Estimation of thermal noise in the mirrors of laser interferometric gravitational wave detectors: Two point correlation function

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A general formula and a computational scheme for estimating the power spectrum of the displacement correlation function of suspended test masses such as those used in interferometric gravitational wave detectors are presented. Unlike previous mode-summation approaches, the fluctuation-dissipation theorem has been applied directly to the displacement correlation. The resulting formula expresses the correlation in terms of material damping parameters and mechanical Green's functions, and provides an efficient and flexible method to compute thermally induced surface displacements of arbitrarily shaped anisotropic elastic bodies. The formula can be used for optimizing the shape and size of test masses in gravitational wave receivers. A simple one-dimensional example is included to clarify the relationship with the modal expansion approach and to illustrate the advantage of the Green's function method. This paper presents the theoretical formulation; numerical evaluations of the formula will be presented elsewhere. © *1997 American Institute of Physics.* [S0034-6748(97)01309-9]

I. INTRODUCTION

There are several programs around the world aiming to detect astrophysical gravitational waves by laser interferometry.[1] The basic idea is to measure the relative displacements of suspended mirrors induced by the passage of gravitational waves using a highly sensitive laser interferometer. Thermally induced surface vibration is a major noise source in such a detector.[2,3] It is crucial, therefore, to understand this important noise source thoroughly and to be able to calculate the noise for different test mass shapes and sizes. The work presented here is motivated by the need for accurate modeling of thermal noise in practical gravitational wave detectors. Specifically, our objective is to establish a general formula and a computational scheme, by which thermally induced elastic vibrations of arbitrarily shaped anisotropic bodies can be computed accurately and efficiently.

Previous authors[2,3] have used an approach based on a modal expansion in terms of the vibrational modes, and applied the fluctuation-dissipation theorem[4–6] to each mode. To recapitulate Saulson's result,[2] let us consider a simple example of a thin elastic rod of length l and linear density $\overline{\rho}$. Let the symmetry axis of the thin rod be the x axis, and the x coordinates of the two ends be 0 and l. For an infinite thin strip or an infinite thin cylinder, the lowest symmetric propagating mode is accurately approximated by a longitudinal displacement parallel to the bounding surfaces and uniform over the cross-section, when the thickness or diameter is a small fraction of the longitudinal wavelength.[7] Under this condition, the problem becomes essentially one-dimensional, and Saulson's result for the power spectrum $\langle u(l)u(l)\rangle_\omega$ of the longitudinal displacement thermal fluctuation $u(l)$ at the rod end l holds, namely

$$\langle u(l)u(l)\rangle_\omega=\frac{2k_BT}{\omega}\cdot\sum_{n>0}\frac{\gamma_n}{(\omega_n^2-\omega^2)^2+\gamma_n^2}\cdot\frac{[\psi_n(l)]^2}{\overline{\rho}},\qquad(1)$$

where k_B and T are the Boltzmann constant and the temperature. In (1), ω_n, γ_n, and $\psi_n(x)$ denote, respectively, the frequency, the damping parameter, and the modal function $\sqrt{2/l}\cos k_nx$ of the nth mode, where $k_n=n\pi/l$, $n=1,2,\cdots$. Our γ_n is related to Saulson's ϕ_n via $\gamma_n=\phi_n\omega_n^2$. We observe that Eq. (1) can be written as

$$\langle u(l)u(l)\rangle_\omega=\frac{2k_BT}{\omega}\cdot\text{Im }G^\omega(l,l),\qquad(2)$$

in terms of an elastic Green's function G^ω where[8]

$$G^\omega(x_1,x_2)\equiv\sum_n\frac{1}{\overline{\rho}}\psi_n(x_1)G_n^\omega(\gamma_n)\psi_n(x_2),$$
$$G_n^\omega(\gamma_n)\equiv(\omega_n^2-\omega^2-i\gamma_n)^{-1}.\qquad(3)$$

From the definition (3), it follows directly that

$$\text{Im }G_n^\omega=G_n^\omega\cdot\gamma_n\cdot G_n^{\omega *}.\qquad(4)$$

In this paper, we will derive the generalization of Eq. (2) for three-dimensional elastic bodies by following the derivation of Ref. 4. In particular, our derivation does not use any modal expansion. In addition, using an analog of the optical theorem, we derive a mode-independent generalization of Eq. (4) which allows computations of Eq. (2) in terms of physical (nonmodal) Green's functions and damping parameters.

The mode-independent expression (2) has advantages over Eq. (1) from both the formulational and calculational point of view. Besides the difficulty in calculating modes for complex mirror and suspension shapes, the modal approach is inconvenient to our problem where typical operating frequencies of gravitational wave receivers are much lower than the modal frequencies. This frequency mismatch forces one to sum over many terms in Eq. (1), each detuned by many linewidths. Another problem is that, since the above damping parameters γ_n are mode-dependent quantities, experimentally determined γ_n's for one specimen geometry are not directly transferable to another. In contrast, the method we present in this paper avoids these problems.

II. FORMULATION

This paper involves linear elasticity of anisotropic materials,[7,9] for which we will generally follow the notation of Ref. 7 and refer to the displacement u_i, the linearized strain S_{ij}, the stress T_{ij}, and the material parameters such as stiffness c_{ijkl}, compliance s_{ijkl}, and viscosity η_{ijkl}. There are however two exceptions. One is that we formulate the lossless anisotropic linear elasticity by starting with the Lagrangian[8]

$$L=\int_V dV\left(\frac{\rho}{2}\dot{u}_i^2-\frac{1}{2}(\partial_i u_j)c_{ijkl}(\partial_k u_l)\right), \tag{5}$$

where ρ is the mass density. The existence of the Lagrangian (5) is important because the resulting canonical formalism ensures the validity of the standard results, particularly the fluctuation-dissipation theorem.[4–6] The second exception concerns the stress-strain relationship, and here we begin with the generalized linear compliance relation

$$S_{ij}(t)=\int_{-\infty}^{\infty}dt'\ s_{ijkl}(t-t')T_{kl}(t'). \tag{6}$$

The time-dependence of the compliance s_{ijkl} accounts for possible time delays of a stress-induced strain by way of the generalized susceptibility.[4,6] Similarly, the stiffness c_{ijkl} is regarded as time-dependent, unless stated otherwise. The usual relationship between s_{ijkl} and c_{ijkl} can be recovered in the frequency domain. Namely, the Fourier transform of Eq. (6) is the algebraic relation $S_{ij}^{\omega}=s_{ijkl}(\omega)T_{kl}^{\omega}$, and its inverse is the generalized Hooke's law $T_{ij}^{\omega}=c_{ijkl}(\omega)S_{kl}^{\omega}$. The generalized stiffness $c_{ijkl}(\omega)$ is complex in general, and we use the standard notation for its real and imaginary parts,[6] namely

$$c_{ijkl}(\omega)\equiv c'_{ijkl}(\omega)-ic''_{ijkl}(\omega). \tag{7}$$

Equations (6) and (7) include, as a special case, the simple lossless stress-strain relationship where c'_{ijkl} are ω-independent constants and $c''_{ijkl}=0$.[7,9] In dispersive media, $c''_{ijkl}(\omega)$ no longer vanish, and both $c'_{ijkl}(\omega)$ and $c''_{ijkl}(\omega)$ become ω-dependent in general. For instance, viscosity-induced c'' can be parametrized as $c''_{ijkl}(\omega)=\omega\eta_{ijkl}$, where the power of ω corresponds to a time-derivative of S in the time domain.[7,9] It is important to note, however, that the c'' of many materials does not vanish at low frequencies $\omega\sim 0$.[2,10,11] Loss of this type is called structural (or hysteretic), and, when present, becomes dominant at $\omega\sim 0$. With these generalizations, most results in Ref. 7 regarding losses remain valid after the formal substitution $\eta\rightarrow c''/\omega$. Our goal is to relate the loss parameters to the power spectrum of the displacement correlation function

$$\langle u_i(\tilde{x}_1)u_j(\tilde{x}_2)\rangle_{\omega}\equiv\int_{-\infty}^{\infty}dt\ e^{i\omega t}\langle u_i(\tilde{x}_1,t)u_j(\tilde{x}_2,0)\rangle \tag{8}$$

between vibrations at different mirror points. Because the laser interferometer is sensitive to surface vibrations, we are mainly concerned with the correlation between two surface points, $\tilde{x}_1$ and $\tilde{x}_2$ in Eq. (8).

We will derive the mode-independent generalization of Eq. (4) first. Basically, we evaluate mechanical surface vibrations of a damped elastic body driven by an external traction. Let us consider the monochromatic driving traction $\mathrm{Re}(T_{in}^{\omega}e^{-i\omega t})$, where $T_{in}^{\omega}=T_{ij}^{\omega}n_j$ is assumed real, which acts on the body through its bounding surface S. The dynamics in question are governed by the field equation

$$-\rho\omega^2u_i^{\omega}-\partial_j c_{ijkl}(\omega)\partial_k u_l^{\omega}=0, \tag{9}$$

which can be derived from Eq. (5), except that the constants c_{ijkl} are replaced by the complex $c_{ijkl}(\omega)$ of Eq. (7). Equation (9) contains neither body forces nor the traction. The traction, instead, should be introduced through the boundary conditions. Since the system is linear, there must exist a response function χ_{ij} so that

$$u_i^{\omega}(\mathbf{x})=\int_S dS_1\chi_{ij}^{\omega}(\mathbf{x},\tilde{x}_1;c)T_{jn}^{\omega}(\tilde{x}_1), \tag{10}$$

where $\mathbf{x}$ may be either a point in the object volume V, or a surface point $\tilde{x}$. In the lossless case, the retardation condition on χ_{ij} should be enforced by the correct choice of ω poles as usual. We now state the generalization of Eq. (4), namely

$$\mathrm{Im}\chi_{ij}^{\omega}(\tilde{x}_1,\tilde{x}_2;c)=\int_V dV[\partial_k\chi_{li}^{\omega}(\mathbf{x},\tilde{x}_1;c)]c''_{klmn}(\omega)\times[\partial_m\chi_{nj}^{\omega}(\mathbf{x},\tilde{x}_2;c)]^{*}. \tag{11}$$

To prove Eq. (11), we use the power balance between external excitation and internal loss. For a lossy elastic medium, when the stationarity condition is met, the total traction-injected power should be balanced in average by the total internal power loss. To formulate the power balance explicitly, we use the acoustic Poynting's theorem [Eq. (5.38) of Ref. 7], insert the monochromatic fields such as $\mathbf{v}(=\mathrm{Re}\{-i\omega\mathbf{u}^{\omega}e^{-i\omega t}\})$ etc. into it, and perform time averages over the period $2\pi/\omega$. The volume energy term dU/dt drops out after averaging, while the body force term is absent in our problem. After eliminating $\mathbf{u}^{\omega}$ via Eq. (10) and expressing the remaining terms in T_{in}^{ω} and χ_{ij}^{ω}, we find that the balance equation takes the form

$$\int_S dS_1\int_S dS_2 T_{in}^{\omega}(\tilde{x}_1)K_{ij}(\tilde{x}_1,\tilde{x}_2)T_{jn}^{\omega}(\tilde{x}_2)=0, \tag{12}$$

where the kernel K is the difference between the left-hand side and the right-hand side of Eq. (11). Since T_{in}^{ω} is arbitrary, Eq. (12) implies that $K_{ij}(\tilde{x}_1,\tilde{x}_2)=0$, thus proving Eq. (11).

The proof of the fluctuation-dissipation theorem by Kubo[6] relies on the canonical formalism, and is in fact applicable straightforwardly to our elasticity problem, as guaranteed by the existence of the Lagrangian (5). A choice specific to our problem is the interaction Hamiltonian

$$H' = -\int_S dS\; u_i(\tilde{x},t) T_{in}(\tilde{x},t) \tag{13}$$

as a perturbation, which is traction-driven in accordance with Eq. (10). Starting with (13), we then follow Kubo's prescription, and evaluate the resulting shift in thermal fluctuation distribution, using statistical mechanics and time-reversal invariance. The result is the specific fluctuation-dissipation relation

$$\langle u_i(\tilde{x}_1) u_j(\tilde{x}_2)\rangle_\omega \approx \frac{2k_BT}{\omega} \mathrm{Im}\, \chi_{ij}^{\omega}(\tilde{x}_1,\tilde{x}_2;c), \tag{14}$$

which is the promised generalization of Eq. (2). It should be remarked that the approximate equality of Eq. (14) is strict only for infinitesimally small c'', and Eq. (14) has an error of $O((c''/c')^2)$. Possible quantum corrections to Eq. (14), which can be accounted for by replacing k_BT with $(\hbar\omega/2)\coth(\hbar\omega/2k_BT)$, are negligibly small for room temperature systems.

From Eqs. (11) and (14), it finally follows that

$$\begin{aligned}\langle u_i(\tilde{x}_1) u_j(\tilde{x}_2)\rangle_\omega &\approx 2k_BT/\omega \int_V dV [\partial_k \chi_{li}^{\omega}(\mathbf{x},\tilde{x}_1;c)] \\ &\quad \times c''_{klpq}(\omega)[\partial_p \chi_{qj}^{\omega}(\mathbf{x},\tilde{x}_2;c)]^* \\ &\approx 2k_BT/\omega \int_V dV [\partial_k \chi_{li}^{static}(\mathbf{x},\tilde{x}_1;c)] \\ &\quad \times c''_{klpq}(\omega)[\partial_p \chi_{qj}^{static}(\mathbf{x},\tilde{x}_2;c)],\end{aligned} \tag{15}$$

where $\chi_{ij}^{static} \equiv \chi_{ij}^{\omega=0}$. The second line of Eq. (15) is valid approximately under the quasi-static condition, namely when ω is significantly smaller than any of the modal frequencies ω_n. Equation (15) constitutes the main result of this paper, and expresses the spectral displacement correlation (representing the level of thermal fluctuations) in terms of the elastic Green's function and the dispersive parts of the elastic constants. From the earlier discussion of viscous and structure losses, it should be noted that (15) exhibits a constant behavior of the thermal fluctuations at low frequencies ω for the viscous case where $c''_{ijkl}/\omega = \eta_{ijkl}$, while a $1/\omega$ increase applies for the structural case. The $1/\omega$ behavior was expected in various contexts,[2,10–12] and in fact observed experimentally.[13]

III. COMPUTATIONAL PROCEDURE AND EXAMPLE

Given Eq. (15), we present the following prescription for computing the power spectrum (8): First, the response function χ_{ij} should be computed, presumably by numerical methods because practical mirrors and suspensions may have complicated shapes. The best numerical algorithm, we believe, is the boundary element method (BEM).[14,15] This method solves the boundary integral equations numerically, and is applicable to anisotropic, piece-wise uniform elasticity. In fact, one of the earliest applications of the BEM was linear elasticity. Second, the quasi-static condition is likely to hold and thus should be exploited in practical cases, requiring that only χ^{static} be computed numerically. This simplifies the computation, since the response function is then ω-independent. Third, the experimental values of c_{ijkl} should be used for the actual mirror and suspension. The materials under consideration for interferometric gravitational wave detectors include fused silica (isotropic), silicon (cubic), and sapphire (trigonal), which require, respectively, two, three, and six components of c_{ijkl}.[7] Both structural and viscous losses may need to be considered. Consequently, the ω dependence of Eq. (15) comes only through the combination $c''_{ijkl}(\omega)/\omega$, where $c''_{ijkl}(\omega)$ are the elastic loss parameters.[2,10,11] Fourth, if necessary, microscopic material noises such as those due to dislocations may be included as a position-dependent c_{ijkl}. The resulting material inhomogeneity can be treated by perturbation theory.

Before concluding this section, we will revisit the one-dimensional example described in Sec. I to illustrate the use of our result. Our formula (15) requires the traction-free Green's function in the interval $0 \leqslant x \leqslant l$,[8]

$$\begin{aligned}\chi_{xx}^{\omega}(x_1,x_2) = -(\overline{\rho v}^2)^{-1}\{&\cos[k(l-|x_1-x_2|)] \\ &+\cos[k(l-x_1-x_2)]\}/(2k\sin kl),\end{aligned} \tag{16}$$

where $k=\omega/v$ while v is the sound velocity. Equation (16) permits explicit evaluation of the integrals in (15), yielding the quasi-static result for comparison with Eq. (1)

$$\langle u_x(l) u_x(l)\rangle_\omega = \frac{2k_BT}{\omega}\cdot\frac{|\overline{c''}_{1111}(\omega)|}{|\overline{c}_{1111}|^2}\cdot\frac{l}{3}, \tag{17}$$

where $\overline{c}_{1111}$ is the normalized elastic constant c_{1111}, so that $\overline{c}_{1111}/\rho = c_{1111}/\rho$. The closed-form result (17) is further evidence in favor of our Green's function approach over the modal result (1). Notice in particular that Eq. (17) requires only one complex elastic constant c_{1111} as input, while Eq. (1) needs many modal constants ω_n and γ_n. For more complicated elastic body shape, several complex elastic constants will be required, but these are all directly determined by the choice of material.

IV. DISCUSSIONS

The main contributions of this paper are Eq. (15) and the accompanying computational strategy described in the subsequent paragraph. More precisely, our basic results are Eqs. (11), (14), and (15) which generalize the previously known Eqs. (4), (2), and (1), respectively. In essence, our generalization has shown that the previous one-dimensional results are correct, and readily applicable to general three-dimensional mirrors and suspensions, except that the generalized formulas should be written in terms of a Green's function. Our Green's function approach is advantageous even for the simple one-dimensional model, since our approach yields a closed-form result (17) instead of the mode-sum result (1). Our master formula (15) is in fact written in terms of physical Green's functions and damping parameters, and allows computation of thermal excitations for any mirror shape and material. Technically, the predictive power of our approach is significant, when augmented by any of the standard numerical methods for computing Green's functions for arbitrarily shaped bodies. Given a mirror material, one must first determine the physical elastic and damping parameters, either by estimation or by experiment with any conveniently shaped specimen. Once the parameters are given, then our computational procedure can predict thermal noise levels for any mirror object design. It can therefore play a useful role in design optimization of test masses in laser-interferrometric gravitational wave receivers.

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