 + \frac{\partial\Phi}{\partial y}^2 + \frac{\partial\Phi}{\partial z}^2}{\|\vec{v}_\infty\|^2} - 1 \right) \right] \\
p &= p_\infty \left[1 - \frac{\gamma-1}{2} M_\infty^2 \left(\frac{\frac{\partial\Phi}{\partial x}^2 + \frac{\partial\Phi}{\partial y}^2 + \frac{\partial\Phi}{\partial z}^2}{\|\vec{v}_\infty\|^2} - 1 \right) \right]^{\frac{\gamma}{\gamma-1}} \\
\rho &= \rho_\infty \left[1 - \frac{\gamma-1}{2} M_\infty^2 \left(\frac{\frac{\partial\Phi}{\partial x}^2 + \frac{\partial\Phi}{\partial y}^2 + \frac{\partial\Phi}{\partial z}^2}{\|\vec{v}_\infty\|^2} - 1 \right) \right]^{\frac{1}{\gamma-1}}
\end{aligned}
$$

- Steady flow (continue)
  - non-conservative form (see later)

$$\boxed{\begin{aligned}&\left(1 - M_x^2\right)\frac{\partial^2\Phi}{\partial x^2} + \left(1 - M_y^2\right)\frac{\partial^2\Phi}{\partial y^2} + \left(1 - M_z^2\right)\frac{\partial^2\Phi}{\partial z^2}\\ &- 2M_xM_y\frac{\partial^2\Phi}{\partial x\,\partial y} - 2M_yM_z\frac{\partial^2\Phi}{\partial y\,\partial z} - 2M_zM_x\frac{\partial^2\Phi}{\partial z\,\partial x} = 0\end{aligned}}$$

where

$$M_x = \frac{1}{c}\frac{\partial\Phi}{\partial x},\;\; M_y = \frac{1}{c}\frac{\partial\Phi}{\partial y},\;\; M_z = \frac{1}{c}\frac{\partial\Phi}{\partial z}$$

are the local Mach components and

$$c = \sqrt{\frac{\gamma p}{\rho}}$$

is the local speed of sound

  - compare the above equation to the Euler equation

$$\frac{\partial W}{\partial t} + \vec{\nabla}\cdot\vec{\mathcal{F}}(W) = (0\;\vec{0}\;0)^T$$

- Steady flow (continue)
  - conservative form (see later)

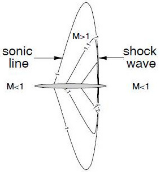
$$\boxed{\frac{\partial\left(\rho\frac{\partial\Phi}{\partial x}\right)}{\partial x}+\frac{\partial\left(\rho\frac{\partial\Phi}{\partial y}\right)}{\partial y}+\frac{\partial\left(\rho\frac{\partial\Phi}{\partial z}\right)}{\partial z}=0}$$

where

$$\rho=\rho_\infty\left[1-\frac{\gamma-1}{2}M_\infty^2\left(\frac{\frac{\partial\Phi}{\partial x}^2+\frac{\partial\Phi}{\partial y}^2+\frac{\partial\Phi}{\partial z}^2}{\|\vec{v}_\infty\|^2}-1\right)\right]^{\frac{1}{\gamma-1}}$$

Additional Assumptions

- Uniform free-stream flow *near Mach one* (say $0.8 \leq M_\infty \leq 1.2 \Rightarrow$ transonic regime)
- Thin body and small angle of attack



$\Longrightarrow$ flow slightly perturbed from the uniform free-stream condition

$$\Longrightarrow \vec{v} = \|\vec{v}_\infty\| \vec{e}_x + \vec{\nabla}\phi$$

where ϕ – **which is not to be confused with Φ**[2] – is referred to as the small-perturbation velocity potential

$$v_x = \|\vec{v}_\infty\| + \frac{\partial \phi}{\partial x}, \qquad v_y = \frac{\partial \phi}{\partial y}, \qquad v_z = \frac{\partial \phi}{\partial z}$$

$$\left|\frac{\partial \phi}{\partial x}\right| << \|\vec{v}_\infty\|, \qquad \left|\frac{\partial \phi}{\partial y}\right| << \|\vec{v}_\infty\|, \qquad \left|\frac{\partial \phi}{\partial z}\right| << \|\vec{v}_\infty\|$$

[2] It can be easily shown that $\Phi = \phi + \|\vec{v}_\infty\| x$

Small-Perturbation Potential Equation

- Steady flow (but the small-perturbation velocity potential approach equally applies to unsteady flows)

$$\boxed{\left(1 - M_\infty^2 - (\gamma+1)M_\infty^2 \frac{\frac{\partial \phi}{\partial x}}{\|\vec{v}_\infty\|}\right)\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0}$$

- The leading term of the above equation cannot be simplified in the transonic regime ($0.8 \leq M_\infty \leq 1.2$)
- The velocity vector is obtained from $\vec{v} = \|\vec{v}_\infty\|\vec{e}_x + \vec{\nabla}\phi$ and the pressure and density from the first-order expansion of the second and third isentropic flow conditions as in the previous case
- The temperature is obtained from $T = T(p, \rho)$ and the total energy per unit mass is obtained from $e = e(T)$

Some Noteworthy Facts

- In the transonic regime, the small-perturbation potential equation is also known as the "transonic small-disturbance equation"
- It is a nonlinear equation of the mixed type
  - elliptic if
$$\left(1 - M_\infty^2 - (\gamma+1)M_\infty^2 \frac{\frac{\partial \phi}{\partial x}}{\|\vec{v}_\infty\|}\right) > 0$$
  - hyperbolic if
$$\left(1 - M_\infty^2 - (\gamma+1)M_\infty^2 \frac{\frac{\partial \phi}{\partial x}}{\|\vec{v}_\infty\|}\right) < 0$$

Additional Assumptions (revisited)

- Uniform free-stream flow ~~*near Mach one* (say $0.8 \leq M_\infty \leq 1.2$)~~
  $\Rightarrow$ subsonic or supersonic regime
- If supersonic, preferrably when $1.2 < M_\infty < 1.3$ (why?)
- Thin body and small angle of attack $\Longrightarrow$ flow slightly perturbed from the uniform free-stream condition

$$\Longrightarrow \vec{v} = \|\vec{v}_\infty\| \vec{e}_x + \vec{\nabla}\phi$$

where ϕ is referred to as the small-perturbation velocity potential

$$v_x = \|\vec{v}_\infty\| + \frac{\partial \phi}{\partial x}, \qquad v_y = \frac{\partial \phi}{\partial y}, \qquad v_z = \frac{\partial \phi}{\partial z}$$

$$\left|\frac{\partial \phi}{\partial x}\right| << \|\vec{v}_\infty\|, \qquad \left|\frac{\partial \phi}{\partial y}\right| << \|\vec{v}_\infty\|, \qquad \left|\frac{\partial \phi}{\partial z}\right| << \|\vec{v}_\infty\|$$

$$\Longrightarrow \left(1 - M_\infty^2 - (\gamma+1) M_\infty^2 \frac{\frac{\partial \phi}{\partial x}}{\|\vec{v}_\infty\|}\right) \approx \left(1 - M_\infty^2\right)$$

Linearized Equation

- Steady flow

$$\boxed{\left(1 - M_\infty^2\right) \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0}$$

Linearized Equation

- Steady flow (continue)
  - the velocity vector is obtained from $\vec{v} = \|\vec{v}_\infty\|\vec{e}_x + \vec{\nabla}\phi$ and the pressure and density from the first-order expansion of the second and third isentropic flow conditions as follows

$$\begin{aligned} p &= p_\infty\left(1 - \gamma M_\infty^2 \frac{\frac{\partial\phi}{\partial x}}{\|\vec{v}_\infty\|}\right) \\ \rho &= \rho_\infty\left(1 - M_\infty^2 \frac{\frac{\partial\phi}{\partial x}}{\|\vec{v}_\infty\|}\right) \end{aligned}$$

  - the temperature is obtained from $T = T(p, \rho)$ and the total energy per unit mass from $e = e(T)$

Some Noteworthy Facts

- The linearized small-perturbation potential equation

$$\boxed{\left(1 - M_\infty^2\right) \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0}$$

  is much easier to solve than the nonlinear transonic small-perturbation potential equation, or the nonlinear full potential equation: it can be recast into Laplace's equation using the simple coordinate stretching in the $\vec{e}_x$ direction

$$\tilde{x} = \frac{x}{\sqrt{(1 - M_\infty^2)}} \qquad \text{(subsonic regime)}$$