IRREDUCIBILITY AND *ANTE REM* MATHEMATICAL STRUCTURES

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Abstract:
Structuralism, as a theory of mathematics, has received much attention over the last several years. One version of structuralism, Stewart Shapiro's ante rem structuralism, displays an interesting feature: structures, as Shapiro conceives of them, appear be highly resistant to philosophical reduction. An argument to this effect is included, and preceded by a brief overview of structuralism.

Despite the fact that we cannot reach out and touch numbers, or spill coffee on them, as Strawson would complain, we can still apparently refer to them. At least at first glance, the very purpose of numerals is nothing other than to denote numbers. More generally, we seem to have the capability to make mathematical statements that are true. The statement "7+5 = 12" has for most people the cognitive appeal of a truth, rather than simply a formal theorem. Embracing these appearances of truth and reference amounts to belief in a mathematical semantics. Such belief, however, commits us also to providing a reasonable epistemology: we must know of a thing in order to make reference to that thing. This too demands explanation.

1. Structuralism

One perspective with considerable currency at the moment is *mathematical structuralism*: the thesis that mathematics is the science not of some mystical objects, or formal manipulations, but rather the study of structures, which are abstract, causally inert entities. These structures are instantiated in myriad systems around us. According to (most) structuralists, construing structures as objects of some kind addresses the problem of semantics; cognitive recognition of their instantiations in the world addresses the problem of epistemology. Stewart Shapiro tells us that "[t]he philosophy of mathematics to be articulated...goes by the name "structuralism", and its slogan is "mathematics is the science of structure." The subject matter of arithmetic is the *natural-number structure*, the pattern common to any system of objects that has a distinguished initial object and a successor relation that satisfies the induction principle."[1]

Structuralism in mathematics is an entire family of views, differing in various aspects. The common ground is some variant of the following: structures are objects made up of featureless internal places

[1] Shapiro, 5.

and relations between them. The places are featureless in the sense of having no distinct properties of their own; their properties are exhausted by the relations that they bear to each other within the structure.

Charles Parsons explains structuralism as "the view that reference to mathematical objects is always in the context of some background structure, and that the objects involved have no more to them than can be expressed in terms of the basic relations of the structure."[2] Quine expressed similar views earlier, without the title to identify them: "Arithmetic is...all there is to number: there is no saying absolutely what the numbers are; there is only arithmetic."[3] The view is frequently traced to Paul Benacerraf's seminal article, "What Numbers Could Not Be":

> [N]umbers are not objects at all, because in giving the properties (that is, necessary and sufficient) of numbers you merely characterize an *abstract structure*—and the distinction lies in the fact that the "elements" of the structure have no properties other than those relating them to other "elements" of the same structure. ... To *be* the number 3 is no more and no less than to be preceded by 2, 1, and possibly 0...[4]

The defining feature here is that talk of individual mathematical objects is meaningless outside of the structure that they belong to. The thing that exists or not, that is talked about or not, is the entire structure itself. In arithmetic, the structure is that of the natural numbers; in real analysis, the structure is that of the real number line; in topology, the structure is that of topological spaces, and so on.

This can be contrasted with traditional Platonism, in which mathematical objects are usually thought to have individual substance or essence. Shapiro claims that

> [t]he Platonist view may be that one can state the *essence* of each number without referring to the other numbers. The essence of 2 does not invoke 6 or any other number... If this notion of independence could be made out, we structuralists would reject it. The essence of a natural number is its *relations* to other natural numbers. The subject matter of arithmetic is a single abstract structure...[5]

Shapiro himself is quite a realist, probably as much so as most Platonists, albeit about a very different sort of entity. Shapiro appropriates a bit of

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[2] Parsons, 272.

[3] Quine, *Ontological Relativity,* 45.

[4] Benacerraf, 70.

[5] Shapiro, 72.

terminology from the long standing debate on universals to describe two sorts of structuralism: A structuralism is *in re* if it claims that structures are ontologically dependent upon their instantiations. A structuralism is *ante rem* if it claims that structures are ontologically independent, "freestanding" entities that exist regardless of instantiation. Shapiro himself defends an *ante rem* structuralism, on the grounds that it is "more perspicuous and less artificial" than other options, including set theory and in re structuralisms.

My present purpose is not to argue for or against structuralism, though I think some form of it is largely correct, but to demonstrate an interesting fact about Shapiro's *ante rem* structures: specifically, that such structures seem immune to reduction in any metaphysically significant sense. The argument for this involves a different twist on a common problem, that of multiple reductions.

2. Structures

First, a word or two about structures. Shapiro presents a formal and axiomatic theory of structure resembling set theory. The technical details will be omitted here. A system is defined as "a collection of objects with certain relations. An extended family is a system of people with blood and marital relationships, a chess configuration is a system of pieces under spatial and "possible-move" relationships..." Systems instantiate structures: "A structure is the abstract form of a system, highlighting the interrelationships among the objects, and ignoring any features of them that do not affect how they relate to other objects in the system"[6] On the *ante rem* approach, all structures are additionally systems, and instantiate themselves. For example, the natural number structure itself instantiates the natural number structure. Notice that structures have no features other than those that are internal and relational. A structure is merely a collection of places and certain relations defined on those places. Notably, there are no one-place relations defined on any of the places of the structure, since "features... that do not affect how they relate to other objects in the system" are ignored. Eliminative varieties of structuralism, as outlined by Shapiro, would typically accept systems and deny structures as separate entities. Discourse that appears to be about structures directly (such as pure arithmetic) is to be interpreted as universally quantified statements about systems that instantiate such structures. Structures are eliminated in favor of sets that instantiate them. Modal structuralisms accomplish the same goal by speaking of possible systems instead of actual structures.

Since the topic of this paper is structures and reduction, it should be noted now that this elimination is different from the sort of reduction to be discussed. Eliminative structuralists do not "reduce"

[6] Shapiro, 74.

structures to sets in the sense employed below. Like *in re* realists about universals, they hold that a structure exists if and only when it is instantiated. Because mathematical statements are to be glossed as universally quantified statements about the instantiations, they have no need to identify a given structure with any particular set; it is sufficient that the structure be instantiated somewhere in the set-theoretic universe and that the statement in question be true for all of the instantiations. The important difference is that eliminative structuralists make incisive changes to *discourse*, not ontology. A similar point is holds for modal structuralists. The same paraphrasis from singular propositions about structures to general propositions about systems is used, as above; additionally, the statements are reinterpreted as modal propositions about the systems. The purpose of this is to avoid a commitment to the systems themselves, and talk only about possibilities. [7] In this case, the modal operator itself is taken as primitive, so "possible structures" are not countenanced. This is one of Shapiro's examples of trading heavier ideology[8] for lighter ontology.

Neither of these programs respect the surface syntax of much of mathematics: simple statements of arithmetic certainly appear to contain genuine singular terms (numerals) and not to contain modal operators. Proponents of these views manage to dispense with structures, but at the cost of an unnatural reinterpretation of the mathematical discourse. When I assert that "2 is the only even prime number," it does not feel as though I am making a universally quantified statement over I know not what. While it is not decisive, this problem of naturalness is one reason why Shapiro prefers the *ante rem* option: it "comes closest to capturing how mathematical theories are conceived." The other reason is that he believes on a fundamental level, all three options are equivalent:

> In a sense, they all say the same thing, using different primitives. The situation with structuralism is analogous to that of geometry. Points can be primitive, or lines can be primitive. It does not matter because, in either case, the same structure is delivered. The same goes for structuralism itself.[9]

All three options use the notions of system and isomorphism together with some other primitive. Eliminative structuralism employs sets, modal structuralism (usually) uses a primitive operator, and *ante rem*

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[7] The phrase "possible system" would make for smoother prose, but implies a *de re* interpretation of the modality, which modal structuralists might not want, and need not accept.

[8] Shapiro here means "ideology" in something like Quine's sense of the term: "the range of severally expressible ideas" in a given language. See Quine, *From A Logical Point of View,* p. 131, and also Quine, *Ontology and Ideology*.

[9] Shapiro, 97.

structuralism takes structures as basic. In any case, all three deliver the same "structure of structures" and are therefore thought to be equivalent. If we agree with Shapiro on this, then we might as well adopt an *ante rem* approach, since it gives a more natural treatment of mathematical language and it is equivalent to the other two anyway. (Hardliners might object that this argument is simply a reification of *ante rem* structures, but this is tangential to the present topic.)

To recapitulate: eliminative and modal structuralisms reinterpret mathematics, suggesting that mathematical statements have the "deep" syntactic structure of universal statements about systems. *Ante rem* structuralism accepts the syntax of mathematical discourse, by accepting structures themselves. Bracketing Shapiro's claim that these three theories are equivalent, the remainder of the present paper deals with *ante rem* structuralism and structures.

3. Reduction

It is common for philosophers to talk about reducing some problematic notion or class of entities to something less problematic, such as the nominalistic reduction of properties to extensions or resemblance classes. It is important to note that these sorts of reductions (and there are others) yield identity statements; e.g., if we reduce a certain property a to a certain resemblance class b, we expect that $a = b$. Call this general version the *reduction-identity principle*: If the Fs are reducible to the Gs, then for all x such that Fx, there is some y such that Gy and $x = y$. (G may of course be some fairly complex property.) This is clearly somewhat rough. Not every sort of philosophical reduction fits this mold: a reduction of metaphysical modality to quantification `a la David Lewis clearly does not, since the notion to be reduced appears as an operator rather than a singular term or predicate. But this is a question of syntax and grammar, not metaphysics. It is still expected that expansion of quantification to other-worldly items somehow is the modality in question. The quantification does more than just play the role of modality. I take it to be a feature of philosophical reductions of this kind that they deliver more than just surrogates; in the appropriate cases, they deliver identity statements.

With this in mind, it might be thought desirable as a matter of ontological housekeeping to pursue such a reduction within *ante rem* structuralism, given its commitment to structures as independent, "freestanding" objects. This way, we could accept structures so as to preserve the surface syntax of mathematics, singular terms and all, while rendering the structures themselves innocuous. However, the problem of multiple reductions, as it is often called, introduces problems for the reductionist.

Obviously, a few remarks about reduction are needed. The term "reduction" itself is overloaded. It can take many related, but differing, meanings. The topic of reduction deserves more space than I have to offer it, but there are a few characteristics of the kind of reduction in question that can be set out briefly. These characteristics, of course, do not amount to an analysis or definition of even one kind of reduction. Instead, they are meant to be certain features of the kind of reduction we have in mind, whatever its nature may turn out to be.

We sometimes talk about reducing one sort of object to another sort of object, e.g., when we talk about reducing all of the *F*s to *G*s. Call the *F*s the *source* entities, and call the *G*s the *target* entities. Usually the target entities will be of a type that we had a prior commitment to, that has some explanatory primacy, is less epistemologically occult, or is more metaphysically basic or general. The reduction is designed to show two things: first, that a separate commitment to *F*s is redundant, and second, that we may continue talking about *F*s for convenience, with the understanding that we are really talking about some of the *G*s. Such is thought to be the case with set-theoretic reductions (the *G*s) of the concept of number (the *F*s). If numbers can be explained and modeled in terms of sets, we need only believe in the existence of sets; a separate belief in the existence of numbers is otiose. Or so the argument goes.

While this is a popular line of reasoning, it faces a strange difficulty, frequently called the problem of multiple reductions.[10] The problem is this: Suppose (as is the case) that Frege, Zermelo and von Neumann each offered differing, incompatible and individually sufficient reductions of the natural numbers to set theory. There are in fact infinitely many such reductions, as will become clear. Unfortunately, because they are all sufficient for the purposes of arithmetic (i.e., they all yield the natural number structure) there seems to be no way to make a principled decision between them. For example, if the natural number structure is to be *reduced* to the Zermelo series, identity must hold between the 2 of the natural number structure and the 2 of the Zermelo series, {{Ø}}. This precludes other reductions of the natural number structure, for example, a reduction to the von Neumann series. Identity with the natural number structure would be conferred by both reductions, and identity is transitive, so we wind up with identity holding between the Zermelo 2 and the von Neumann 2, which obviously cannot be: the von Neumann 2 is {Ø,{Ø}}. This is, in a nutshell, the problem of multiple reductions. Structuralism is widely understood to provide a (dis)solution for this problem. The natural number structure is instantiated by all of the above set-theoretic constructions, and there is little more to the matter than that. None of the three are to be preferred over the others, since all three exemplify the natural number structure equally. For an *ante rem* structuralist, the

[10] Benacerraf, "What Numbers Could Not Be," is the seminal treatment of this issue.

explanation can end there, since structures themselves are countenanced.

4. Irreducibility

There is however, another aspect to this problem and its structuralist solution. Suppose we agree with the *ante rem* structuralist. We might wish to adopt *ante rem* structuralism because of its perspicuity and naturalness; perhaps we find the extreme paraphrases of eliminative structuralisms unacceptable. At the same time, we might be underwhelmed by our new ontological commitment to a realm of structures. As a result, we might try to retain the theoretical framework of *ante rem* structuralism while providing an ontological reduction of structures themselves.[11] It is my contention that this move is doomed to fail. For any structure *S*, and any alleged reduction *R* of *S*, *R* must instantiate *S* in order to for the reduction to be successful. If *R* failed to instantiate *S*, there would be features (places, relations) of *S* that were not found in *R*; hence *R* would fail as a reduction of *S*. Where structures are concerned, instantiation is a necessary condition for reduction. However, instantiation is not *equivalent* to reduction, in the relevant sense; this is a lesson from the problem of multiple reductions. It is compatible for both the Zermelo and von Neumann ordinal systems to instantiate the natural number structure; in fact, they do. It is incompatible for both to reduce the natural number structure. Reduction of mathematical structures must amount to instantiation plus something else. Therefore, we can always ask whether *R* in fact *reduces S,* or merely *instantiates S*. We need some criteria by which we can tell the difference in any given case. Consider the following two claims:

1. *R* instantiates *S*;
2. *R* is a reduction of *S*.

Both the reductionist and the realist can agree on 1, but the reductionist will also assert 2, which the realist denies. Since reduction and instantiation are not equivalent, 1 alone is not sufficient for 2: reduction is instantiation plus something else. What could that something possibly be? We cannot discriminate between instantiations of *S* on the basis of any properties of *S*. Since *S* has only structural properties,[12] and by

[11] It is of course optimal to have one's cake and eat it too. This general strategy is not unheard of in cases where the source entities are adduced for their explanatory value, at some ontological cost. Consider the debate over Fregean senses. Senses are (for some people) theoretically desirable because of the elegance with which they explain certain well-known semantic puzzles. They are undesirable because we cannot say exactly what they are. If they could be reduced to some other class of entities, their theoretical benefit could be retained, at no significant ontological cost. One could plausibly take a similar attitude toward *ante rem* structures.

[12] It might be suggested here that S does in fact have some non-structural property, perhaps even essentially: the property of being a structure. However, such a proposal would

definition any instantiation has all of *S*'s structural properties, all of *S*'s properties are shared by all of its instantiations. Since there are no properties of *S* by which we can pick out the correct reduction, it might be thought that there are properties of the reductions themselves that could serve the purpose. For example, imagine an infinite series of small stones laid out in a spatial sequence. Each stone certainly has the properties of being extended and made of stone. The natural number 2 is neither extended nor made of stone; similarly, the natural number sequence is not a sequence of extended stones. Therefore, an infinite sequence of small stones can be ruled out as a possible reduction of the natural number sequence. While this is a relatively silly example, the reductionist may still claim some small headway in discriminating between reductions.

The problem here is again that the structure to be reduced, *S*, has no properties other than its structural properties, whereas any target entity proposed as a reduction of *S* will invariably have properties other than the structural properties shared with *S*. If this assertion seems suspicious, consider the following: if the target entity had no other properties than these, it would be indistinguishable from *S*. We could not realize it to be a distinct thing from *S*; it is difficult to see how we could ever be in a position to propose it as a reduction of S. Anything that could be a candidate for a reduction of *S* will have properties not shared by *S*. And, if identity is treated normally, it will fail for this reason. Again, suppose we reduce the natural number structure to a Zermelo series. This tells us that identity holds between the 2 of the natural numbers and the Zermelo 2, i.e., 2 = {{Ø}}. The Zermelo 2 has the property of being a set; by Leibniz's Law, it seems that the number 2 also has the property of being a set. It first appeared that the number 2 had only structural properties; it is difficult to see why we should accept that it also has the property of being a set. More importantly however, we cannot appeal to this property of the proposed reduction as a reason to favor it over other proposed reductions; we had no prior belief that the number 2 had the property of being a set. It is not as though we found something that we were looking for.

5. Conclusion

The foregoing should not be taken as an argument for the existence of structures, only for their irreducibility. That is, it follows from the definition of the term 'structure' that structures are irreducible. This feature is *not* shared with many other abstract entities proposed by

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wind up being either question-begging or flimsy. If the property of being a structure is taken as substantive or irreducible, then it begs the question against the reductionist. If it is taken in a theory-neutral sense, i.e., if it can be explained away by the reduction along with everything else, then it cuts no ice to begin with.

philosophers, such as properties or Fregean senses. Irreducibility is not entailed by the very concept of a property or a Fregean sense; therefore there is some hope of building a theory around such entities and later targeting them for reduction. In contrast, if we subscribe to *ante rem* structuralism, we take on a provisional belief in structures which we cannot later divest ourselves of. This sort of provisional belief is natural in the process of ontological reduction; Quine compares it to *reductio ad absurdum*, "where we assume a falsehood that we are out to disprove. If what we want to show is that the universe *U* is excessive... then we are quite within our rights to assume all of *U* for the space of the argument."[13] The problem with the case of structures, to continue Quine's analogy, is that we cannot close this branch of the proof; we cannot successfully rid ourselves of the supposition of structures. The cause of this difficulty is, as noted, the requirement that successful reductions yield identity statements in the normal, philosophical sense, between the source and the target entities. I believe that the reduction-identity principle is justified for anything deserving of the name "reduction" in the philosophical sense. If the process is indeterminate, it plays havoc with reference; if we cannot tell which set-theoretic reduction of the natural numbers is the correct one, we cannot specify the referent of numerals such as '2'. Any alleged reduction that leaves our language in such a state cannot be considered successful; it is incomplete, at best.

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[13] Quine, *Ontological Relativity*, 58.