Accuracy of Ritz values from a given subspace

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The rate of convergence of iterative methods has always been one of the interests of the Numerical Analysis group at Stanford. For example I read "On the asymptotic directions of the s-dimensional optimum gradient method" by George Forsythe (1968) while refereeing a paper for Gene Golub when I visited the group in 1972 as an early guest of Gene.

In this talk we generalize a well-known eigenvalue result. If $x, y \in \mathrm{C}^n$ are unit-length vectors ($x^H x = y^H y = 1$), where y is an approximation to an eigenvector x of $A = A^H \in \mathrm{C}^{n\times n}$ with $Ax = x\lambda$, $\lambda = x^H Ax \in \mathrm{R}$, then the Rayleigh quotient $y^H Ay$ satisfies

$$|\lambda - y^H Ay| \leq \sin^2 \theta(x, y).\mathrm{spread}(A). \tag{1}$$

Here, if $\lambda_1(A) \geq \cdots \geq \lambda_n(A)$ are the eigenvalues of A in descending order then $\mathrm{spread}(A) \equiv \lambda_1(A) - \lambda_n(A)$, and $\theta(x, y) \equiv \cos^{-1} |x^H y| \in [0, \pi/2]$ is the acute angle between x and y.

We generalize this result to a higher-dimensional subspace $\mathcal{Y}$ approximating an invariant subspace $\mathcal{X}$ of A. Let $X, Y \in \mathrm{C}^{n\times k}$ be such that $X^H X = Y^H Y = I_k$, where $\mathcal{Y} \equiv \mathrm{range}(Y)$ is an approximation to the invariant subspace $\mathcal{X} \equiv \mathrm{range}(X)$ of A, so that $AX = X.X^H AX$. Let $\lambda(X^H AX)$ and $\lambda(Y^H AY) \in \mathrm{R}^k$ be the vectors of eigenvalues in descending order of $X^H AX$ and $Y^H AY$ respectively. The elements of $\lambda(Y^H AY)$ are called Ritz values in the Rayleigh-Ritz method for approximating the eigenvalues $\lambda(X^H AX)$ of A. Such approximations can be computed for example via the Lanczos or block-Lanczos methods for the Hermitian eigenproblem. Here we obtain new bounds on $\lambda(X^H AX) - \lambda(Y^H AY)$ of a form paralleling (1). We then ask whether such results might contribute to useful "rate of convergence" analyses for iterative eigenproblem methods for large sparse matrices.