The University of Nottingham

SCHOOL OF MATHEMATICAL SCIENCES

Discontinuous Galerkin FEMs for CFD: A Posteriori Error Estimation and Adaptivity

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Funded by EPSRC

Papers available from: http://www.maths.nottingham.ac.uk/∼ph

Overview

1. Adaptive mesh generation.
2. Discontinuous Galerkin methods.
3. First–order hyperbolic conservation laws.
   - 3.1 Discretization.
   - 3.2 A posteriori error estimation.
   - 3.3 Numerical examples.
4. Second–order PDEs.
   - 4.1 Discretization.
   - 4.2 A posteriori error estimation.
   - 4.3 Numerical examples.
5. High–order/hp–adaptive DG methods.
6. Summary and outlook.

1. Adaptive Mesh Generation

Adaptive Mesh Generation

Let $\mathcal{L} : D(\mathcal{L}) \subset \mathcal{H} \to \mathcal{H}$. Given $f \in \mathcal{H}$, find $u \in D(\mathcal{L})$:

$$\boxed{\mathcal{L}u = f}$$

- *Assumption:* $\mathcal{L}$ is a *sufficiently* accurate physical model.
- Modelling error control
  Stein & Ohnimus 1999, Actis, Szabo, & Schwab 1999, Oden & Vemaganti 2000, 2001, Fatone, Gervasio, & Quarteroni 2001, Braack & Ern 2004, Hartmann & H. (in preparation)

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$\Rightarrow$ Finite dimensional problem

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Given a finite dimensional space S_N, find $u_N \in S_N$ such that

$$\boxed{\Pi_N \, \mathcal{L} u_N = \Pi_N \, f}$$

Galerkin Orthogonality: $\Pi_N \underbrace{(f - \mathcal{L}u_N)}_{\text{Residual}} \equiv \Pi_N \, r_N = 0.$

Adaptive Mesh Generation

1. Given a metric $\mathcal{D}(\cdot)$, design S_N such that

$$\min \mathcal{D}(u - u_N)$$

   subject to a given fixed computational cost (memory, cpu time).

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Or ...

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$$\min \text{Computational Cost}$$

subject to the constraint

$$\mathcal{D}(u - u_N) \leq \text{TOL},$$

where $\text{TOL} > 0$.

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$$\mathcal{D}(u - u_N) \leq \text{TOL}, \Rightarrow \text{Reliable Error Control}$$

   where $\text{TOL} > 0$.

Adaptive Algorithms

- Computable bound on $\mathcal{D}(u - u_N)$ $\Rightarrow$ A posteriori error estimation

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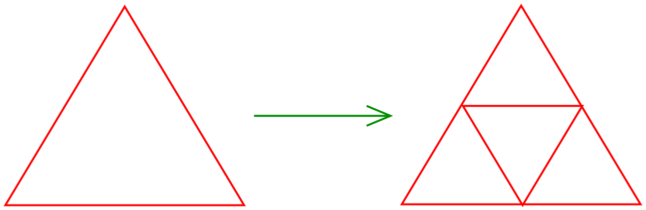

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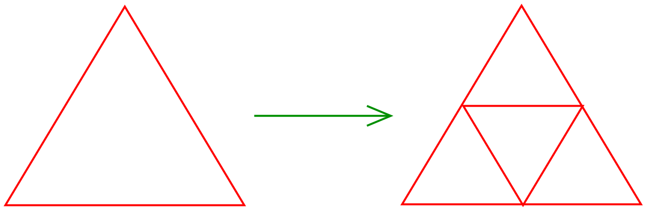
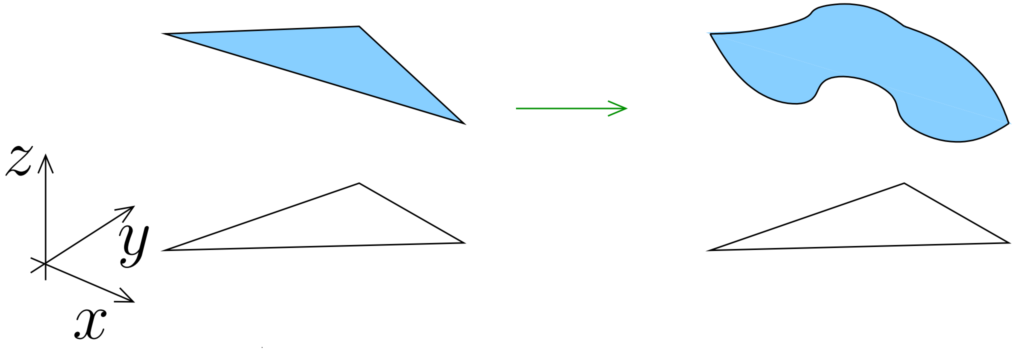


  - Local polynomial enrichment (p–refinement)



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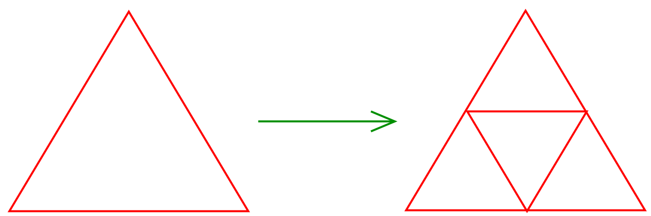

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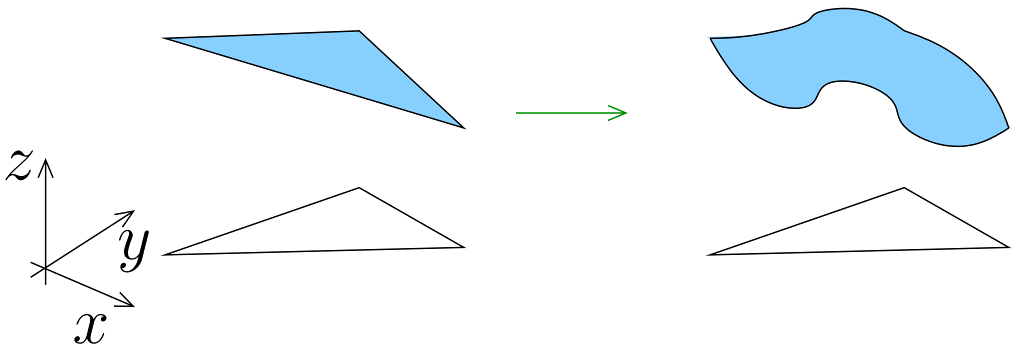


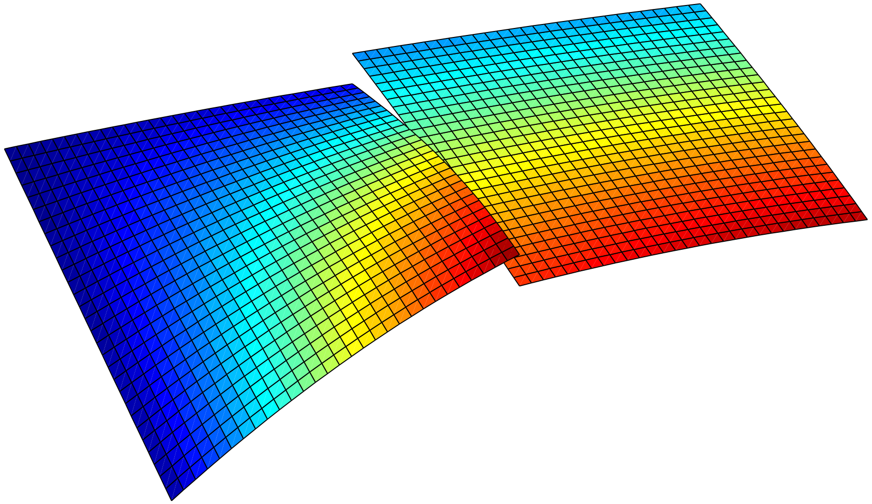
- Commercial/free software

  ALBERTA, ALGOR, ABAQUS, CLAWPACK, DEAL II, Diffpack, ELFEN, FEATFLOW, FEFLOW, FEMLAB, FesaWin, FLUENT, HiFlow, MADNESS, MC, PLTMG, VisualFEA, . . .

2. Discontinuous Galerkin Methods

Discontinuous Galerkin Methods

- Method Construction
  - Employ local spaces of discontinuous piecewise polynomials;
  - Inter-element continuity weakly enforced.



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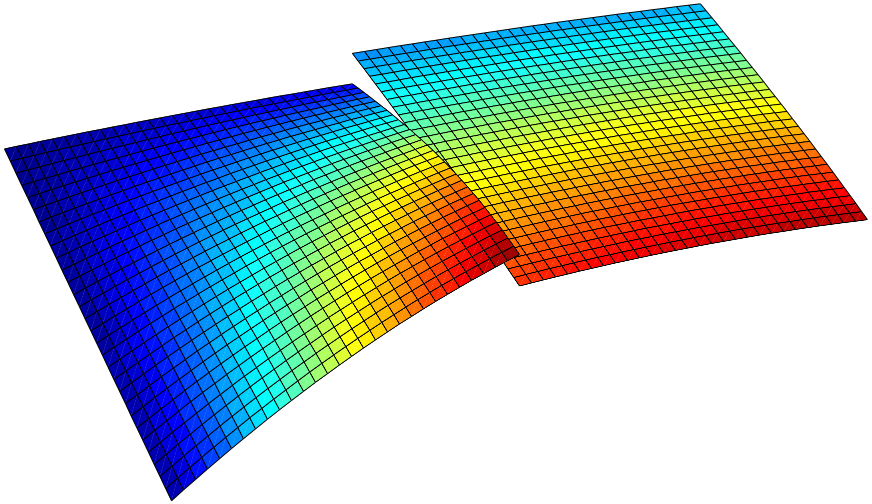

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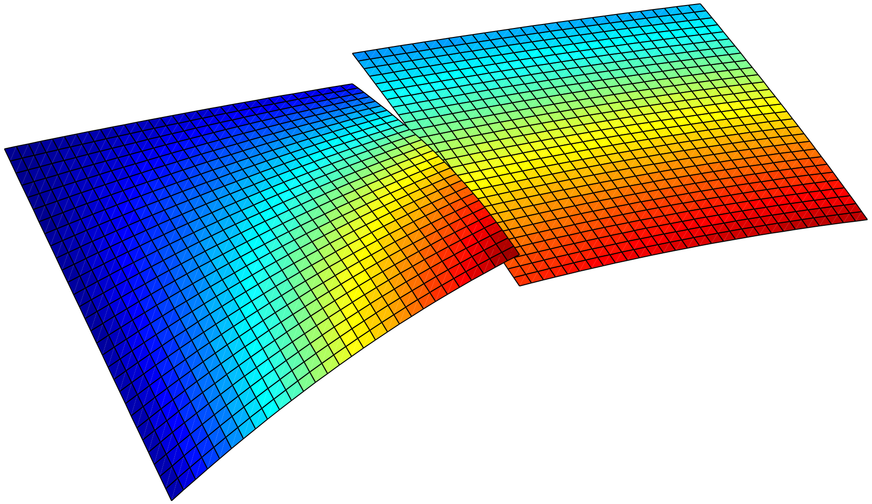

$\Rightarrow$ Hybrid FE/FV Method

- Background
  - Elliptic PDEs

    Pian 1965, Nitsche 1971, Wheeler 1978, Arnold 1982, ...
  - Hyperbolic PDEs

    Reed & Hill 1973, Lesaint & Raviart 1974, Johnson, Nävert & Pitkaranta 1984, ...

Discontinuous Galerkin Methods

- Recent Upsurge in Interest
  - Applications

    Linear elliptic/parabolic/hyperbolic PDEs, Fokker–Planck equations, Incompressible/ Compressible fluid flows, Turbulent flows, Non-Newtonian flows, Time and frequency domain Maxwell's equations, MHD, ...
  - References

    Cockburn & Shu 1990→, Bassi & Rebay 1997→, H., Schwab & Süli 1998→, Baumann & Oden 1999, Riviere, Wheeler & Girault 1999, Barth & Larson 2000, Prudhomme, Pascal, Oden & Romkes 2000, Wihler & Schwab 2000, Arnold, Brezzi, Cockburn & Marini 2002, Hesthaven & Warburton 2001, Hartmann & H. 2002→, Dolejsi, Feistauer & Schwab 2002, H., Perugia & Schötzau 2004→, ...

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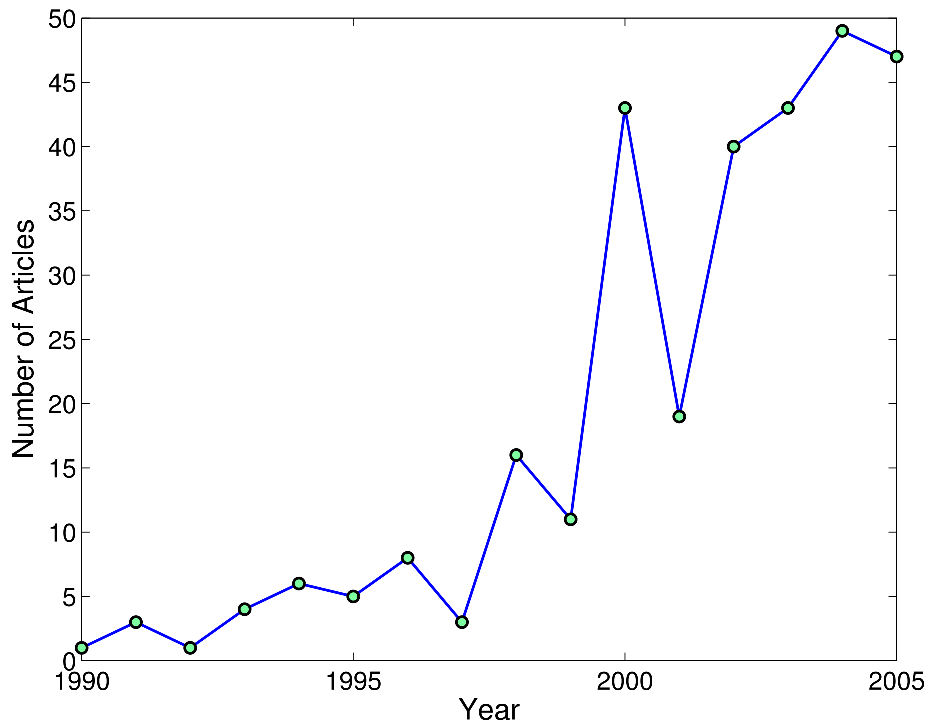
  - References

    MathSciNet: 313 articles (Discontinuous Galerkin OR DG)



Discontinuous Galerkin Methods

- Robustness/stability;
- Locally conservative;
- Ease of implementation;
- Highly parallelizable;
- Flexible mesh design (hybrid grids, non-matching grids, non-uniform/anisotropic polynomial degrees);
- Wider choice of stable FE spaces for mixed problems;
- Unified treatment of a wide range of PDEs.

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- Unified treatment of a wide range of PDEs.
- Computational overhead/efficiency (increase in DoFs).
  [Parallel efficiency gains, optimized quadrature]

3. First–Order Hyperbolic Conservation Laws

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3.1 Discretization

Model Problem

Conservation Law: Given $\Omega \subset \mathbb{R}^n$, find $\boldsymbol{u} : \Omega \to \mathbb{R}^m$, such that

$$\mathrm{div}\mathcal{F}(\boldsymbol{u}) = \boldsymbol{f} \quad \text{in } \Omega,$$

where $\mathcal{F}(\boldsymbol{u}) = (\mathcal{F}_1(\boldsymbol{u}), \ldots, \mathcal{F}_n(\boldsymbol{u}))$ and $\boldsymbol{f} \in [L_2(\Omega)]^m$.

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Boundary conditions: e.g., at inflow/outflow

$$B^-(\boldsymbol{u}, \boldsymbol{n})(\boldsymbol{u} - \boldsymbol{g}) = \boldsymbol{0} \quad \text{on } \partial\Omega$$

where

$$\begin{aligned} B(\boldsymbol{u}, \boldsymbol{n}) &= \sum_{i=1}^{n} \boldsymbol{n}_i \nabla_{\boldsymbol{u}} \mathcal{F}_i(\boldsymbol{u}) \\ B^{\pm}(\boldsymbol{u}, \boldsymbol{n}) &= X(\boldsymbol{u}, \boldsymbol{n})^{-1} \Lambda_{\pm} X(\boldsymbol{u}, \boldsymbol{n}). \end{aligned}$$

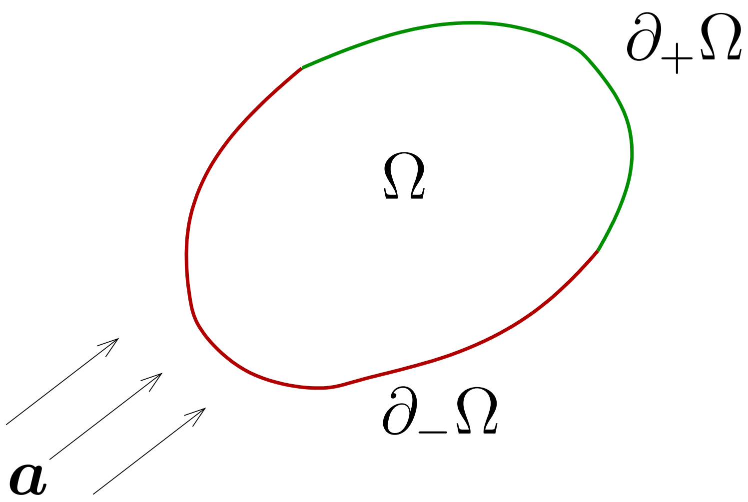
Examples

- Linear (scalar) advection problem

$$\begin{aligned} \nabla \cdot (\boldsymbol{a}u) &= f \text{ in } \Omega, \\ u &= g \text{ on } \partial_-\Omega, \end{aligned}$$

where

$$\begin{aligned} \partial_-\Omega &= \{\boldsymbol{x} \in \partial\Omega : \boldsymbol{a} \cdot \boldsymbol{n} < 0\}, \\ \partial_+\Omega &= \{\boldsymbol{x} \in \partial\Omega : \boldsymbol{a} \cdot \boldsymbol{n} > 0\}. \end{aligned}$$



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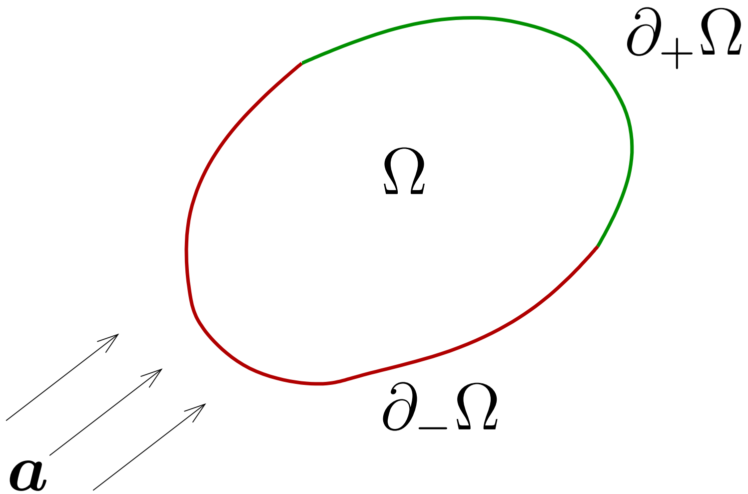

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- Compressible Euler Equations ($\Omega \subset \mathbb{R}^2$)

$$\begin{aligned} \boldsymbol{u} &= [\rho, \rho v_1, \rho v_2, \rho E]^\top, \\ \mathcal{F}_i(\boldsymbol{u}) &= [\rho v_i, \rho v_1 v_i + \delta_{1i} p, \rho v_2 v_i + \delta_{2i} p, \rho H v_i]^\top, \quad i = 1, 2, \end{aligned}$$

ρ: density, (v_1, v_2): velocity, E: energy, p: pressure, H: enthalpy.

Discontinuous Galerkin Method

- $\mathcal{T}_h = \{\kappa\}$ is a non-degenerate mesh;
- Finite element space

$$S_{h,\boldsymbol{p}} = \{\boldsymbol{v} \in [L_2(\Omega)]^m : \boldsymbol{v}|_\kappa \in [\mathcal{S}_{p_\kappa}]^m \quad \forall \kappa \in \mathcal{T}_h\},$$

where, for each κ, we define

$$\mathcal{S}_\ell = \begin{cases} \mathcal{P}_\ell = \text{span}\,\{x^\alpha : 0 \leq |\alpha| \leq \ell\} & \text{(simplex)}, \\ \mathcal{Q}_\ell = \text{span}\,\{x^\alpha : 0 \leq \alpha_i \leq \ell,\ 1 \leq i \leq n\} & \text{(hypercube)}, \end{cases}$$

for $\ell \geq 0$.

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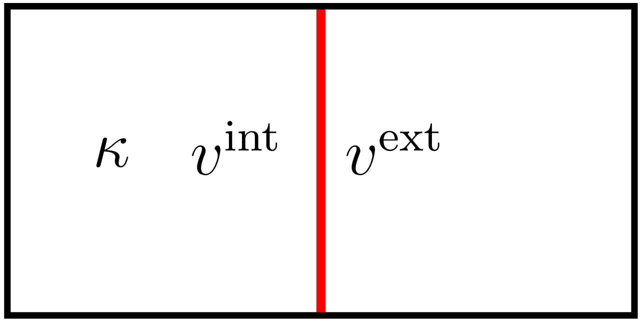
- h-Version: For $p_\kappa = p$ for all κ in $\mathcal{T}_h$, we write $S_{h,p}$.

Discontinuous Galerkin Method

For $v \in H^1(\kappa)$, we write

$$v^{\text{int}} = \text{interior trace of } v \text{ on } \partial\kappa \text{ taken from within } \kappa$$

$$v^{\text{ext}} = \text{exterior trace of } v \text{ on } \partial\kappa \text{ taken from outside } \kappa$$



Discontinuous Galerkin Method

- Local weak formulation

  On each $\kappa \in \mathcal{T}_h$: find $\boldsymbol{u}$ such that

$$-\int_\kappa \mathcal{F}(\boldsymbol{u}) : \nabla \boldsymbol{v} \mathrm{d}\boldsymbol{x} + \int_{\partial\kappa} (\mathcal{F}(\boldsymbol{u}^{\mathrm{int}}) \cdot \boldsymbol{n}_\kappa) \cdot \boldsymbol{v}^{\mathrm{int}} \, \mathrm{d}s = \int_\kappa \boldsymbol{f} \cdot \boldsymbol{v} \mathrm{d}\boldsymbol{x}$$

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- Weakly enforce the boundary conditions

$$-\int_\kappa \mathcal{F}(\boldsymbol{u}) : \nabla \boldsymbol{v} \mathrm{d}\boldsymbol{x} + \int_{\partial\kappa} \mathcal{H}(\boldsymbol{u}^{\mathrm{int}}, \boldsymbol{u}^{\mathrm{ext}}, \boldsymbol{n}_\kappa) \cdot \boldsymbol{v}^{\mathrm{int}} \, \mathrm{d}s = \int_\kappa \boldsymbol{f} \cdot \boldsymbol{v} \mathrm{d}\boldsymbol{x}$$

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- Add artificial viscosity

$$\begin{aligned} -\int_\kappa \mathcal{F}(\boldsymbol{u}) : \nabla \boldsymbol{v} \mathrm{d}\boldsymbol{x} \quad &+ \quad \int_{\partial\kappa} \mathcal{H}(\boldsymbol{u}^{\mathrm{int}}, \boldsymbol{u}^{\mathrm{ext}}, \boldsymbol{n}_\kappa) \cdot \boldsymbol{v}^{\mathrm{int}} \,\mathrm{d}s \\ &+ \quad \int_\kappa \varepsilon_h(\boldsymbol{u}) \nabla \boldsymbol{u} : \nabla \boldsymbol{v} \mathrm{d}\boldsymbol{x} = \int_\kappa \boldsymbol{f} \cdot \boldsymbol{v} \mathrm{d}\boldsymbol{x} \end{aligned}$$

Discontinuous Galerkin method

$$\begin{aligned}\mathcal{N}(\boldsymbol{u}_h, \boldsymbol{v}_h) &= \sum_{\kappa \in \mathcal{T}_h} \Big\{ -\int_\kappa \mathcal{F}(\boldsymbol{u}_h) : \nabla \boldsymbol{v}_h \mathrm{d}\boldsymbol{x} \\ &\quad + \int_{\partial\kappa} \mathcal{H}(\boldsymbol{u}_h^{\mathrm{int}}, \boldsymbol{u}_h^{\mathrm{ext}}, \boldsymbol{n}_\kappa) \cdot \boldsymbol{v}_h^{\mathrm{int}} \mathrm{d}s + \int_\kappa \varepsilon_h(\boldsymbol{u}_h) \nabla \boldsymbol{u}_h : \nabla \boldsymbol{v}_h \mathrm{d}\boldsymbol{x} \Big\} \\ \ell(\boldsymbol{v}_h) &= \sum_{\kappa \in \mathcal{T}_h} \int_\kappa \boldsymbol{f} \cdot \boldsymbol{v}_h \mathrm{d}\boldsymbol{x} \\ \varepsilon_h(\boldsymbol{u}_h) &= C_\varepsilon \, h_\kappa^{2-\beta} \, |\boldsymbol{f} - \mathrm{div}\mathcal{F}(\boldsymbol{u}_h)|,\end{aligned}$$

where $C_\varepsilon \geq 0$ and $\beta = 1/10$.

Jaffre, Johnson, & Szepessy 1995, Cockburn & Gremaud 1996

Discontinuous Galerkin method

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\end{aligned}$$

where $C_\varepsilon \geq 0$ and $\beta = 1/10$.

Jaffre, Johnson, & Szepessy 1995, Cockburn & Gremaud 1996

DGFEM: Find $\boldsymbol{u}_h \in S_{h,p}$ such that

$$\mathcal{N}(\boldsymbol{u}_h, \boldsymbol{v}_h) = \ell(\boldsymbol{v}_h) \qquad \forall \boldsymbol{v}_h \in S_{h,p}.$$

Numerical Flux Function

$\mathcal{H}(\cdot,\cdot,\boldsymbol{n})$ is a Lipschitz continuous, consistent and conservative flux function.

- Lax–Friedrichs flux

$$\mathcal{H}(\boldsymbol{u}^{\rm int},\boldsymbol{u}^{\rm ext},\boldsymbol{n}) = \frac{1}{2}\left(\left(\mathcal{F}(\boldsymbol{u}^{\rm int})\cdot\boldsymbol{n} + \mathcal{F}(\boldsymbol{u}^{\rm ext})\cdot\boldsymbol{n}\right) - \alpha(\boldsymbol{u}^{\rm ext} - \boldsymbol{u}^{\rm int})\right),$$

  where $\alpha = max_i|\Lambda_{ii}|$.

- Roe's flux

$$\begin{aligned}\mathcal{H}(\boldsymbol{u}^{\rm int},\boldsymbol{u}^{\rm ext},\boldsymbol{n}) &= \frac{1}{2}\left(\left(\mathcal{F}(\boldsymbol{u}^{\rm int})\cdot\boldsymbol{n} + \mathcal{F}(\boldsymbol{u}^{\rm ext})\cdot\boldsymbol{n}\right)\right.\\ &\qquad \left. - |B(\tilde{\boldsymbol{u}},\boldsymbol{n})|(\boldsymbol{u}^{\rm ext} - \boldsymbol{u}^{\rm int})\right),\end{aligned}$$

  where $|B(\tilde{\boldsymbol{u}},\boldsymbol{n})| = B^+(\tilde{\boldsymbol{u}},\boldsymbol{n}) - B^-(\tilde{\boldsymbol{u}},\boldsymbol{n})$.

- Vijayasundaram flux

$$\mathcal{H}(\boldsymbol{u}^{\rm int},\boldsymbol{u}^{\rm ext},\boldsymbol{n}) = B^+(\tilde{\boldsymbol{u}},\boldsymbol{n})\boldsymbol{u}^{\rm int} + B^-(\tilde{\boldsymbol{u}},\boldsymbol{n})\boldsymbol{u}^{\rm ext}.$$

Galerkin Orthogonality

- Consistency

  Assuming $\boldsymbol{u}$ is *sufficiently* regular then

$$\mathcal{N}(\boldsymbol{u}, \boldsymbol{v}_h) = \ell(\boldsymbol{v}_h) \qquad \forall \boldsymbol{v}_h \in S_{h,p}.$$

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$$\mathcal{N}(\boldsymbol{u}, \boldsymbol{v}_h) - \mathcal{N}(\boldsymbol{u}_h, \boldsymbol{v}_h) = 0 \qquad \forall \boldsymbol{v}_h \in S_{h,p}.$$

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[Linear case: $\mathcal{N}(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}_h) = 0 \qquad \forall \boldsymbol{v}_h \in S_{h,p}.$]

3.2 A Posteriori Error Estimation

Measurement Problem

Given a user–defined tolerance $\texttt{TOL} > 0$, can we efficiently design $S_{h,p}$ such that

- Norm control

$$\|\boldsymbol{u} - \boldsymbol{u}_h\| \leq \texttt{TOL};$$

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- Norm control

$$\|\boldsymbol{u} - \boldsymbol{u}_h\| \leq \mathtt{TOL};$$

- Functional control

$$|J(\boldsymbol{u}) - J(\boldsymbol{u}_h)| \leq \mathtt{TOL}.$$

Examples of Target Functionals

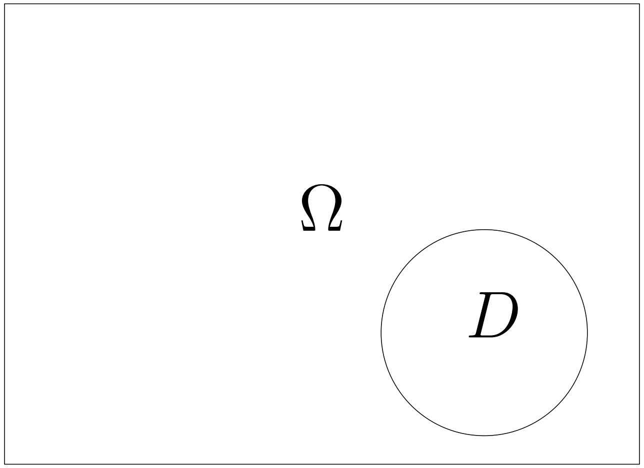
1. Flux through the boundary

$$J(\boldsymbol{u}) = \int_{\Gamma_0} (\mathcal{F}(\boldsymbol{u}) \cdot \boldsymbol{n}) \cdot \boldsymbol{\psi}(s) \mathrm{d}s.$$

Examples of Target Functionals

2. Mean value over subdomain

$$J(\boldsymbol{u}) = \int_D \boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{\psi}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}$$



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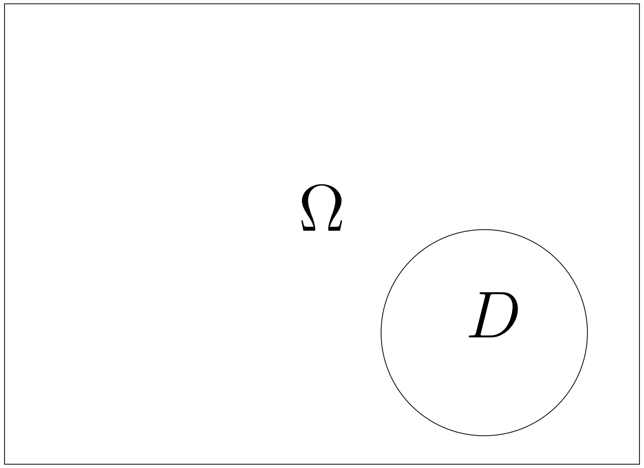
2. Mean value over subdomain

$$J(\boldsymbol{u}) = \int_D \boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{\psi}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}$$



3. Other examples:

Fluid dynamics:	Drag and lift coefficients.
Electromagnetics:	Far field pattern.
Elasticity theory:	Stress intensity factor.
Other examples:	Point value, etc.

A Posteriori Error Estimation

Structure of the proofs:

A Posteriori Error Estimation

Structure of the proofs:

- Derivation of an error representation formula using duality;

A Posteriori Error Estimation

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$$|J(\boldsymbol{u}) - J(\boldsymbol{u}_h)| \leq \sum_{\kappa \in \mathcal{T}_h} |(r_h, \boldsymbol{z} - \boldsymbol{z}_h)_\kappa|$$

$\Rightarrow$ (Weighted) Type I Error Bound

Becker & Rannacher 1996, Rannacher *et al.* 1996 $\longrightarrow$

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- Local interpolation error estimates for the dual solution;
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Structure of the proofs:

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$$|J(\boldsymbol{u}) - J(\boldsymbol{u}_h)| \leq \sum_{\kappa \in \mathcal{T}_h} |(r_h, \boldsymbol{z} - \boldsymbol{z}_h)_\kappa|$$

$\Rightarrow$ (Weighted) Type I Error Bound

Becker & Rannacher 1996, Rannacher *et al.* 1996 $\longrightarrow$

- Local interpolation error estimates for the dual solution;
- Stability estimates for the dual problem.

$$|J(\boldsymbol{u}) - J(\boldsymbol{u}_h)| \leq C_{\text{int}} C_{\text{stab}} \| h^s r_h \|, \quad s > 0$$

$\Rightarrow$ (Unweighted) Type II Error Bound

Johnson *et al.* 1995 $\longrightarrow$

Linear Advection

Consider the following problem

$$\mathcal{L}u \equiv \nabla \cdot (\boldsymbol{a}u) = f \text{ in } \Omega, \qquad u = g \text{ on } \partial_{-}\Omega,$$

where

$$\begin{aligned} \partial_{-}\Omega &= \{\boldsymbol{x} \in \partial\Omega : \boldsymbol{a} \cdot \boldsymbol{n} < 0\}\,, \\ \partial_{+}\Omega &= \{\boldsymbol{x} \in \partial\Omega : \boldsymbol{a} \cdot \boldsymbol{n} > 0\}\,. \end{aligned}$$

Linear Advection

DGFEM: Find $u_h \in S_{h,p}$ such that

$$\mathcal{B}(u_h, v_h) \equiv \mathcal{N}(u_h, v_h) = \ell(v_h) \qquad \forall v_h \in S_{h,p},$$

where, selecting,

$$\mathcal{H}(u_h^{\text{int}}, u_h^{\text{ext}}, \boldsymbol{n}_\kappa)|_{\partial\kappa} = \boldsymbol{a} \cdot \boldsymbol{n}_\kappa \lim_{s \to 0^+} u_h(\boldsymbol{x} - s\boldsymbol{a}) \quad \text{for } \kappa \in \mathcal{T}_h,$$

we have

$$\begin{aligned} \mathcal{B}(u_h, v_h) &= \sum_{\kappa \in \mathcal{T}_h} \left\{ -\int_\kappa (\boldsymbol{a} u_h) \cdot \nabla v_h \mathrm{d}\boldsymbol{x} \right. \\ &\quad \left. + \int_{\partial_+\kappa} (\boldsymbol{a} \cdot \boldsymbol{n}_\kappa)\, u_h^{\text{int}} v_h^{\text{int}} \mathrm{d}s + \int_{\partial_-\kappa \setminus \partial\Omega} (\boldsymbol{a} \cdot \boldsymbol{n}_\kappa)\, u_h^{\text{ext}} v_h^{\text{int}} \mathrm{d}s \right\}, \\ \ell(v_h) &= \sum_{\kappa \in \mathcal{T}_h} \left\{ \int_\kappa f v_h \mathrm{d}\boldsymbol{x} - \int_{\partial_-\kappa \cap \partial\Omega} (\boldsymbol{a} \cdot \boldsymbol{n}_\kappa)\, g\, v_h^{\text{int}} \mathrm{d}s \right\}. \end{aligned}$$

A Posteriori Error Estimation

Dual problem: find z such that

$$\mathcal{B}(v, z) = J(v) \quad \forall v,$$

where $J(\cdot)$ is a given *linear functional*.

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Examples:

1. Outflow flux: $J(u) = \int_{\partial_+\Omega} (\boldsymbol{a} \cdot \boldsymbol{n} u)\psi \, \mathrm{d}s$, $\psi \in L_2(\partial_+\Omega)$.

$$\begin{aligned} \mathcal{L}^* z \equiv -\boldsymbol{a} \cdot \nabla z &= 0 \;\; \text{in } \Omega\,, \\ z &= \psi \;\; \text{on } \partial_+\Omega\,. \end{aligned}$$

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Error Representation Formula

$$J(u) - J(u_h)$$

$$= J(u - u_h) \qquad \text{[Linearity]}$$

Error Representation Formula

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J(u) - J(u_h) & \\
&= J(u - u_h) && \text{[Linearity]} \\
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&= \ell(z - z_h) - \mathcal{B}(u_h, z - z_h) && \text{[Consistency]} \\
&= \sum_{\kappa \in \mathcal{T}_h} \left\{ \int_\kappa (f - \nabla \cdot (\boldsymbol{a} u))(z - z_h) \mathrm{d}\boldsymbol{x} \right. \\
&\quad + \int_{\partial_- \kappa \setminus \partial\Omega} \boldsymbol{a} \cdot \boldsymbol{n}_\kappa (u^{\mathrm{int}} - u^{\mathrm{ext}})(z - z_h)^{\mathrm{int}} \mathrm{d}s \\
&\quad \left. + \int_{\partial_- \kappa \cap \partial\Omega} \boldsymbol{a} \cdot \boldsymbol{n}_\kappa (u^{\mathrm{int}} - g)(z - z_h)^{\mathrm{int}} \mathrm{d}s \right\} \equiv \sum_{\kappa \in \mathcal{T}_h} \eta_\kappa .
\end{aligned}
$$

Residuals

Internal Residual

$$\boldsymbol{r}_h\big|_\kappa = \left(\boldsymbol{f} - \mathrm{div}\mathcal{F}(\boldsymbol{u}_h)\right)\big|_\kappa$$

Residuals

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Flux Residual

$$\boldsymbol{\sigma}_h|_{\partial\kappa} = \left(\mathcal{F}(\boldsymbol{u}_h^{\text{int}}) \cdot \boldsymbol{n}_\kappa - \mathcal{H}(\boldsymbol{u}_h^{\text{int}}, \boldsymbol{u}_h^{\text{ext}}, \boldsymbol{n}_\kappa)\right)|_{\partial\kappa}$$

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AVIS Residual

$$\boldsymbol{\alpha}_h\big|_\kappa = (\varepsilon_h(\boldsymbol{u}_h)\nabla \boldsymbol{u}_h)\,\big|_\kappa$$

Linearization

- Gâteaux derivative of $J(\cdot)$:

$$J'[\boldsymbol{w}](\boldsymbol{v}) = \lim_{\epsilon \to 0} \frac{J(\boldsymbol{w} + \epsilon \boldsymbol{v}) - J(\boldsymbol{w})}{\epsilon}.$$

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- Mean–value linearization of $J(\cdot)$:

$$\begin{aligned} \bar{J}(\boldsymbol{u}, \boldsymbol{u}_h; \boldsymbol{u} - \boldsymbol{u}_h) &= J(\boldsymbol{u}) - J(\boldsymbol{u}_h) \\ &= \int_0^1 J'[\theta \boldsymbol{u} + (1-\theta)\boldsymbol{u}_h](\boldsymbol{u} - \boldsymbol{u}_h) \mathrm{d}\theta. \end{aligned}$$

Linearization

- Gâteaux derivative of $\mathcal{N}(\cdot,\cdot)$:

$$\mathcal{N}'_{\boldsymbol{u}}[\boldsymbol{w}](\boldsymbol{v},\cdot) = \lim_{\epsilon \to 0} \frac{\mathcal{N}(\boldsymbol{w}+\epsilon\boldsymbol{v},\cdot) - \mathcal{N}(\boldsymbol{w},\cdot)}{\epsilon}.$$

Linearization

- Gâteaux derivative of $\mathcal{N}(\cdot,\cdot)$:

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- Mean–value linearisation of $\mathcal{N}(\cdot,\cdot)$:

$$\begin{aligned} \mathcal{M}(\boldsymbol{u},\boldsymbol{u}_h;\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{v}) &= \mathcal{N}(\boldsymbol{u},\boldsymbol{v}) - \mathcal{N}(\boldsymbol{u}_h,\boldsymbol{v}) \\ &= \int_0^1 \mathcal{N}'_{\boldsymbol{u}}[\theta\boldsymbol{u}+(1-\theta)\boldsymbol{u}_h](\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{v})\mathrm{d}\theta. \end{aligned}$$

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- Galerkin orthogonality:

$$\mathcal{N}(\boldsymbol{u},\boldsymbol{v}_h) - \mathcal{N}(\boldsymbol{u}_h,\boldsymbol{v}_h) = \mathcal{M}(\boldsymbol{u},\boldsymbol{u}_h;\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{v}_h) = 0$$

$$\forall \boldsymbol{v}_h \in S_{h,p}.$$

A Posteriori Error Analysis

Dual problem: find $\boldsymbol{z}$ such that

$$\mathcal{M}(\boldsymbol{u}, \boldsymbol{u}_h; \boldsymbol{w}, \boldsymbol{z}) = \bar{J}(\boldsymbol{u}, \boldsymbol{u}_h; \boldsymbol{w}) \quad \forall \boldsymbol{w}.$$

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Well–posedness (?)

Tadmor 1991, Bouchut & James 1998, Godlewski, Olazabal, & Raviart 1999, Ulbrich 2001, 2002, 2003

Error Representation Formula

$$
\begin{aligned}
&J(\boldsymbol{u}) - J(\boldsymbol{u}_h) \\
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\end{aligned}
$$

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&\quad = \sum_{\kappa \in \mathcal{T}_h} \bigg(\int_\kappa \boldsymbol{r}_h \cdot (\boldsymbol{z} - \boldsymbol{z}_h) \mathrm{d}\boldsymbol{x} + \int_{\partial\kappa} \boldsymbol{\sigma}_h \cdot (\boldsymbol{z} - \boldsymbol{z}_h)^{\mathrm{int}} \mathrm{d}s \\
&\qquad - \int_\kappa \boldsymbol{\alpha}_h : \nabla(\boldsymbol{z} - \boldsymbol{z}_h) \mathrm{d}\boldsymbol{x} \bigg) \equiv \sum_{\kappa \in \mathcal{T}_h} \eta_\kappa .
\end{aligned}
$$

Type I *A Posteriori* Error Bound

Theorem

Assuming the dual problem is well–posed, the following result holds:

$$|J(\boldsymbol{u}) - J(\boldsymbol{u}_h)| \leq \sum_{\kappa \in \mathcal{T}_h} \eta_\kappa^{(\mathsf{I})},$$

where $\eta_\kappa^{(\mathsf{I})} = |\eta_\kappa|$,

$$\begin{aligned}
\eta_\kappa &= \int_\kappa \boldsymbol{r}_h \cdot (\boldsymbol{z} - \boldsymbol{z}_h)\mathrm{d}\boldsymbol{x} + \int_{\partial\kappa} \boldsymbol{\sigma}_h \cdot (\boldsymbol{z} - \boldsymbol{z}_h)^{\mathrm{int}}\mathrm{d}s \\
&\quad - \int_\kappa \boldsymbol{\alpha}_h : \nabla(\boldsymbol{z} - \boldsymbol{z}_h)\mathrm{d}\boldsymbol{x}, \\
\boldsymbol{r}_h|_\kappa &= (\boldsymbol{f} - \mathrm{div}\mathcal{F}(\boldsymbol{u}_h))\,|_\kappa, \\
\boldsymbol{\sigma}_h|_{\partial\kappa} &= \left(\mathcal{F}(\boldsymbol{u}_h^{\mathrm{int}}) \cdot \boldsymbol{n}_\kappa - \mathcal{H}(\boldsymbol{u}_h^{\mathrm{int}}, \boldsymbol{u}_h^{\mathrm{ext}}, \boldsymbol{n}_\kappa)\right)|_{\partial\kappa}, \\
\boldsymbol{\alpha}_h|_\kappa &= (\varepsilon_h(\boldsymbol{u}_h)\nabla \boldsymbol{u}_h)\,|_\kappa.
\end{aligned}$$

Approximation Theory

Lemma

Given $\kappa \in \mathcal{T}_h$, suppose that $v|_\kappa \in H^{k_\kappa}(\kappa)$, $0 \le k_\kappa \le p+1$. Then, there exists Πv in the finite element space $S_{h,p}$, such that

$$\begin{aligned}
\|v - \Pi v\|_{L_2(\kappa)} &+ h_\kappa \|\nabla (v - \Pi v)\|_{L_2(\kappa)} \\
&+ h_\kappa^{1/2} \|v - \Pi v\|_{L_2(\partial\kappa)} \le C_{\text{int}} h_\kappa^{k_\kappa} \|v\|_{H^{k_\kappa}(\kappa)}.
\end{aligned}$$

Proof: See Babuška & Suri 1987, for example.

Approximation Theory

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Proof: See Babuška & Suri 1987, for example.

Type II *a posteriori* bound:

Set $\boldsymbol{z}_h = \Pi \boldsymbol{z}$ and apply the above result, together with stability results for the dual.

Type II *A Posteriori* Error Bound

Theorem

Assuming

$$\|\boldsymbol{z}\|_{H^s(\Omega)} \leq C_{\mathrm{stab}}\,, \quad 1 \leq s \leq p+1\,,$$

we have that

$$|J(\boldsymbol{u}) - J(\boldsymbol{u}_h)| \leq C_{\mathrm{int}} C_{\mathrm{stab}} \left(\sum_{\kappa \in \mathcal{T}_h} \left(\eta_\kappa^{(\mathrm{II})}\right)^2 \right)^{1/2}\,,$$

where

$$\eta_\kappa^{(\mathrm{II})} = h^s \|\boldsymbol{r}_h\|_{L_2(\kappa)} + h^{s-1/2} \|\boldsymbol{\sigma}_h\|_{L_2(\partial\kappa)} + h^{s-1} \|\boldsymbol{\alpha}_h\|_{L_2(\kappa)}.$$

Implementation Aspects

Type I bound:

- Right–hand side is not computable

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  - z must be numerically approximated.

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- Right–hand side is not computable
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Linearization: find $\hat{\boldsymbol{z}}$ such that

$$\hat{\mathcal{N}}'_{\boldsymbol{u}}[\boldsymbol{u}_h](\boldsymbol{w}, \hat{\boldsymbol{z}}) = J'[\boldsymbol{u}_h](\boldsymbol{w}) \quad \forall \boldsymbol{w}.$$

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Type I bound:

- Right–hand side is not computable
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$$\hat{\mathcal{N}}'_{\boldsymbol{u}}[\boldsymbol{u}_h](\boldsymbol{w}, \hat{\boldsymbol{z}}) = J'[\boldsymbol{u}_h](\boldsymbol{w}) \quad \forall \boldsymbol{w}.$$

Discretization: find $\hat{\boldsymbol{z}}_{h',p'}$ such that

$$\hat{\mathcal{N}}'_{\boldsymbol{u}}[\boldsymbol{u}_h](\boldsymbol{w}, \hat{\boldsymbol{z}}_{h',p'}) = J'[\boldsymbol{u}_h](\boldsymbol{w}_{h',p'}) \quad \forall \boldsymbol{w}_{h',p'} \in S_{h',p'}.$$

In practice, we set $S_{h',p'} = S_{h,p+1}$.

Implementation Aspects

Type II bound:

- Computation of $C_{\rm int}$
  - Estimates are available from approximation theory.

Implementation Aspects

Type II bound:

- Computation of C_{int}
  - Estimates are available from approximation theory.
- Computation of C_{stab}
  - Theoretical bounds are typically over-pessimistic, if available at all.
  - The dual problem must be solved a (large) number of times with *typical* data.
  - Library of stability constants may be generated.

3.3 Numerical Examples

Burgers' Equation

$$u_t + \left((1/2)\, u^2\right)_x = 0,$$

$$u(x,0) = 2/(1 + x^3)\ \sin^2(\pi x)\ .$$

$$\boxed{J(u) \equiv u(1.95, 1.35) = 0.451408206331223\ .}$$

Burgers' Equation

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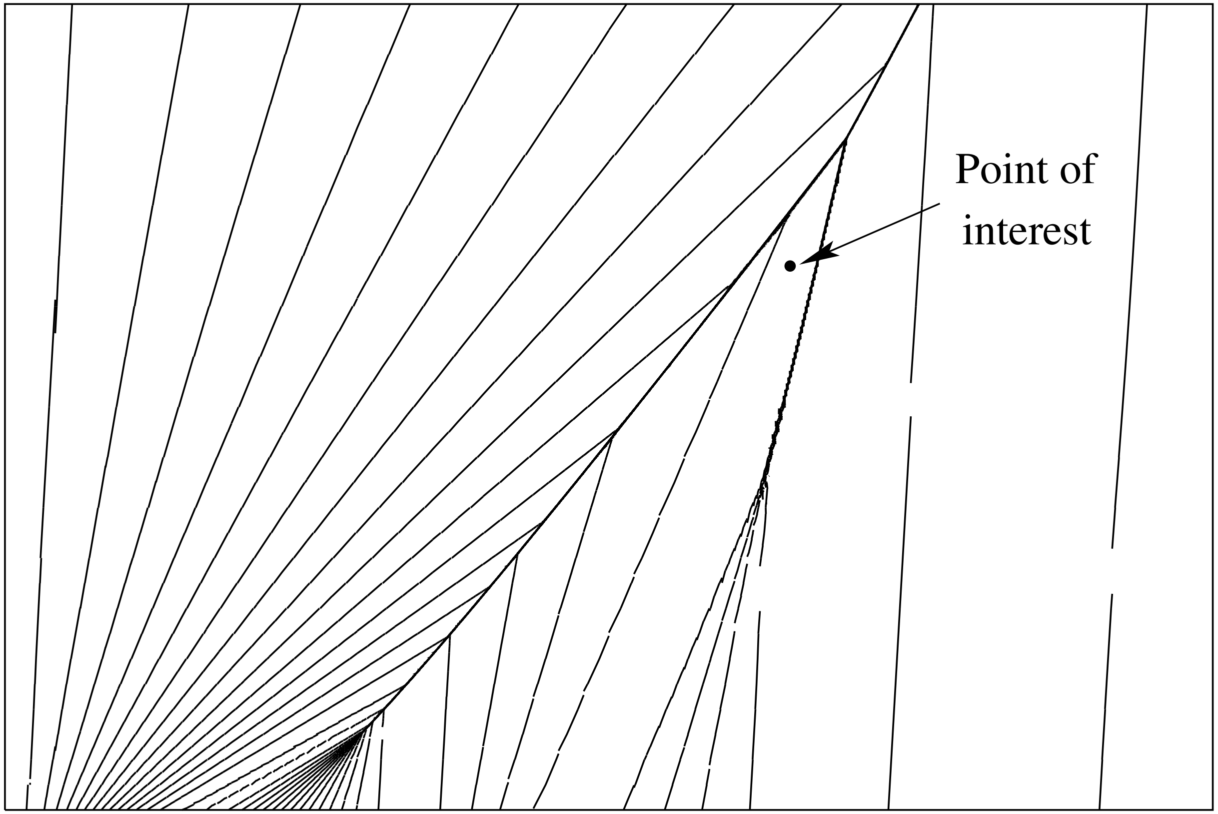
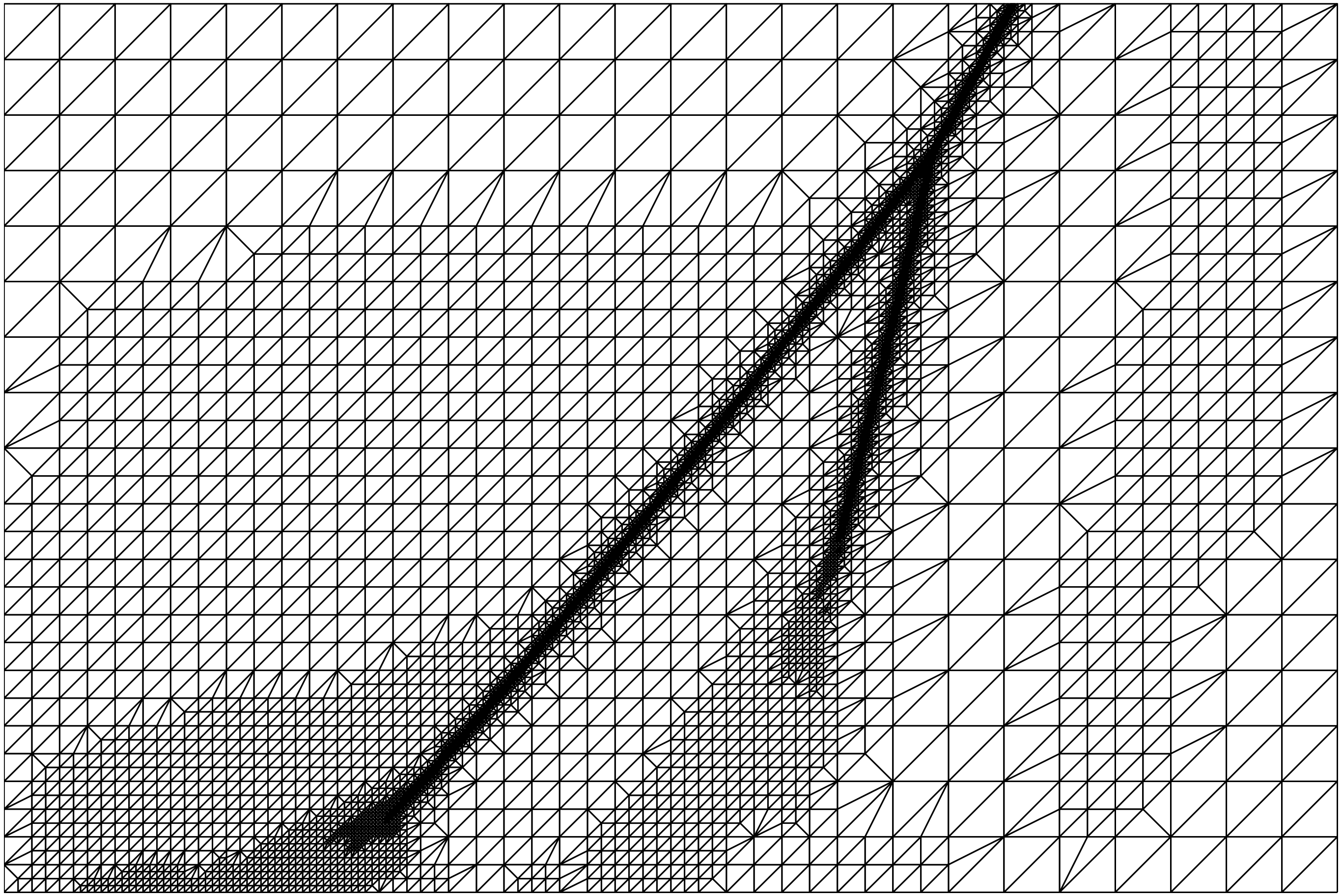


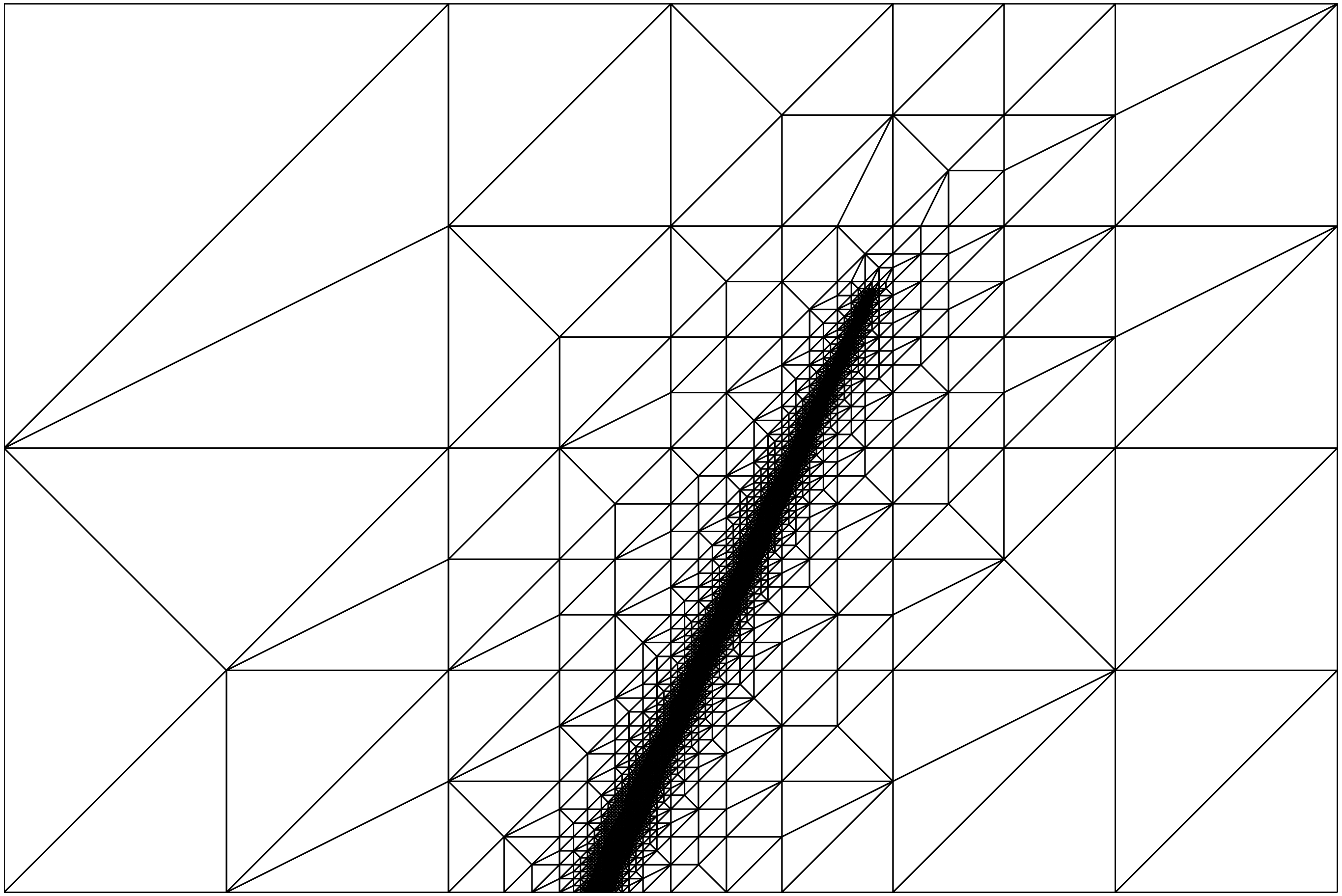
Burgers' Equation



Mesh designed using Type II indicator

20186 elements, $|J(u) - J(u_h)| = 4.015 \times 10^{-4}$

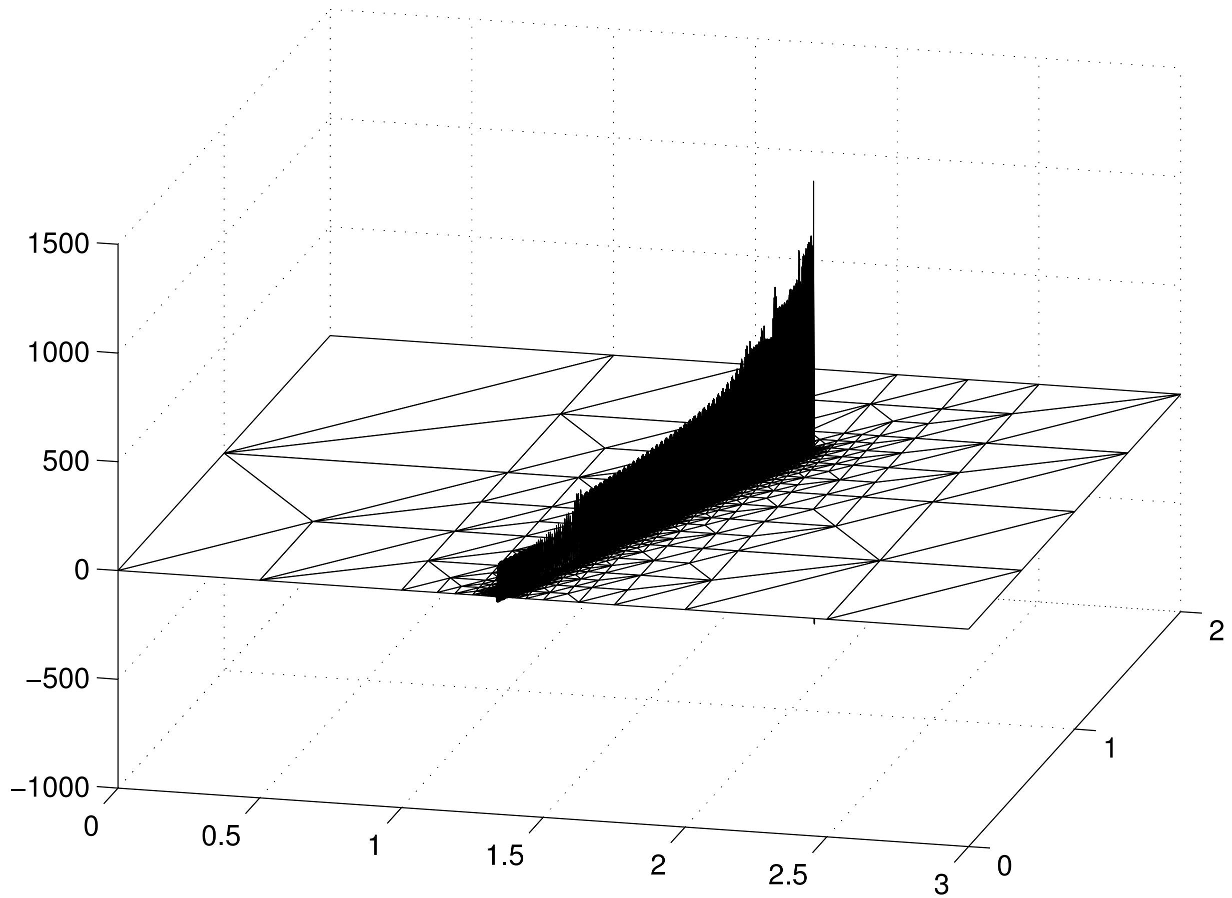
Burgers' Equation



Mesh designed using Type I indicator

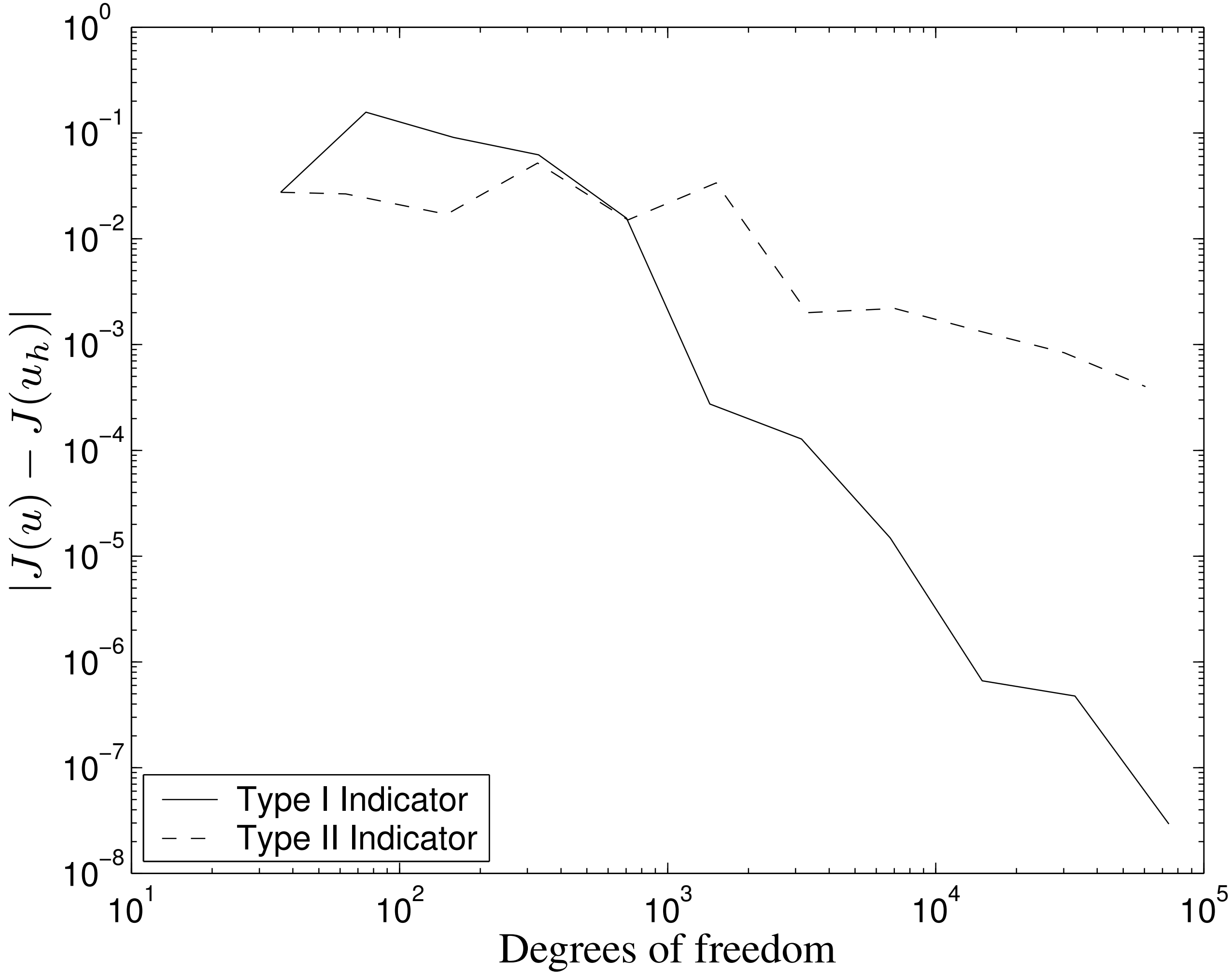
24662 elements, $|J(u) - J(u_h)| = 2.934 \times 10^{-8}$

Burgers' Equation

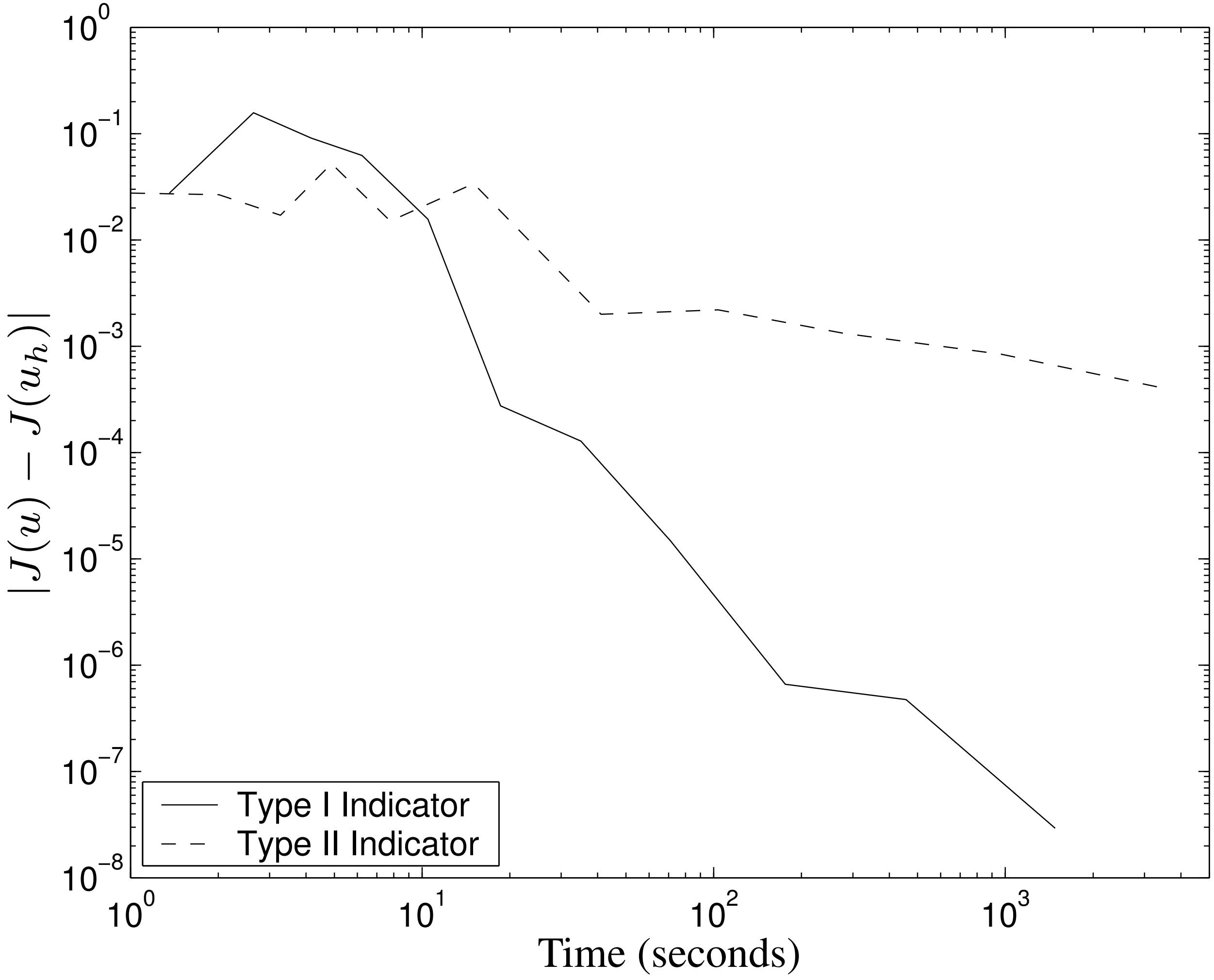


Dual Solution

Burgers' Equation



Burgers' Equation

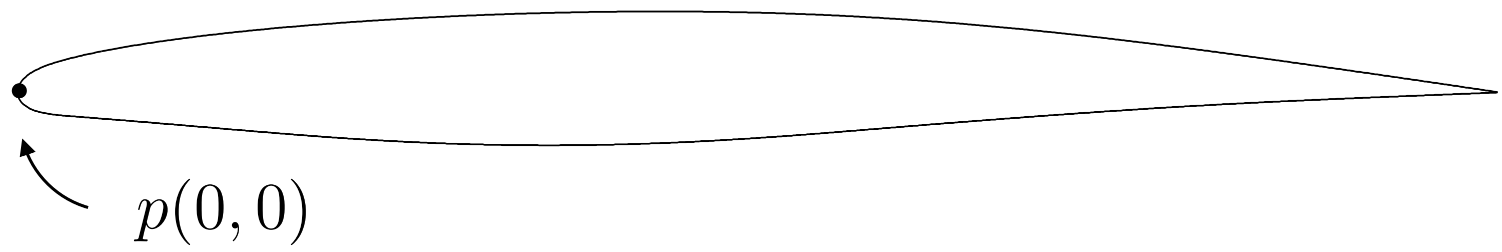


Burgers' Equation

Elements	$J(u - u_h)$	$\sum_\kappa \eta_\kappa$	θ_1	$\sum_\kappa \lvert\eta_\kappa\rvert$	θ_2
53	9.09e-2	-3.19e-2	-0.35	2.24e-1	2.47
110	6.23e-2	5.20e-2	0.83	1.21e-1	1.94
234	-1.57e-2	-3.27e-3	0.21	3.88e-2	2.48
479	2.75e-4	3.30e-4	1.20	1.08e-2	39.48
1053	1.28e-4	1.21e-4	0.95	3.27e-3	25.54
2256	1.48e-5	1.46e-5	0.98	1.86e-3	125.1
4968	-6.63e-7	-7.16e-7	1.08	7.76e-4	1171
11003	-4.76e-7	-4.78e-7	1.01	3.20e-4	672.0
24662	2.93e-8	2.91e-8	0.99	1.28e-4	4357

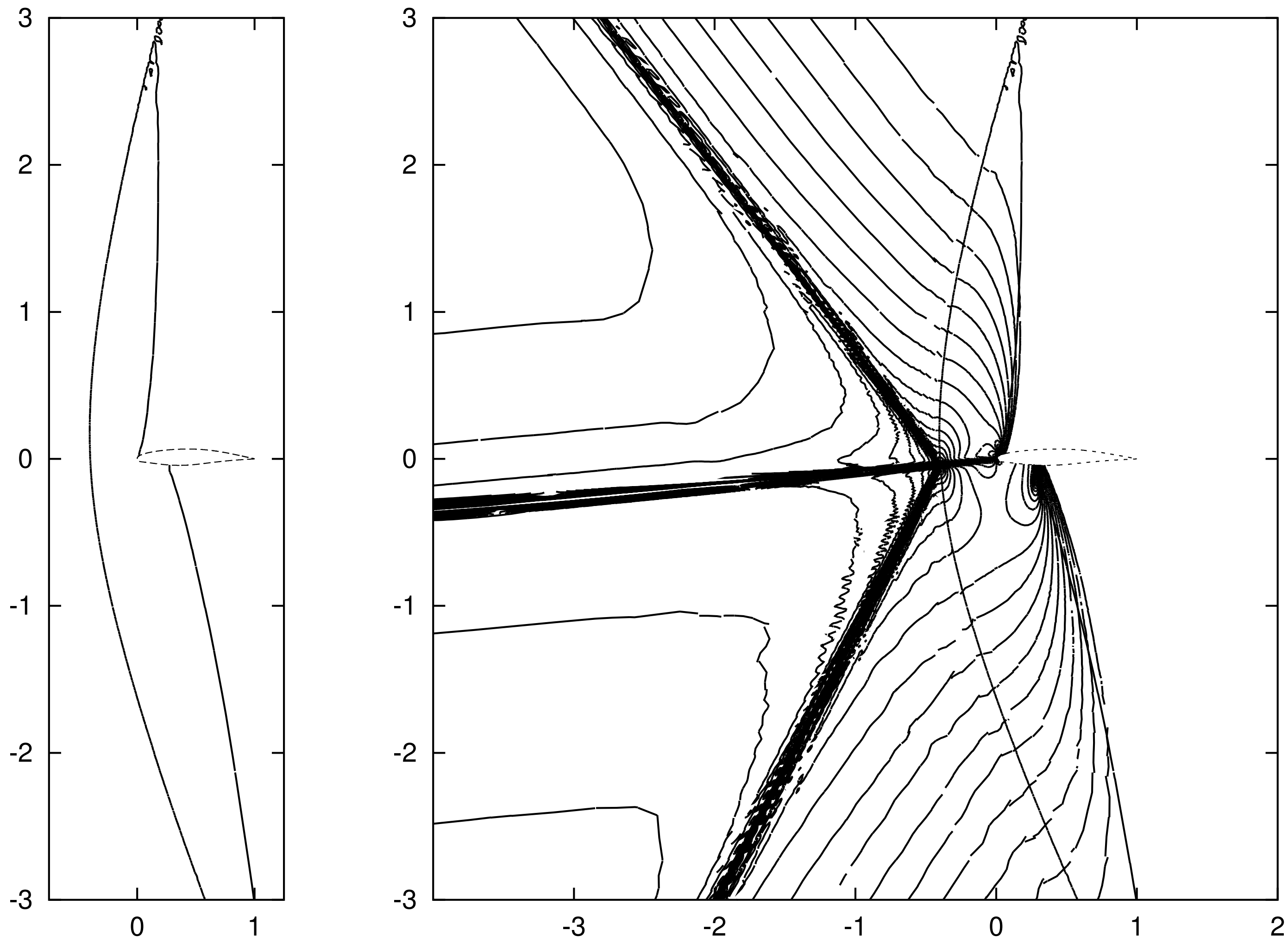
BAC3-11 Airfoil

$\mathrm{Ma} = 1.2$, $\alpha = 5^\circ$, $\rho = 1$ and pressure $p = 1$

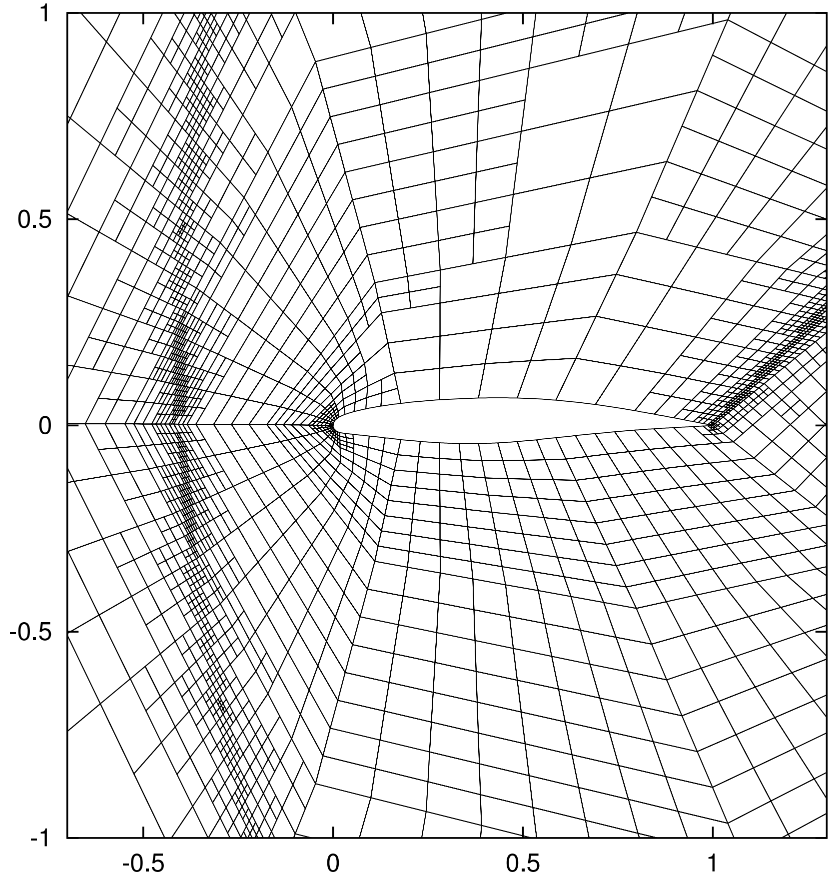


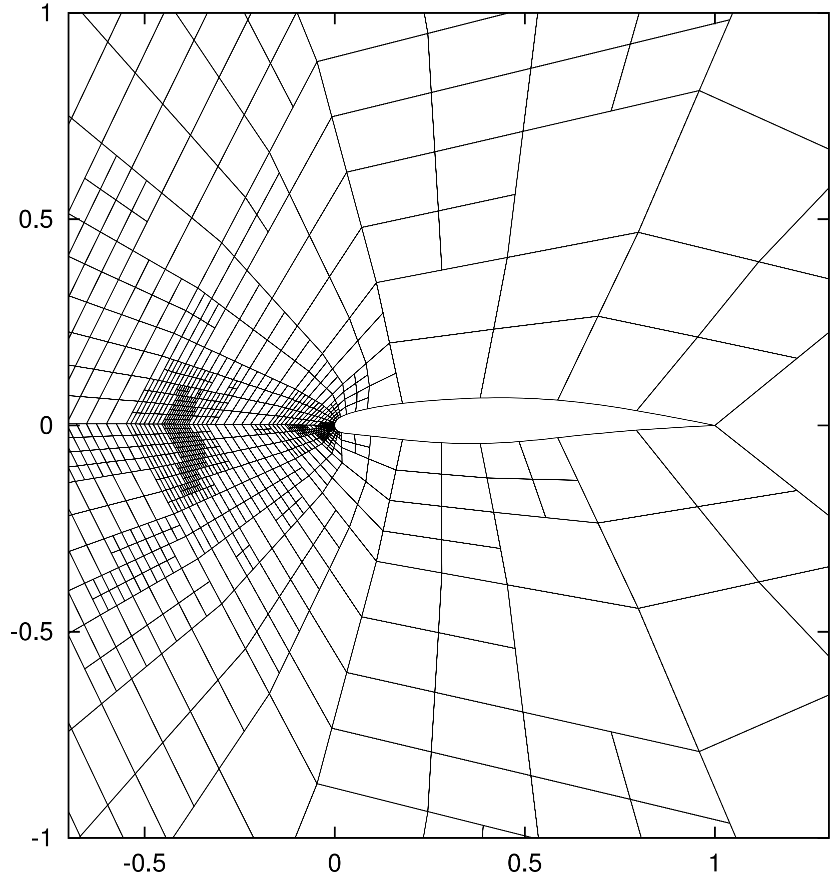
$$J(u) \equiv p(0,0) \approx 2.393 \ .$$

BAC3-11 Airfoil



BAC3-11 Airfoil





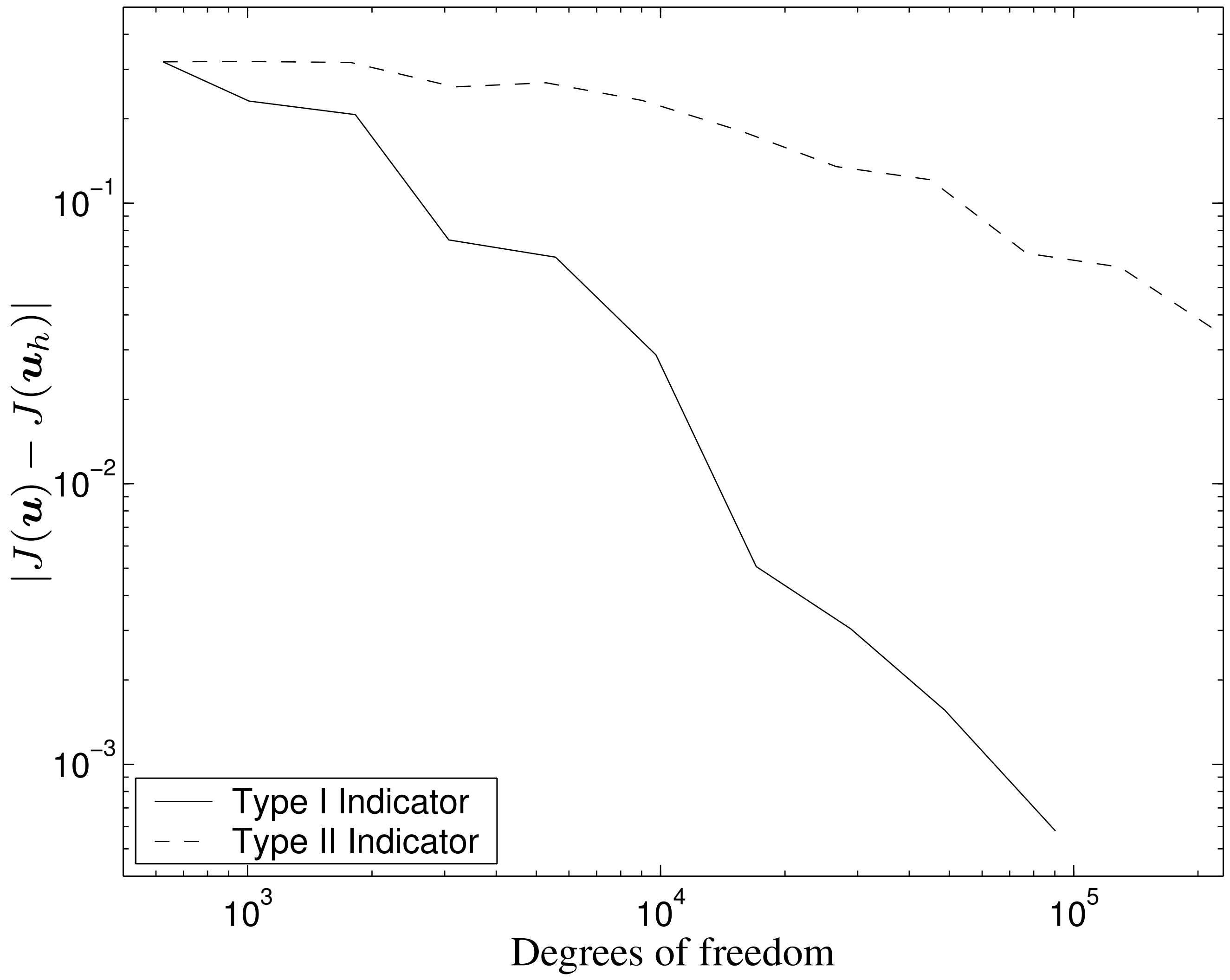
Type II: 13719 elements

$$|J(\boldsymbol{u}) - J(\boldsymbol{u}_h)| = 3.542 \times 10^{-2}$$

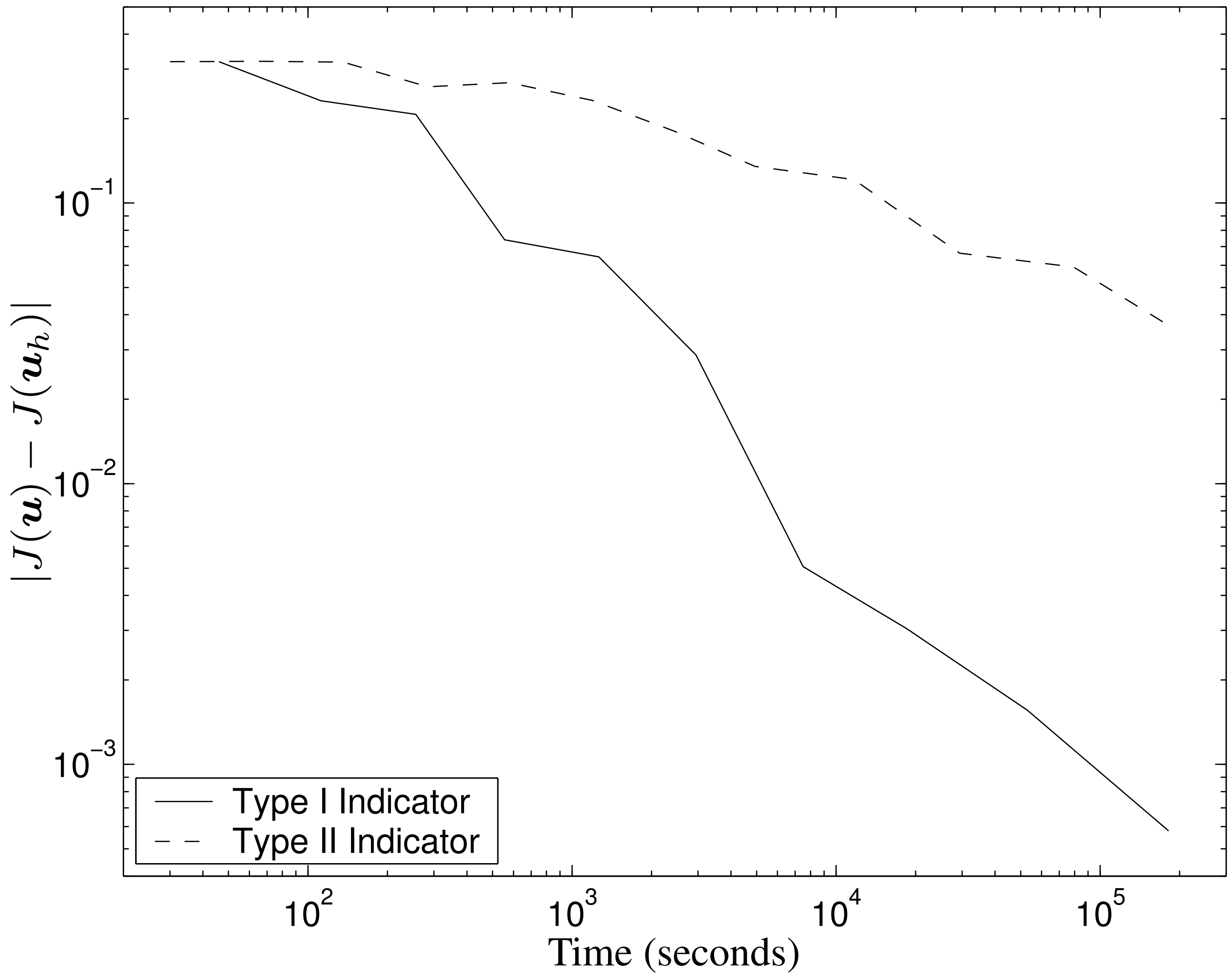
Type I: 1803 elements

$$|J(\boldsymbol{u}) - J(\boldsymbol{u}_h)| = 3.042 \times 10^{-3}$$

BAC3-11 Airfoil



BAC3-11 Airfoil



BAC3-11 Airfoil

Elements	$J(\boldsymbol{u}) - J(\boldsymbol{u}_h)$	$\sum_\kappa \eta_\kappa$	θ_1	$\sum_\kappa \lvert\eta_\kappa\rvert$	θ_2
63	2.31e-1	-1.50e-2	-0.06	2.00e-1	0.87
114	2.07e-1	7.28e-2	0.35	3.50e-1	1.69
192	7.40e-2	5.40e-2	0.73	2.68e-1	3.62
348	6.43e-2	2.70e-2	0.42	2.12e-1	3.30
609	2.88e-2	1.39e-2	0.48	1.84e-1	6.39
1065	5.07e-3	7.60e-3	1.50	1.17e-1	23.11
1803	3.04e-3	2.87e-3	0.94	1.03e-1	33.78
3045	1.56e-3	2.80e-3	1.79	1.07e-1	68.39
5643	5.79e-4	5.79e-4	1.00	5.55e-2	95.88

4. Second–Order PDEs

4. Second–Order PDEs
4.1 Discretization

Model Problem

Second–Order Semilinear System:

Given $\Omega \subset \mathbb{R}^n$ and $\boldsymbol{f} \in [L_2(\Omega)]^m$, find $\boldsymbol{u} : \Omega \rightarrow \mathbb{R}^m$, such that

$$\operatorname{div}(\mathcal{F}^c(\boldsymbol{u}) - \mathcal{F}^v(\boldsymbol{u}, \nabla \boldsymbol{u})) = \boldsymbol{f} \quad \text{in } \Omega.$$

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$$\mathrm{div}(\mathcal{F}^c(\boldsymbol{u}) - \mathcal{F}^v(\boldsymbol{u}, \nabla \boldsymbol{u})) = \boldsymbol{f} \quad \text{in } \Omega.$$

Writing

$$\mathcal{F}_i^v(\boldsymbol{u}, \nabla \boldsymbol{u}) = G_{ij}(\boldsymbol{u}) \partial \boldsymbol{u} / \partial x_j, \qquad i = 1, \ldots, n,$$

where $G_{ij}(\boldsymbol{u}) = \partial \mathcal{F}_i^v(\boldsymbol{u}, \nabla \boldsymbol{u}) / \partial \boldsymbol{u}_{x_j}$, $i, j = 1, \ldots, n$, gives

$$\frac{\partial}{\partial x_i}\left(\mathcal{F}_i^c(\boldsymbol{u}) - G_{ij}(\boldsymbol{u}) \frac{\partial \boldsymbol{u}}{\partial x_j}\right) \quad = \quad 0 \quad \text{in } \Omega.$$

Model Problem

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Given $\Omega \subset \mathbb{R}^n$ and $\boldsymbol{f} \in [L_2(\Omega)]^m$, find $\boldsymbol{u} : \Omega \to \mathbb{R}^m$, such that

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$$\frac{\partial}{\partial x_i}\left(\mathcal{F}_i^c(\boldsymbol{u}) - G_{ij}(\boldsymbol{u}) \frac{\partial \boldsymbol{u}}{\partial x_j}\right) = 0 \quad \text{in } \Omega,$$

$$\boldsymbol{u} = \boldsymbol{g}_{\mathrm{D}} \quad \text{on } \partial\Omega_{\mathrm{D}}, \qquad \mathcal{F}^v(\boldsymbol{u}, \nabla \boldsymbol{u}) \cdot \boldsymbol{n} = \boldsymbol{g}_{\mathrm{N}} \text{ on } \partial\Omega_{\mathrm{N}},$$

where $\partial\Omega = \partial\Omega_{\mathrm{D}} \cup \partial\Omega_{\mathrm{N}}$.

Model Problem

Second–Order Semilinear System:

Given $\Omega \subset \mathbb{R}^n$ and $\boldsymbol{f} \in [L_2(\Omega)]^m$, find $\boldsymbol{u} : \Omega \to \mathbb{R}^m$, such that

$$\boxed{\mathrm{div}(\mathcal{F}^c(\boldsymbol{u}) - \mathcal{F}^v(\boldsymbol{u}, \nabla \boldsymbol{u})) = \boldsymbol{f} \quad \text{in } \Omega.}$$

Writing

$$\mathcal{F}_i^v(\boldsymbol{u}, \nabla \boldsymbol{u}) = G_{ij}(\boldsymbol{u}) \partial \boldsymbol{u} / \partial x_j, \qquad i = 1, \ldots, n,$$

where $G_{ij}(\boldsymbol{u}) = \partial \mathcal{F}_i^v(\boldsymbol{u}, \nabla \boldsymbol{u}) / \partial \boldsymbol{u}_{x_j}$, $i, j = 1, \ldots, n$, gives

$$\frac{\partial}{\partial x_i}\left(\mathcal{F}_i^c(\boldsymbol{u}) - G_{ij}(\boldsymbol{u}) \frac{\partial \boldsymbol{u}}{\partial x_j}\right) = 0 \quad \text{in } \Omega,$$

$$\boldsymbol{u} = \boldsymbol{g}_{\mathrm{D}} \quad \text{on } \partial\Omega_{\mathrm{D}}, \qquad \mathcal{F}^v(\boldsymbol{u}, \nabla \boldsymbol{u}) \cdot \boldsymbol{n} = \boldsymbol{g}_{\mathrm{N}} \text{ on } \partial\Omega_{\mathrm{N}}.$$

Second–Order Quasi-linear PDEs:

Bustinza & Gatica 2004, Gatica, González, & Meddahi 2004, H. Robson, & Süli 2005.

Examples

- Linear (degenerate) convection–diffusion problem, $a \geq 0$

$$-\nabla \cdot (a \nabla u) + \nabla \cdot (\boldsymbol{b} u) + cu = f \quad \text{in } \Omega,$$

$$u = g_{\mathrm{D}} \quad \text{on } \partial\Omega_{\mathrm{D}}, \qquad \boldsymbol{n} \cdot (a \nabla u) = g_{\mathrm{N}} \quad \text{on } \partial\Omega_{\mathrm{N}}.$$

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$$-\nabla \cdot (a\nabla u) + \nabla \cdot (\boldsymbol{b}u) + cu = f \quad \text{in } \Omega,$$

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- Compressible Navier–Stokes Equations ($\Omega \subset \mathbb{R}^2$)

$$\boldsymbol{u} = [\rho, \rho v_1, \rho v_2, \rho E]^\top,$$

$$\mathcal{F}_i^c(\boldsymbol{u}) = [\rho v_i, \rho v_1 v_i + \delta_{1i} p, \rho v_2 v_i + \delta_{2i} p, \rho H v_i]^\top, \quad i = 1, 2,$$

$$\mathcal{F}_i^v(\boldsymbol{u}, \nabla \boldsymbol{u}) = [0, \tau_{1i}, \tau_{2i}, \tau_{ij} v_j + \mathcal{K} T_{x_i}]^\top, \quad i = 1, 2,$$

ρ: density, $\boldsymbol{v} = (v_1, v_2)$: velocity, E: energy, p: pressure, H: enthalpy, T: temperature, $\mathcal{K}$: thermal conductivity coefficient,

$$\tau = \mu \left(\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^\top - \tfrac{2}{3} (\nabla \cdot \boldsymbol{v}) I \right),$$

and μ: dynamic viscosity coefficient.

DGFEM for Diffusion

Babuška & Zlámal (1973)

(Symmetric) Interior Penalty Method; Douglas & Dupont (1976)

Bassi & Rebay (1997)

Bassi, Rebay, Mariotti, Pedinotti & Savini (1997)

Local DG Method; Cockburn & Shu (1998)

Brezzi, Manzini, Marini, Pietra and Russo (1999)

Baumann & Oden (1999)

(Non-Symmetric) IP Method; Riviere, Wheeler & Girault (1999)

Brezzi, Manzini, Marini, Pietra and Russo (2000)

Arnold, Brezzi, Cockburn & Marini (2002)

Prudhomme, Pascal, Oden & Romkes (2000)

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Poisson's Equation

Given $\Omega \subset \mathbb{R}^2$ and $f \in L_2(\Omega)$, find u such that

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

Discontinuous Galerkin Method

- $\mathcal{T}_h = \{\kappa\}$ is a non-degenerate mesh;
- Finite element space

$$S_{h,p} = \{\boldsymbol{v} \in [L_2(\Omega)]^m : \boldsymbol{v}|_\kappa \in [\mathcal{S}_p]^m \quad \forall \kappa \in \mathcal{T}_h\},$$

where, for each κ, we define

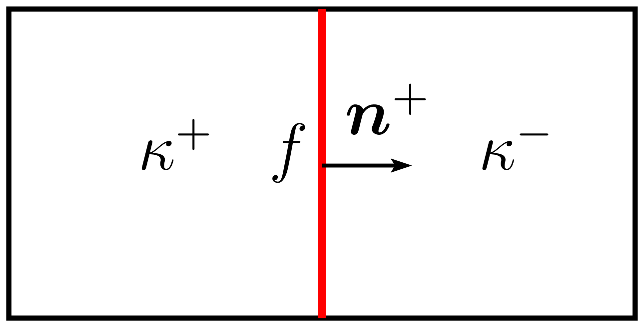
$$\mathcal{S}_\ell = \begin{cases} \mathcal{P}_\ell = \text{span}\,\{x^\alpha : 0 \leq |\alpha| \leq \ell\} & \text{(simplex)}, \\ \mathcal{Q}_\ell = \text{span}\,\{x^\alpha : 0 \leq \alpha_i \leq \ell,\ 1 \leq i \leq n\} & \text{(hypercube)}, \end{cases}$$

for $\ell \geq 1$.

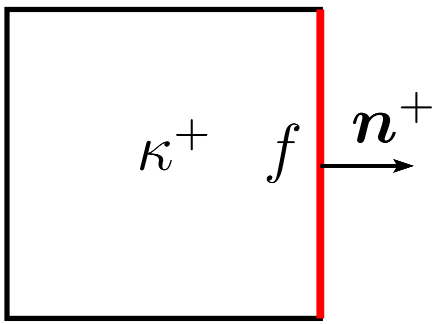
Discontinuous Galerkin Method

- $\mathcal{F}_h = \mathcal{F}_h^{\mathcal{I}} \cup \mathcal{F}_h^{\partial}$ set of all faces in the mesh $\mathcal{T}_h$.

$f \subset \mathcal{F}_h^{\mathcal{I}}$



$f \subset \mathcal{F}_h^{\partial}$



$$
\begin{aligned}
[\![v]\!] &= v^+ \boldsymbol{n}^+ + v^- \boldsymbol{n}^- \\
\underline{[\![\boldsymbol{q}]\!]} &= \boldsymbol{q}^+ \otimes \boldsymbol{n}^+ + \boldsymbol{q}^- \otimes \boldsymbol{n}^- \\
[\![\underline{\tau}]\!] &= \underline{\tau}^+ \boldsymbol{n}^+ + \underline{\tau}^- \boldsymbol{n}^- \\
\{\!\{ v \}\!\} &= (v^+ + v^-)/2
\end{aligned}
$$

$$
\begin{aligned}
[\![v]\!] &= v \boldsymbol{n} \\
\underline{[\![\boldsymbol{q}]\!]} &= \boldsymbol{q} \otimes \boldsymbol{n} \\
\{\!\{ v \}\!\} &= v
\end{aligned}
$$

Discontinuous Galerkin Method

- Rewrite as a first–order system:

$$\boldsymbol{s} - \nabla u = 0, \qquad -\nabla \cdot \boldsymbol{s} = f, \qquad \text{in } \Omega.$$

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- Elemental weak formulation: Find $(\boldsymbol{s}, u)$ such that

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$$\int_\kappa \boldsymbol{s} \cdot \boldsymbol{\tau} \, d\boldsymbol{x} + \int_\kappa \nabla \cdot \boldsymbol{\tau} u \, d\boldsymbol{x} - \int_{\partial\kappa} \boldsymbol{\tau} \cdot \boldsymbol{n}_\kappa u \, ds \quad = \quad 0,$$

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for all $(\boldsymbol{\tau}, v)$.

Discontinuous Galerkin Method

- Rewrite as a first–order system:

$$\boldsymbol{s} - \nabla u = 0, \qquad -\nabla\cdot\boldsymbol{s} = f, \qquad \text{in } \Omega.$$

- Elemental weak formulation: Find $(\boldsymbol{s}, u)$ such that

$$\begin{aligned}
\int_\kappa \boldsymbol{s}\cdot\boldsymbol{\tau}\,d\boldsymbol{x} + \int_\kappa \nabla\cdot\boldsymbol{\tau} u\,d\boldsymbol{x} - \int_{\partial\kappa} \boldsymbol{\tau}\cdot\boldsymbol{n}_\kappa \widehat{u}\,ds &= 0,\\
\int_\kappa \boldsymbol{s}\cdot\nabla v\,d\boldsymbol{x} - \int_{\partial\kappa} \widehat{\boldsymbol{s}}\cdot\boldsymbol{n}_\kappa v\,ds &= \int_\kappa f v\,d\boldsymbol{x}
\end{aligned}$$

for all $(\boldsymbol{\tau}, v)$.

Discontinuous Galerkin Method

- Numerical Fluxes:

$$\begin{aligned} \widehat{u} &= \{\!\{u\}\!\}, \\ \widehat{\boldsymbol{s}} &= \{\!\{\nabla_h u\}\!\} - \mathtt{a}[\![u]\!], \end{aligned}$$

where $\mathtt{a}$ denotes the interior penalty parameter.

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Select $\boldsymbol{\tau} = \nabla_h v$ and integrate by parts.

Discontinuous Galerkin Method

- Numerical Fluxes:

$$\widehat{u} = \{\!\{u\}\!\},$$
$$\widehat{\boldsymbol{s}} = \{\!\{\nabla_h u\}\!\} - \mathsf{a}[\![u]\!],$$

where a denotes the interior penalty parameter.

- Eliminate the auxiliary variables:

Select $\boldsymbol{\tau} = \nabla_h v$ and integrate by parts.

- Sum over the elements and restrict to FE spaces

Discontinuous Galerkin Method

Interior Penalty Method: Find $u_h \in S_{h,p}$ such that

$$\mathcal{B}(u_h, v_h) = \ell(v_h) \quad \forall v_h \in S_{h,p},$$

where

$$\begin{aligned}
\mathcal{B}(u_h, v_h) &= \int_\Omega \nabla_h u_h \cdot \nabla_h v_h \,\mathrm{d}\boldsymbol{x} - \int_{\mathcal{F}_h} [\![v_h]\!] \cdot \{\!\{ \nabla_h u_h \}\!\} \mathrm{d}s \\
&\quad - \int_{\mathcal{F}_h} [\![u_h]\!] \cdot \{\!\{ \nabla_h v_h \}\!\} \mathrm{d}s + \int_{\mathcal{F}_h} \mathtt{a} [\![u_h]\!] \cdot [\![v_h]\!] \mathrm{d}s, \\
\ell(v_h) &= \int_\Omega f v_h \,\mathrm{d}\boldsymbol{x} - \int_{\mathcal{F}_h^\partial} g(\nabla_h v_h \cdot \boldsymbol{n} - \mathtt{a} v_h) \mathrm{d}s.
\end{aligned}$$

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SIP: $\theta = -1$; IIP: $\theta = 0$; NIP: $\theta = 1$.

Discontinuous Galerkin Method

Interior Penalty Method: Find $u_h \in S_{h,p}$ such that

$$\mathcal{B}(u_h, v_h) = \ell(v_h) \quad \forall v_h \in S_{h,p}.$$

Penalty Parameter: Given $\gamma > 0$, we define

$$\mathsf{a}|_f = \gamma \frac{\{\!\{p\}\!\}^2}{h_f} \quad \text{for } f \in \mathcal{F}_h.$$

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Penalty Parameter: Given $\gamma > 0$, we define

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Galerkin Orthogonality: Assuming $u \in H^2(\Omega)$, we have

$$\mathcal{B}(u - u_h, v_h) = 0 \quad \forall v_h \in S_{h,p}.$$

Semilinear PDEs in 2D

$$
\begin{aligned}
&\mathcal{N}(\boldsymbol{u}_h, \boldsymbol{v}_h) \\
&= \sum_{\kappa \in \mathcal{T}_h} \left\{ -\int_\kappa \mathcal{F}^c(\boldsymbol{u}_h) : \nabla \boldsymbol{v}_h \mathrm{d}\boldsymbol{x} + \int_\kappa \mathcal{F}^v(\boldsymbol{u}_h, \nabla_h \boldsymbol{u}_h) : \nabla_h \boldsymbol{v}_h \mathrm{d}\boldsymbol{x} \right. \\
&\qquad \left. + \int_{\partial\kappa} \mathcal{H}(\boldsymbol{u}_h^{\mathrm{int}}, \boldsymbol{u}_h^{\mathrm{ext}}, \boldsymbol{n}_\kappa)\, \boldsymbol{v}_h^{\mathrm{int}} \mathrm{d}s \right\} \\
&- \int_{\mathcal{F}_h^{\mathcal{I}}} \{\!\{\mathcal{F}^v(\boldsymbol{u}_h, \nabla_h \boldsymbol{u}_h)\}\!\} : \underline{[\![\boldsymbol{v}_h]\!]} \mathrm{d}s - \int_{\partial\Omega \setminus \partial\Omega_N} \mathcal{F}^v(\boldsymbol{u}_h^{\mathrm{int}}, \nabla_h \boldsymbol{u}_h^{\mathrm{int}}) : \underline{[\![\boldsymbol{v}_h]\!]} \mathrm{d}s \\
&- \int_{\partial\Omega_{\mathrm{N}}} \boldsymbol{g}_{\mathrm{N}} \cdot \boldsymbol{v}_h \mathrm{d}s - \int_{\mathcal{F}_h^{\mathcal{I}}} \{\!\{ \left(G_{i1}^\top \partial_h \boldsymbol{v}_h / \partial x_i, G_{i2}^\top \partial_h \boldsymbol{v}_h / \partial x_i \right) \}\!\} : \underline{[\![\boldsymbol{u}_h]\!]} \mathrm{d}s \\
&- \int_{\partial\Omega \setminus \partial\Omega_{\mathrm{N}}} \left(G_{i1}^\top(\boldsymbol{u}_h^{\mathrm{int}}) \partial_h \boldsymbol{v}_h^{\mathrm{int}} / \partial x_i, G_{i2}^\top(\boldsymbol{u}_h^{\mathrm{int}}) \partial_h \boldsymbol{v}_h^{\mathrm{int}} / \partial x_i \right) : \left(\boldsymbol{u}_h^{\mathrm{int}} - \boldsymbol{u}_h^{\mathrm{ext}} \right) \otimes \boldsymbol{n} \mathrm{d}s \\
&+ \int_{\mathcal{F}_h^{\mathcal{I}}} \mathtt{a} \underline{[\![\boldsymbol{u}_h]\!]} : \underline{[\![\boldsymbol{v}_h]\!]} \mathrm{d}s + \int_{\partial\Omega \setminus \partial\Omega_N} \mathtt{a} \left(\boldsymbol{u}_h^{\mathrm{int}} - \boldsymbol{u}_h^{\mathrm{ext}} \right) \cdot \boldsymbol{v}_h^{\mathrm{int}} \mathrm{d}s.
\end{aligned}
$$

Semilinear PDEs in 2D

DGFEM: Find $\boldsymbol{u}_h \in S_{h,p}$ such that

$$\mathcal{N}(\boldsymbol{u}_h, \boldsymbol{v}_h) = \ell(\boldsymbol{v}_h) \qquad \forall \boldsymbol{v}_h \in S_{h,p},$$

where

$$\ell(\boldsymbol{v}_h) = \sum_{\kappa \in \mathcal{T}_h} \int_\kappa \boldsymbol{f} \cdot \boldsymbol{v}_h \mathrm{d}\boldsymbol{x}.$$

4.2 A Posteriori Error Estimation

Dual Problem

- Mean–value linearization of $J(\cdot)$:

$$\begin{aligned}\bar{J}(\boldsymbol{u},\boldsymbol{u}_h;\boldsymbol{u}-\boldsymbol{u}_h) &= J(\boldsymbol{u})-J(\boldsymbol{u}_h)\\ &= \int_0^1 J'[\theta\boldsymbol{u}+(1-\theta)\boldsymbol{u}_h](\boldsymbol{u}-\boldsymbol{u}_h)\mathrm{d}\theta.\end{aligned}$$

Dual Problem

- Mean–value linearization of $J(\cdot)$:

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- Mean–value linearisation of $\mathcal{N}(\cdot,\cdot)$:

$$\begin{aligned}\mathcal{M}(\boldsymbol{u}, \boldsymbol{u}_h; \boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{v}) &= \mathcal{N}(\boldsymbol{u}, \boldsymbol{v}) - \mathcal{N}(\boldsymbol{u}_h, \boldsymbol{v}) \\ &= \int_0^1 \mathcal{N}'_{\boldsymbol{u}}[\theta\boldsymbol{u} + (1-\theta)\boldsymbol{u}_h](\boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{v})\mathrm{d}\theta.\end{aligned}$$

Dual Problem

- Mean–value linearization of $J(\cdot)$:

$$\begin{aligned}\bar{J}(\boldsymbol{u}, \boldsymbol{u}_h; \boldsymbol{u} - \boldsymbol{u}_h) &= J(\boldsymbol{u}) - J(\boldsymbol{u}_h) \\ &= \int_0^1 J'[\theta \boldsymbol{u} + (1-\theta)\boldsymbol{u}_h](\boldsymbol{u} - \boldsymbol{u}_h)\mathrm{d}\theta.\end{aligned}$$

- Mean–value linearisation of $\mathcal{N}(\cdot, \cdot)$:

$$\begin{aligned}\mathcal{M}(\boldsymbol{u}, \boldsymbol{u}_h; \boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}) &= \mathcal{N}(\boldsymbol{u}, \boldsymbol{v}) - \mathcal{N}(\boldsymbol{u}_h, \boldsymbol{v}) \\ &= \int_0^1 \mathcal{N}'_{\boldsymbol{u}}[\theta \boldsymbol{u} + (1-\theta)\boldsymbol{u}_h](\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v})\mathrm{d}\theta.\end{aligned}$$

- Dual problem: find $\boldsymbol{z}$ such that

$$\mathcal{M}(\boldsymbol{u}, \boldsymbol{u}_h; \boldsymbol{w}, \boldsymbol{z}) = \bar{J}(\boldsymbol{u}, \boldsymbol{u}_h; \boldsymbol{w}) \quad \forall \boldsymbol{w}.$$

Error Representation Formula

$$J(\boldsymbol{u}) - J(\boldsymbol{u}_h) = \sum_{\kappa \in \mathcal{T}_h} \eta_\kappa,$$

$$\eta_\kappa = \int_\kappa \boldsymbol{r}_h \cdot \boldsymbol{\omega}_\hbar \mathrm{d}\boldsymbol{x} + \int_{\partial\kappa} (\mathcal{F}^c(\boldsymbol{u}_h) \cdot \boldsymbol{n}_\kappa - \mathcal{H}(\boldsymbol{u}_h^{\mathrm{int}}, \boldsymbol{u}_h^-, \boldsymbol{n}_\kappa)) \cdot \boldsymbol{\omega}_\hbar \mathrm{d}s$$

$$+ \frac{1}{2} \int_{\partial\kappa \backslash \partial\Omega} \Big(\big(G_{i1}^\top \partial_h \boldsymbol{\omega}_\hbar / \partial x_i, G_{i2}^\top \partial_h \boldsymbol{\omega}_\hbar / \partial x_i \big) : \underline{[\![\boldsymbol{u}_h]\!]} - [\![\mathcal{F}^v(\boldsymbol{u}_h, \nabla \boldsymbol{u}_h)]\!] \cdot \boldsymbol{\omega}_\hbar \Big) \mathrm{d}s$$

$$+ \int_{\partial\kappa \cap (\partial\Omega \backslash \partial\Omega_{\mathrm{N}})} \big(G_{i1}^\top (\boldsymbol{u}_h^{\mathrm{int}}) \partial_h \boldsymbol{\omega}_\hbar / \partial x_i, G_{i2}^\top (\boldsymbol{u}_h^{\mathrm{int}}) \partial_h \boldsymbol{\omega}_\hbar / \partial x_i \big) : \big(\boldsymbol{u}_h^{\mathrm{int}} - \boldsymbol{u}_h^{\mathrm{ext}} \big) \otimes \boldsymbol{n} \mathrm{d}s$$

$$- \int_{\partial\kappa \cap \partial\Omega_{\mathrm{N}}} (\mathcal{F}^v(\boldsymbol{u}_h^{\mathrm{int}}, \nabla \boldsymbol{u}_h^{\mathrm{int}}) \cdot \boldsymbol{n}_\kappa - \boldsymbol{g}_{\mathrm{N}}) \cdot \boldsymbol{\omega}_\hbar \mathrm{d}s$$

$$- \int_{\partial\kappa \backslash \partial\Omega} \mathsf{a} \underline{[\![\boldsymbol{u}_h]\!]} : \boldsymbol{\omega}_\hbar \otimes \boldsymbol{n}_\kappa \mathrm{d}s - \int_{\partial\kappa \cap (\partial\Omega \backslash \partial\Omega_N)} \mathsf{a} \left(\boldsymbol{u}_h^{\mathrm{int}} - \boldsymbol{u}_h^{\mathrm{ext}} \right) \cdot \boldsymbol{\omega}_\hbar \mathrm{d}s,$$

$\boldsymbol{r}_h = \boldsymbol{f} - \nabla \cdot \mathcal{F}^c(\boldsymbol{u}_h) + \nabla \cdot \mathcal{F}^v(\boldsymbol{u}_h, \nabla \boldsymbol{u}_h)$, and $\boldsymbol{\omega}_\hbar = \boldsymbol{z} - \boldsymbol{z}_h$, $\boldsymbol{z}_h \in S_{h,p}$.

A Posteriori Error Bounds

Theorem [Type I Bound]

Assuming the dual problem is well–posed, the following result holds:

$$|J(\boldsymbol{u}) - J(\boldsymbol{u}_h)| \leq \sum_{\kappa \in \mathcal{T}_h} \eta_\kappa^{(\mathsf{I})},$$

where $\eta_\kappa^{(\mathsf{I})} = |\eta_\kappa|$.

A Posteriori Error Bounds

Theorem [Type I I Bound]

Assuming $\|\boldsymbol{z}\|_{H^s(\Omega)} \le C_{\text{stab}}$, $2 \le s \le p+1$, we have that

$$|J(\boldsymbol{u}) - J(\boldsymbol{u}_h)| \le C \left(\sum_{\kappa \in \mathcal{T}_h} \left(\eta_\kappa^{(\mathrm{II})}\right)^2 \right)^{1/2},$$

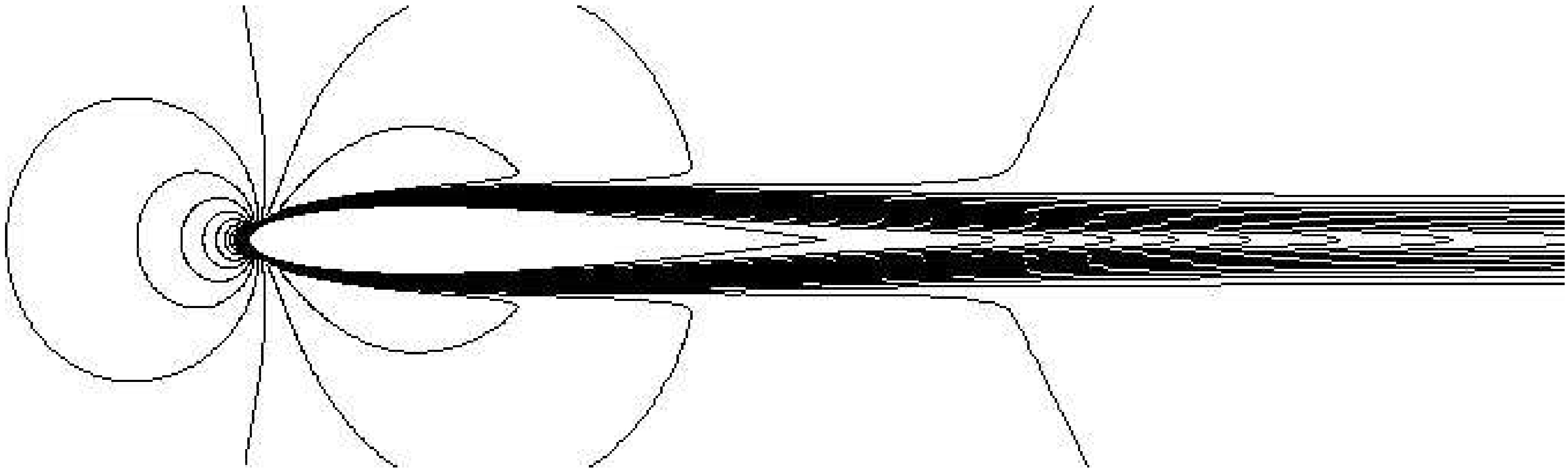
$$\begin{aligned}
\eta_\kappa^{(\mathrm{II})} \quad = \quad & h^s \|\boldsymbol{r}_h\|_{L_2(\kappa)} + h^{s-1/2} \|\mathcal{F}^c(\boldsymbol{u}_h) \cdot \boldsymbol{n}_\kappa - \mathcal{H}(\boldsymbol{u}_h^{\text{int}}, \boldsymbol{u}_h^{\text{ext}}, \boldsymbol{n}_\kappa)\|_{L_2(\partial\kappa)} \\
& + h^{s-3/2} \|G_{\cdot j} \underline{[\![\boldsymbol{u}_h]\!]}_j \|_{L_2(\partial\kappa \setminus \partial\Omega)} + h^{s-1/2} \|[\![\mathcal{F}^v(\boldsymbol{u}_h, \nabla \boldsymbol{u}_h)]\!]\|_{L_2(\partial\kappa \setminus \partial\Omega)} \\
& + h^{s-1/2} \|\mathcal{F}^v(\boldsymbol{u}_h^{\text{int}}, \nabla \boldsymbol{u}_h^{\text{int}}) \cdot \boldsymbol{n}_\kappa - \boldsymbol{g}_{\mathrm{N}}\|_{L_2(\partial\kappa \cap \partial\Omega_{\mathrm{N}})} \\
& + h^{s-3/2} \|G_{\cdot j}(\boldsymbol{u}_h^{\text{int}}) \left[\left(\boldsymbol{u}_h^{\text{int}} - \boldsymbol{u}_h^{\text{ext}}\right) \otimes \boldsymbol{n}\right]_j \|_{L_2(\partial\kappa \cap (\partial\Omega \setminus \partial\Omega_{\mathrm{N}}))} \\
& + h^{s-1/2} \|\mathsf{a} \left(\boldsymbol{u}_h^{\text{int}} - \boldsymbol{u}_h^{\text{ext}}\right)\|_{L_2(\partial\kappa \setminus \partial\Omega_{\mathrm{N}})},
\end{aligned}$$

and $\|G_{\cdot j} \underline{[\![\boldsymbol{u}_h]\!]}_j \|_{L_2(\partial\kappa)} = (\sum_{i=1}^2 \int_{\partial\kappa} |G_{ij} \underline{[\![\boldsymbol{u}_h]\!]}_j|^2 \mathrm{d}s)^{\frac{1}{2}}$.

4.3 Numerical Examples

NACA0012: $\mathrm{Ma} = 0.5$, $\mathrm{Re} = 5000$, $\alpha = 0^\circ$

$\mathrm{Ma} = 0.5$, $\mathrm{Re} = 5000$, $\alpha = 0^\circ$ and adiabatic wall condition.



Mach isolines

NACA0012: Ma $= 0.5$, Re $= 5000$, $\alpha = 0^\circ$

Drag coefficients:

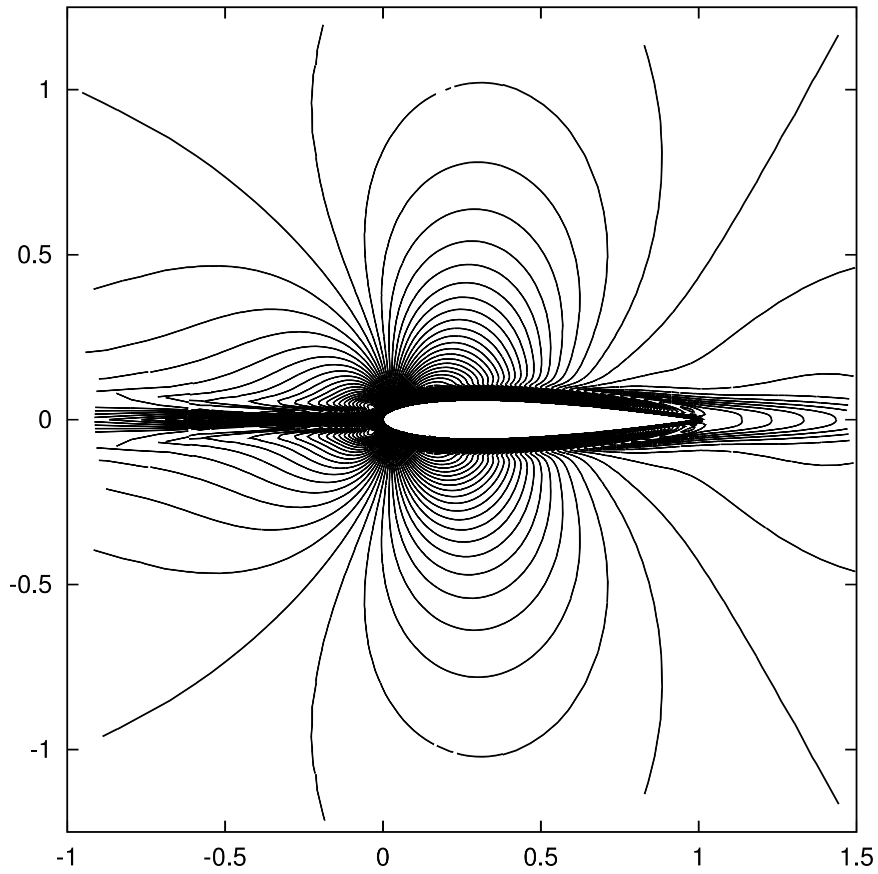
$$J_{c_{\mathrm{dp}}}(\boldsymbol{u}) = \frac{2}{l\bar{\rho}|\bar{\boldsymbol{v}}|^2}\int_S p\,(\boldsymbol{n}\cdot\psi_d)\mathrm{d}s, \qquad J_{c_{\mathrm{df}}}(\boldsymbol{u}) = \frac{2}{l\bar{\rho}|\bar{\boldsymbol{v}}|^2}\int_S (\boldsymbol{\tau}\,\boldsymbol{n})\cdot\psi_d\mathrm{d}s,$$

where

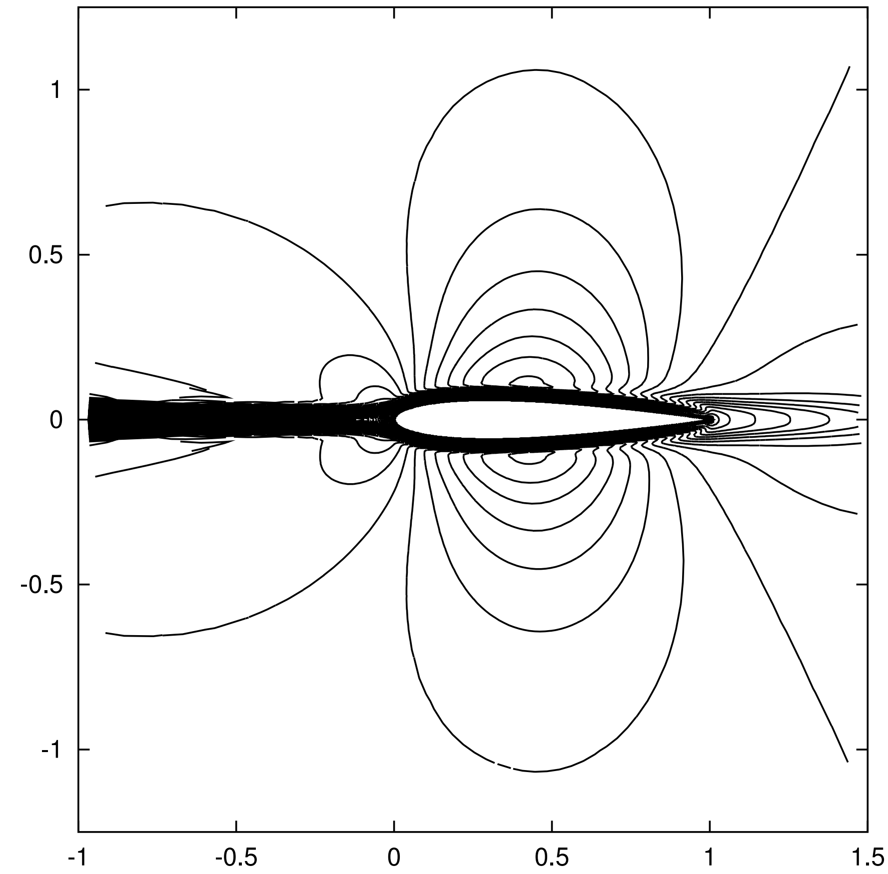
$$\psi_d = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

$$J_{c_{\mathrm{dp}}}(\boldsymbol{u}) \approx 0.0222875, \qquad J_{c_{\mathrm{df}}}(\boldsymbol{u}) \approx 0.032535.$$

NACA0012: $\mathrm{Ma} = 0.5$, $\mathrm{Re} = 5000$, $\alpha = 0°$

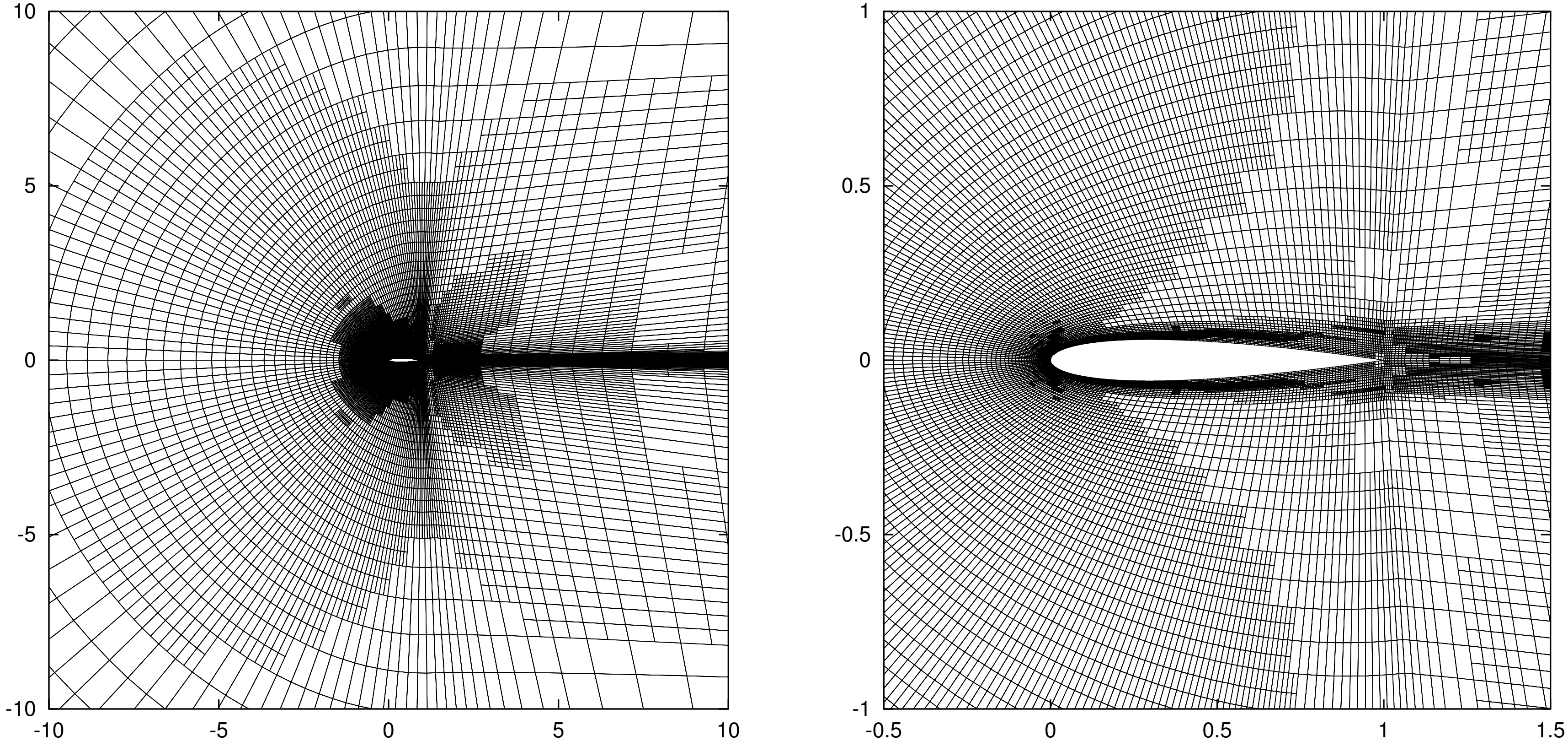


Dual solution for $J_{c_{\mathrm{dp}}}$



Dual solution for $J_{c_{\mathrm{df}}}$

NACA0012: Ma = 0.5, Re = 5000, $\alpha = 0°$

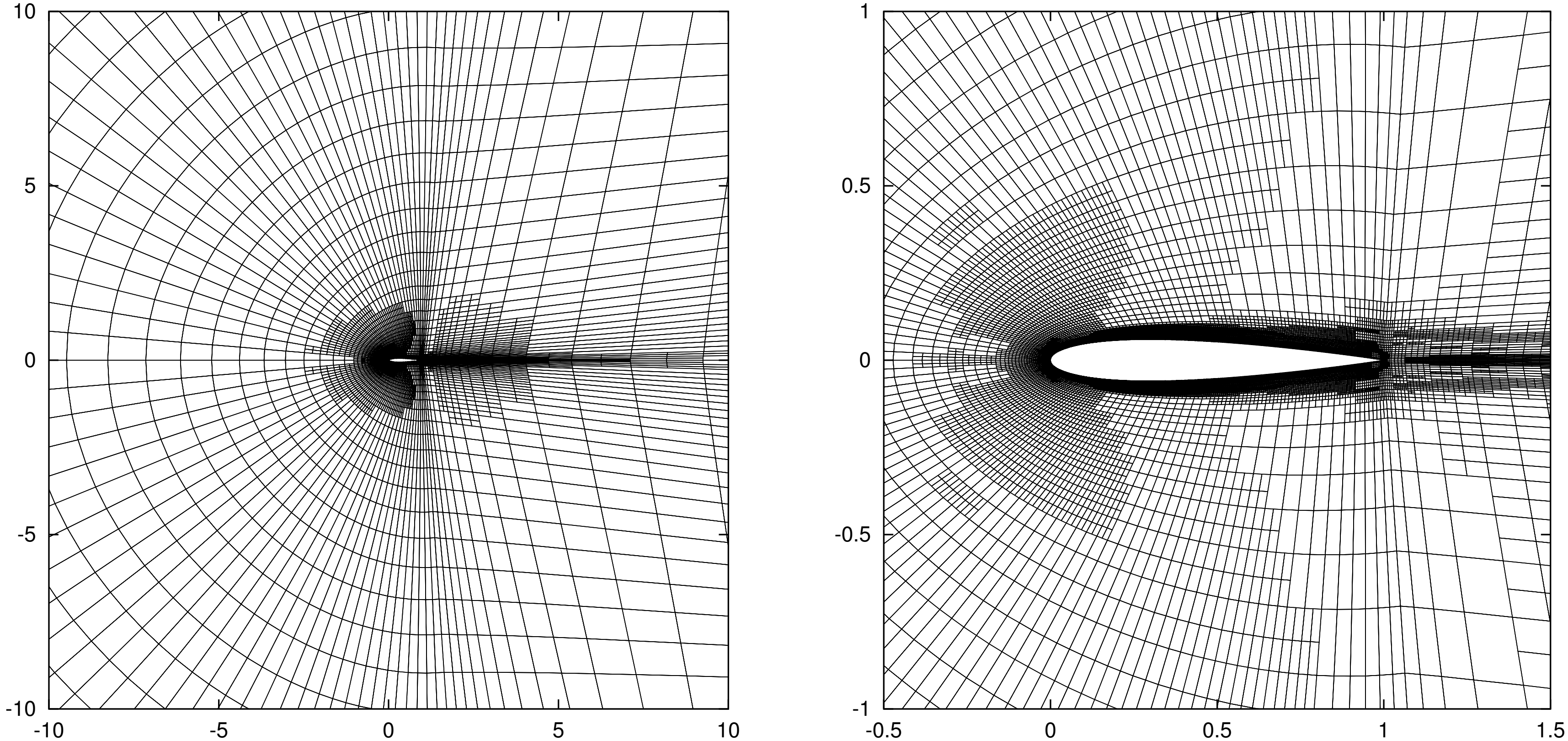


Mesh designed using Type II indicator

34458 elements, $|J_{c_{\mathrm{dp}}}(\boldsymbol{u}) - J_{c_{\mathrm{dp}}}(\boldsymbol{u}_h)| = 3.151 \times 10^{-4}$,

$$|J_{c_{\mathrm{df}}}(\boldsymbol{u}) - J_{c_{\mathrm{df}}}(\boldsymbol{u}_h)| = 1.090 \times 10^{-3}$$

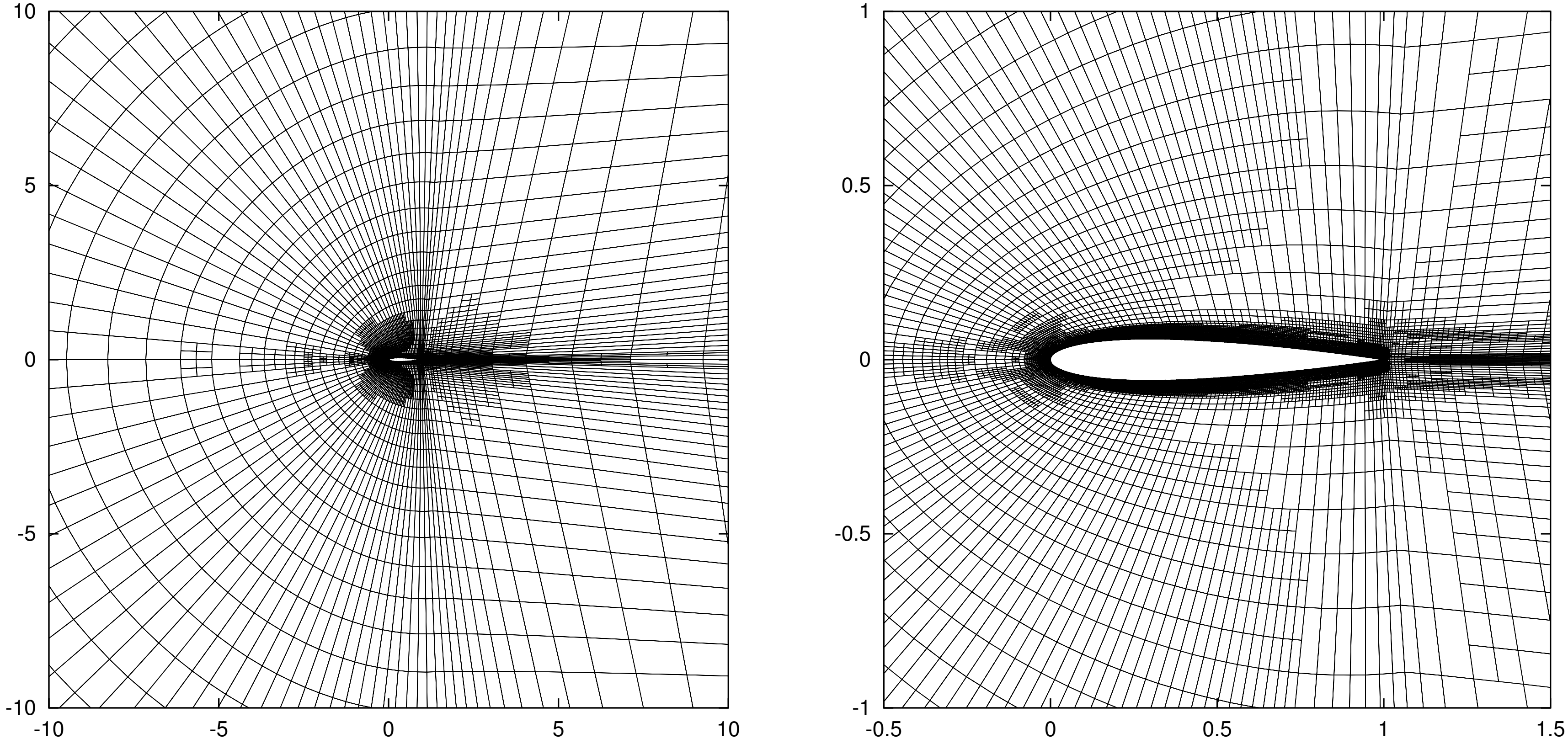
NACA0012: $\mathrm{Ma} = 0.5$, $\mathrm{Re} = 5000$, $\alpha = 0^\circ$



Mesh designed using Type I indicator

35610 elements, $|J_{c_{\mathrm{dp}}}(\boldsymbol{u}) - J_{c_{\mathrm{dp}}}(\boldsymbol{u}_h)| = 2.54 \times 10^{-5}$.

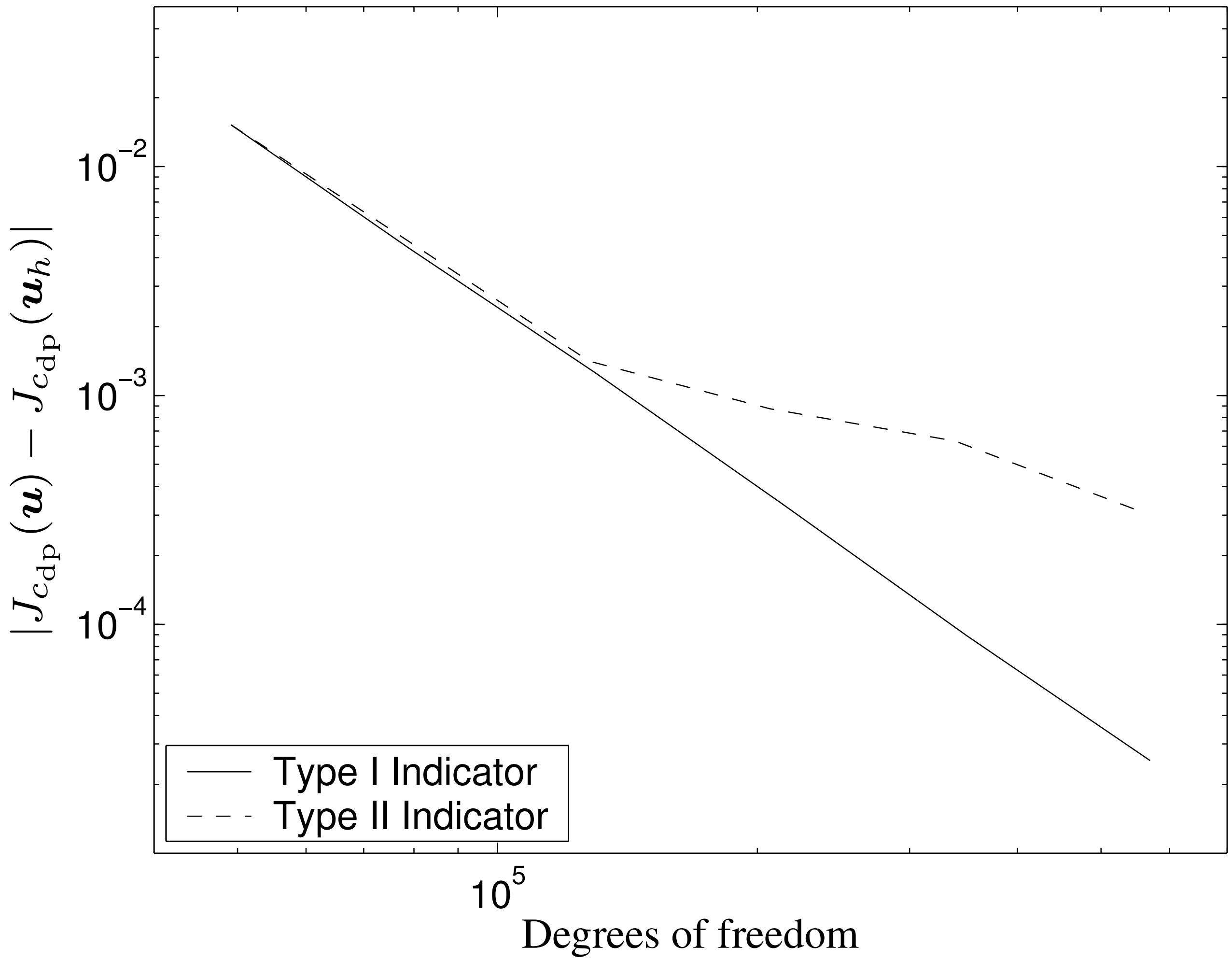
NACA0012: $\mathrm{Ma} = 0.5$, $\mathrm{Re} = 5000$, $\alpha = 0^\circ$



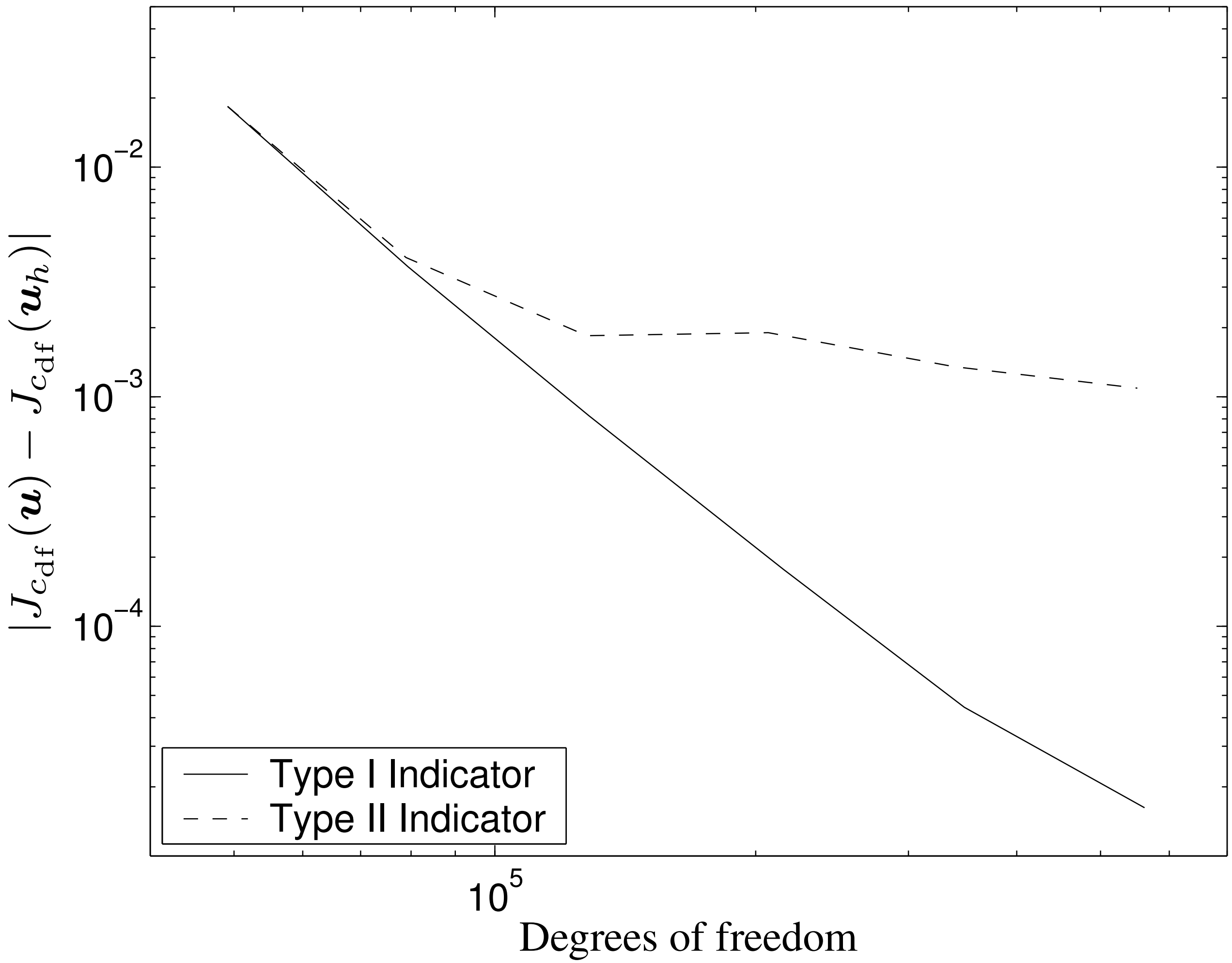
Mesh designed using Type I indicator

35610 elements, $|J_{c_{\mathrm{df}}}(\boldsymbol{u}) - J_{c_{\mathrm{df}}}(\boldsymbol{u}_h)| = 1.62 \times 10^{-5}$.

NACA0012: $\mathrm{Ma} = 0.5$, $\mathrm{Re} = 5000$, $\alpha = 0°$



NACA0012: $\mathrm{Ma} = 0.5$, $\mathrm{Re} = 5000$, $\alpha = 0^\circ$



NACA0012: $\mathrm{Ma} = 0.5$, $\mathrm{Re} = 5000$, $\alpha = 0°$

Elements	$J_{c_{\mathrm{dp}}}(\boldsymbol{u}) - J_{c_{\mathrm{dp}}}(\boldsymbol{u}_h)$	$\sum_{\kappa\in\mathcal{T}_h} \eta_\kappa$	θ_1	$\sum_{\kappa\in\mathcal{T}_h} \lvert\eta_\kappa\rvert$	θ_2
3072	1.522e-02	1.040e-02	0.68	1.963e-02	1.29
4929	4.410e-03	3.839e-03	0.87	6.659e-03	1.51
8097	1.262e-03	1.156e-03	0.92	2.208e-03	1.75
13467	3.285e-04	3.106e-04	0.95	7.156e-04	2.18
21846	8.918e-05	8.675e-05	0.97	2.725e-04	3.06
35610	2.536e-05	2.530e-05	1.00	1.253e-04	4.94

NACA0012: Ma $= 0.5$, Re $= 5000$, $\alpha = 0^\circ$

Elements	$J_{c_{\mathrm{df}}}(\boldsymbol{u}) - J_{c_{\mathrm{df}}}(\boldsymbol{u}_h)$	$\sum_{\kappa\in\mathcal{T}_h} \eta_\kappa$	θ_1	$\sum_{\kappa\in\mathcal{T}_h} \lvert\eta_\kappa\rvert$	θ_2
3072	-1.839e-02	-1.274e-02	0.69	2.430e-02	1.32
4962	-3.680e-03	-3.239e-03	0.88	9.399e-03	2.55
8028	-8.246e-04	-7.596e-04	0.92	4.209e-03	5.10
13446	-1.773e-04	-1.680e-04	0.95	2.067e-03	11.65
21750	-4.444e-05	-4.258e-05	0.96	1.044e-03	23.48
35118	-1.624e-05	-1.626e-05	1.00	5.328e-04	32.82

5. High–Order/hp–Adaptive DG Methods

High–order/hp–FEM

- High–order/variable order FEMs
  Babuška, Szabo, & Katz 1981, Gui & Babuška 1986, Schwab 1998

High–order/hp–FEM

- High–order/variable order FEMs
  Babuška, Szabo, & Katz 1981, Gui & Babuška 1986, Schwab 1998
- Approximation theory
  - Suppose that $u|_\Omega \in H^k(\Omega)$

$$\|u - \Pi_{hp} u\|_{H^s(\Omega)} \leq C \, \frac{h^{\min(p+1,k)-s}}{p^{k-s}} \, \|u\|_{H^k(\Omega)},$$

$0 \leq s \leq \min(p+1, k)$.

Babuška & Suri 1987

High–order/hp–FEM

- High–order/variable order FEMs
  Babuška, Szabo, & Katz 1981, Gui & Babuška 1986, Schwab 1998

- Approximation theory
  - Suppose that $u|_\Omega \in H^k(\Omega)$

$$\|u - \Pi_{hp} u\|_{H^s(\Omega)} \leq C \, \frac{h^{\min(p+1,k)-s}}{p^{k-s}} \, \|u\|_{H^k(\Omega)},$$

$0 \leq s \leq \min(p+1, k)$. Babuška & Suri 1987

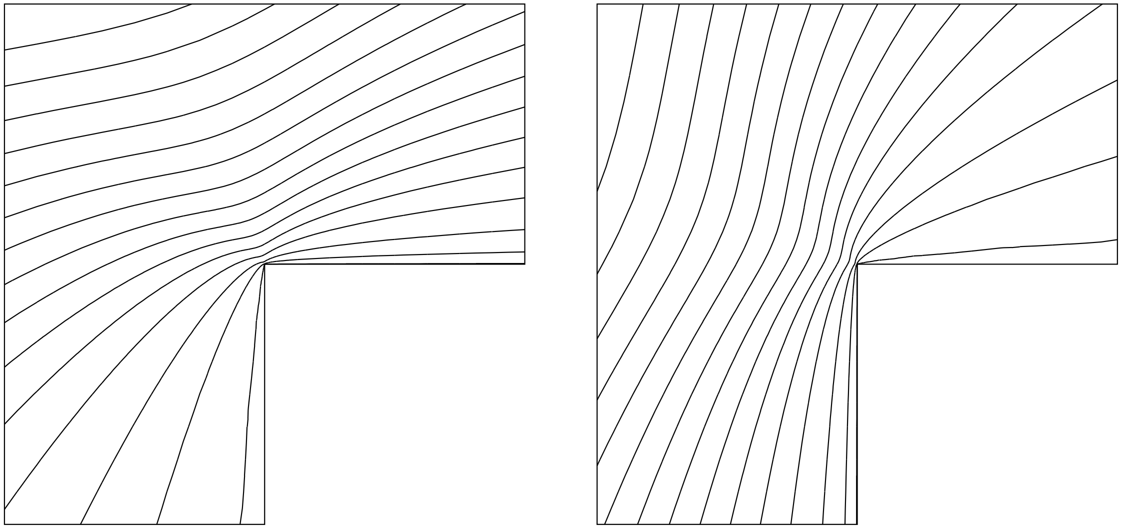
  - If u is a real analytic function on Ω, then

$$\|u - \Pi_{hp} u\|_{H^s(\Omega)} \leq C(u) \, h^{p+1-s} \, \mathrm{e}^{-bp}, \quad b > 0,$$

$\Rightarrow$ Exponential convergence Schwab 1998

High–order/hp–FEM

- Example: 2D Stokes equations in an L-shaped domain.



$\boldsymbol{u}_1$ $\boldsymbol{u}_2$

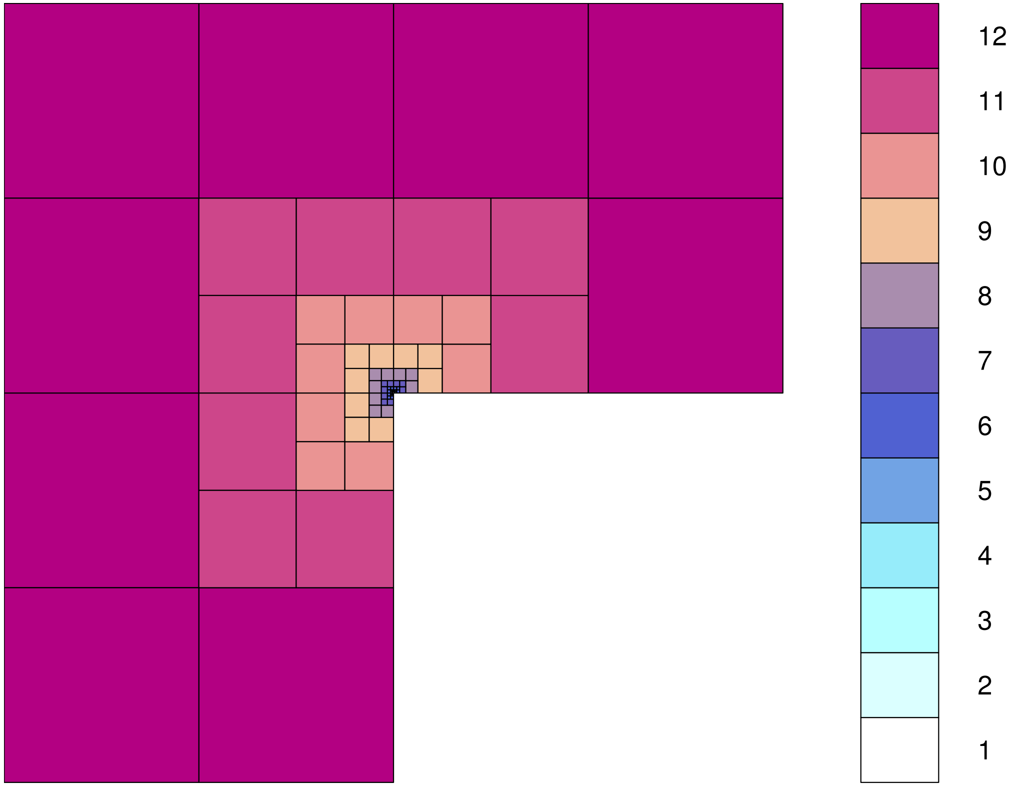
In particular,

$$(\boldsymbol{u}, p) \in H^{1+\varepsilon}(\Omega)^2 \times H^{\varepsilon}(\Omega), \quad \varepsilon > 0.$$

Dauge 1989

High–order/hp–FEM

- Example: 2D Stokes equations in an L-shaped domain.
- hp–Refinement



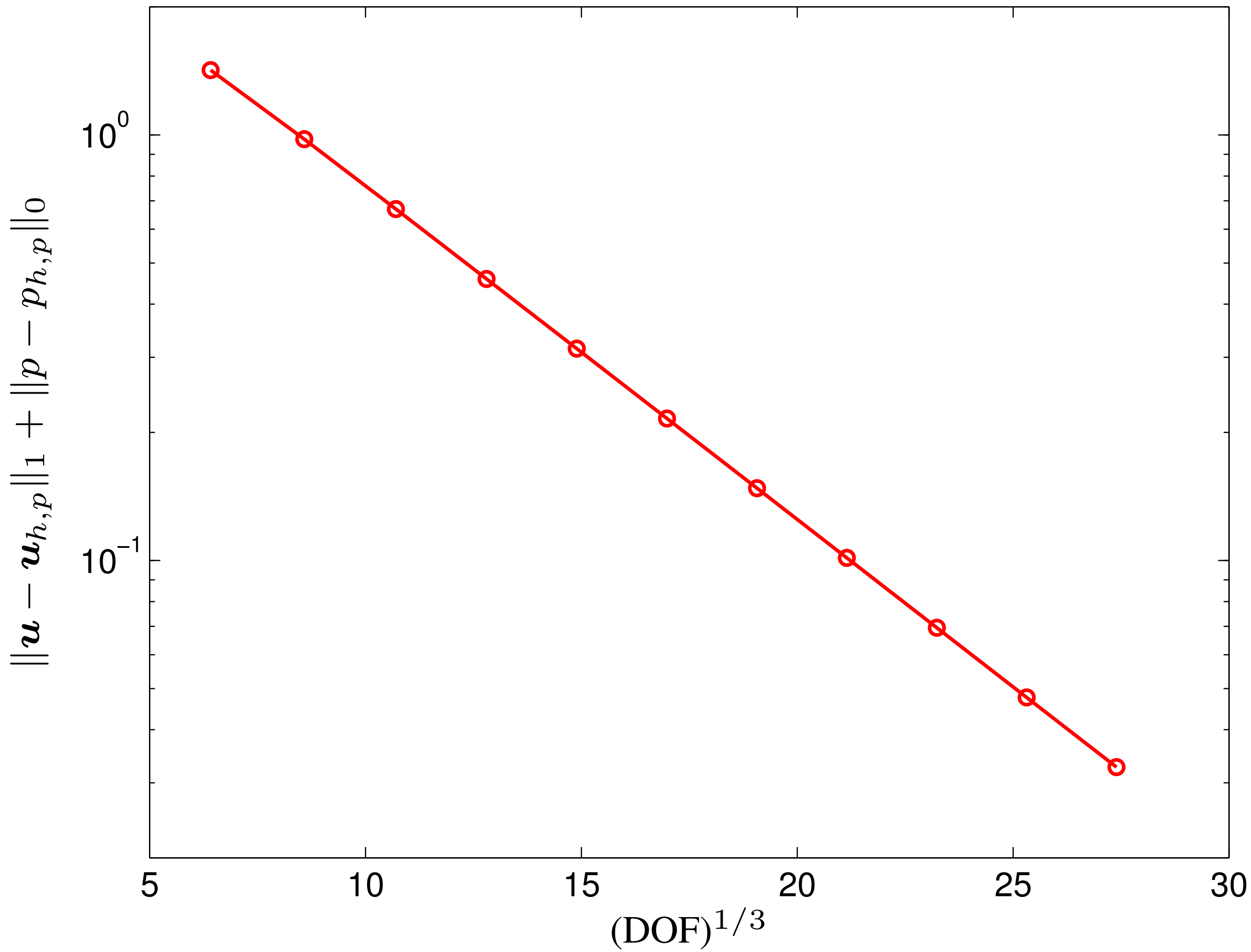
High–order/hp–FEM

- Example: 2D Stokes equations in an L-shaped domain.
- hp–Refinement

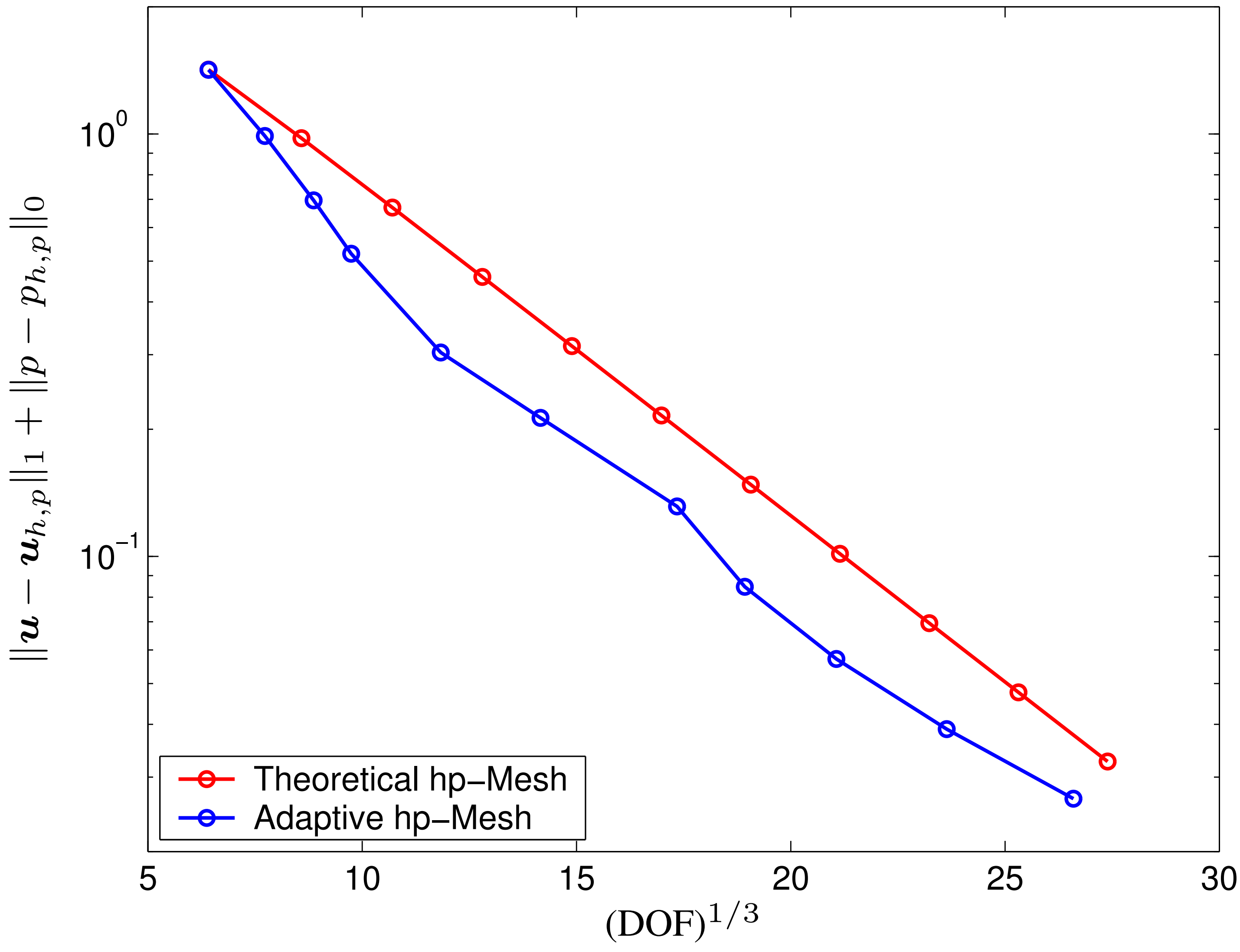
$$\|\boldsymbol{u} - \boldsymbol{u}_{h,p}\|_{H^1(\Omega)} + \|p - p_{h,p}\|_{L_2(\Omega)} \leq C \exp(-\gamma \, (\text{DOF})^{1/3}), \quad \gamma > 0.$$

Houston, Schötzau, & Wihler 2003

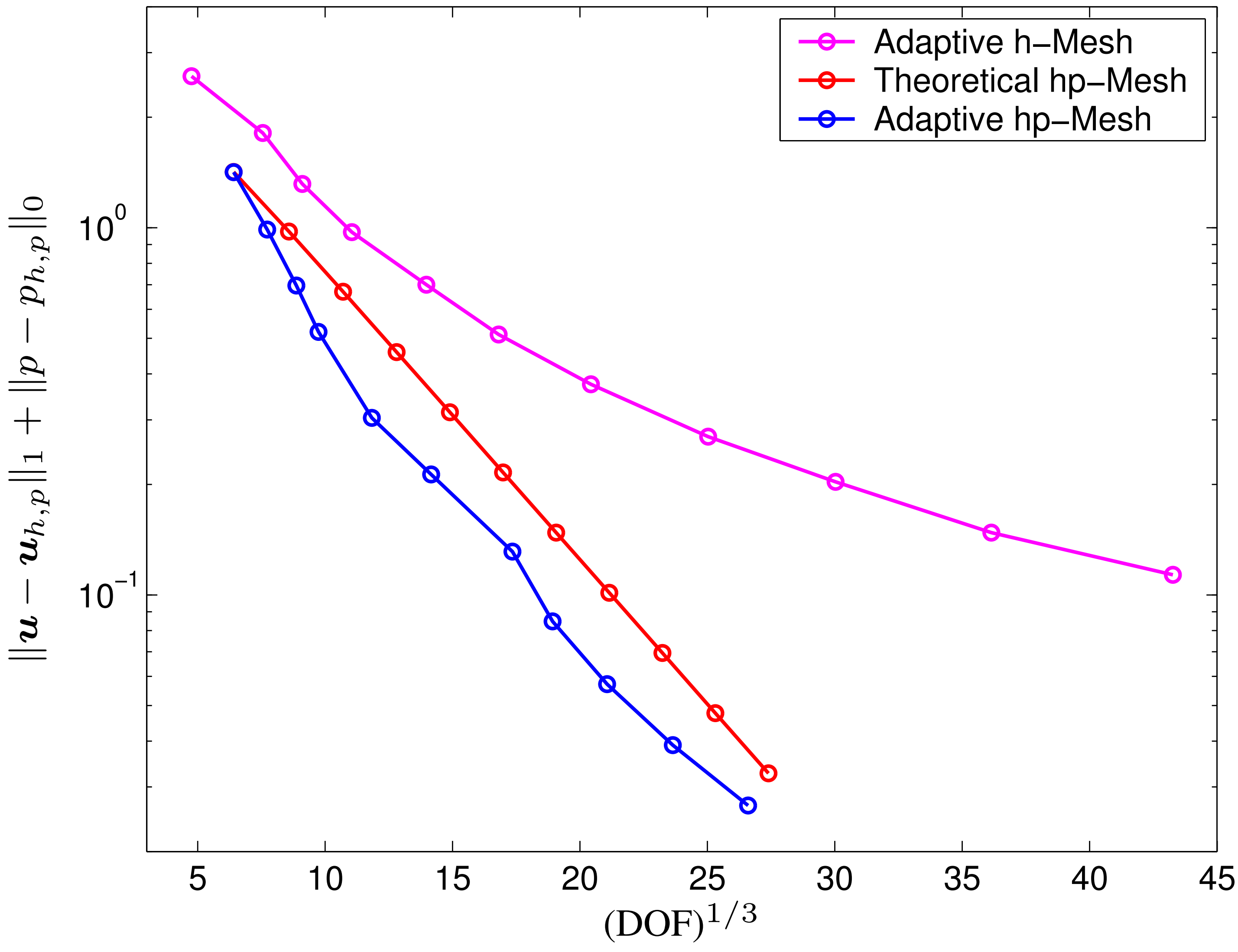
2D Stokes Equations



2D Stokes Equations



2D Stokes Equations



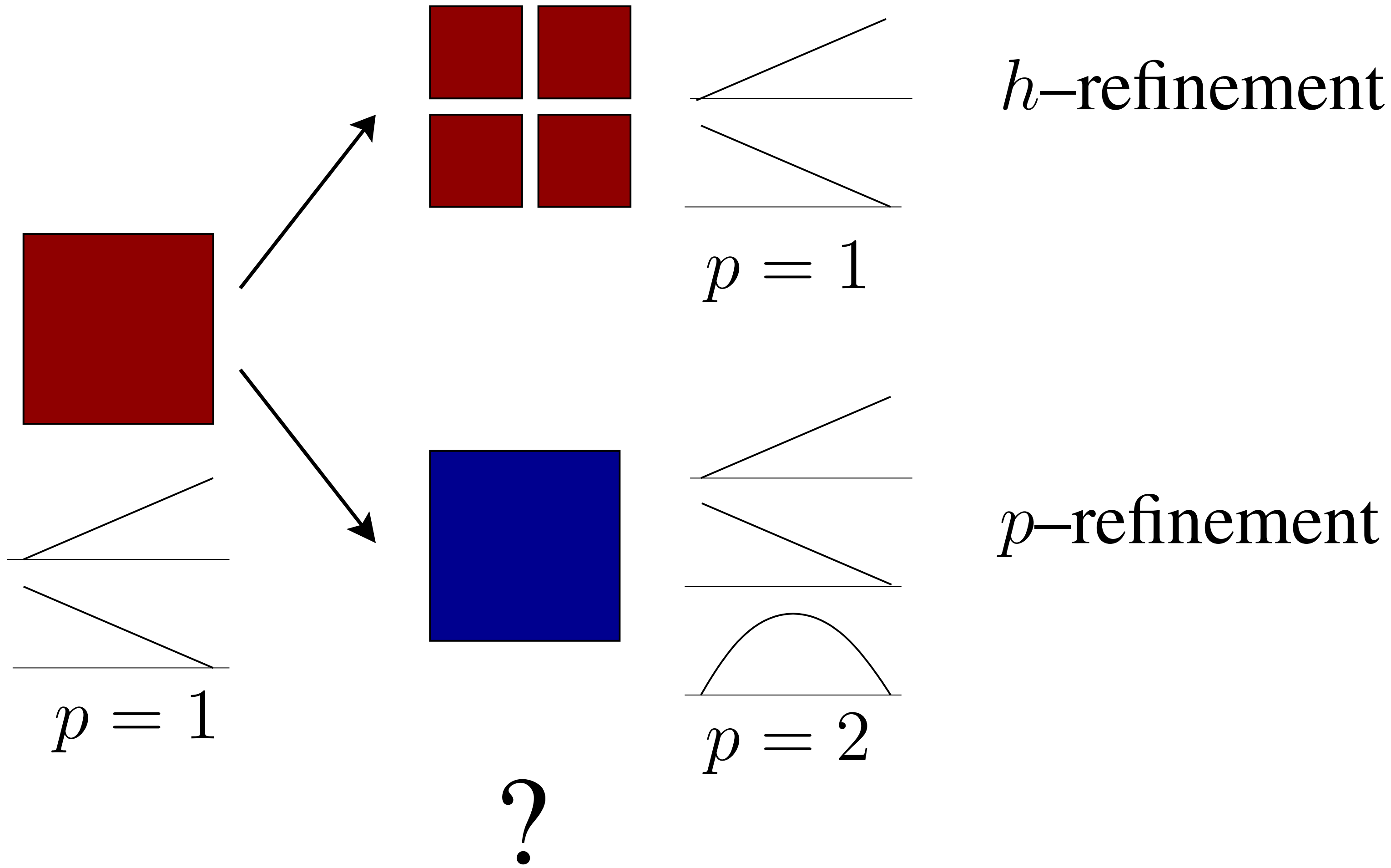
Flow Over a Flat Plate

$\mathrm{Ma} = 0.01$, $\mathrm{Re} = 10000$, $\alpha = 0^\circ$

	$p = 1$	$p = 2$	$p = 3$
Elements	36	5	3
DoFs	72	15	12

Resolution of boundary layer needed to approximate $J_{c_{\mathrm{df}}}(\boldsymbol{u})$ to 5%.

Mesh Modification Strategy



A Priori Error Analysis

Linear (degenerate) convection–diffusion problem: find u such that

$$
\begin{aligned}
-\nabla \cdot (a\nabla u) + \nabla \cdot (\boldsymbol{b} u) + cu &= f \quad \text{in } \Omega, \\
u = g_{\mathrm{D}} \quad \text{on } \partial\Omega_{\mathrm{D}}, \qquad \boldsymbol{n} \cdot (a\nabla u) &= g_{\mathrm{N}} \quad \text{on } \partial\Omega_{\mathrm{N}}.
\end{aligned}
$$

A Priori Error Analysis

Linear (degenerate) convection–diffusion problem: find u such that

$$
\begin{aligned}
-\nabla \cdot (a\nabla u) + \nabla \cdot (\boldsymbol{b}u) + cu &= f \quad \text{in } \Omega, \\
u = g_{\mathrm{D}} \quad \text{on } \partial\Omega_{\mathrm{D}}, \qquad \boldsymbol{n} \cdot (a\nabla u) &= g_{\mathrm{N}} \quad \text{on } \partial\Omega_{\mathrm{N}}.
\end{aligned}
$$

Theorem Harriman, H., Senior, & Süli 2003

For $u|_\kappa \in H^{k_\kappa}(\kappa)$, $k_\kappa \geq 2$, and $z|_\kappa \in H^{l_\kappa}(\kappa)$, $l_\kappa \geq 2$, $\forall \kappa$ in $\mathcal{T}_h$:

$$
\begin{aligned}
&|J(u) - J(u_{h,p})|^2 \\
&\leq C \sum_{\kappa \in \mathcal{T}_h} \left(\|a\|_{L_\infty(\kappa)} \frac{h_\kappa^{2(\tau_\kappa - 1)}}{p_\kappa^{2(k_\kappa - 3/2)}} + \|\boldsymbol{b}\|_{L_\infty(\kappa)} \frac{h_\kappa^{2\tau_\kappa - 1}}{p_\kappa^{2k_\kappa - 1}} \right) \|u\|^2_{H^{k_\kappa}(\kappa)} \\
&\quad \times \sum_{\kappa \in \mathcal{T}_h} \left(\|a\|_{L_\infty(\kappa)} \frac{h_\kappa^{2(\theta_\kappa - 1)}}{p_\kappa^{2(k_\kappa - 3/2)}} + \|\boldsymbol{b}\|_{L_\infty(\kappa)} \frac{h_\kappa^{2\theta_\kappa - 1}}{p_\kappa^{2k_\kappa - 1}} \right) \|z\|^2_{H^{l_\kappa}(\kappa)},
\end{aligned}
$$

where $\tau_\kappa = \min(p_\kappa + 1, k_\kappa)$ and $\theta_\kappa = \min(p_\kappa + 1, l_\kappa)$ for all $\kappa \in \mathcal{T}_h$.

hp–Refinement

Basic Strategy:

	High–Error	Low–Error
Solution Smooth	$p \rightarrow p+1$	$h \rightarrow 2h$
Solution Nonsmooth	$h \rightarrow h/2$	$p \rightarrow p-1$

hp–Refinement

Basic Strategy:

	High–Error	Low–Error
Solution Smooth	$p \rightarrow p+1$	$h \rightarrow 2h$
Solution Nonsmooth	$h \rightarrow h/2$	$p \rightarrow p-1$

hp–Adaptive Strategy:

	Refinement	Derefinement
u or z smooth	$p_\kappa \rightarrow p_\kappa+1$	$h_\kappa \rightarrow 2h_\kappa$
Otherwise	$h_\kappa \rightarrow h_\kappa/2$	$p_\kappa \rightarrow p_\kappa-1$

hp–Mesh Refinement

- ‘Texas 3–Step’
  Oden, Patra & Feng 1992, Bey, Oden & Patra 1995, ...
- A Priori Information
  Bernardi & Raugel 1981, Valenciano & Owens 2000, Bernardi, Fiétier & Owens 2001
- Type Parameter
  Gui & Babuška 1986, Adjerid, Aiffa & Flaherty 1998
- Predicted Error Reduction
  Melenk & Wohlmuth 2001, Heuveline & Rannacher 2003
- Mesh Optimisation Strategy
  Rachowicz, Demkowicz & Oden 1989, Demkowicz, Rachowicz & Devloo 2001
- Estimate decay rates of Legendre coefficients
  Mavriplis 1994
- Local Regularity Estimation
  Ainsworth & Senior 1998, Houston & Süli 2000, Houston, Senior & Süli 2001, 2002

Analyticity Estimation

Given $u \in L_2(-1,1)$, we have that

$$u(\xi) = \sum_{i=0}^{\infty} a_i L_i(\xi), \qquad a_i = \frac{2i+1}{2} \int_{-1}^{1} u(\xi) L_i(\xi) \mathrm{d}\xi \, .$$

Analyticity Estimation

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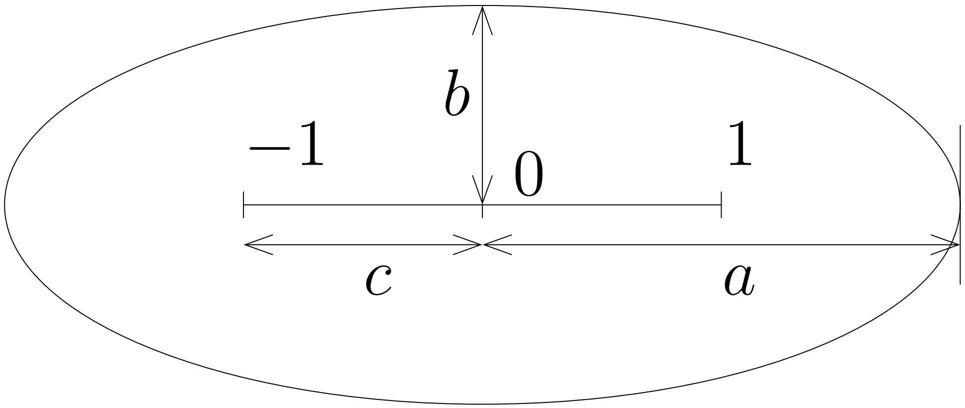
Bernstein Ellipse $\hat{\mathcal{E}}_\rho$
with radius $\rho = (a+b)/c$



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Bernstein Ellipse $\hat{\mathcal{E}}_\rho$
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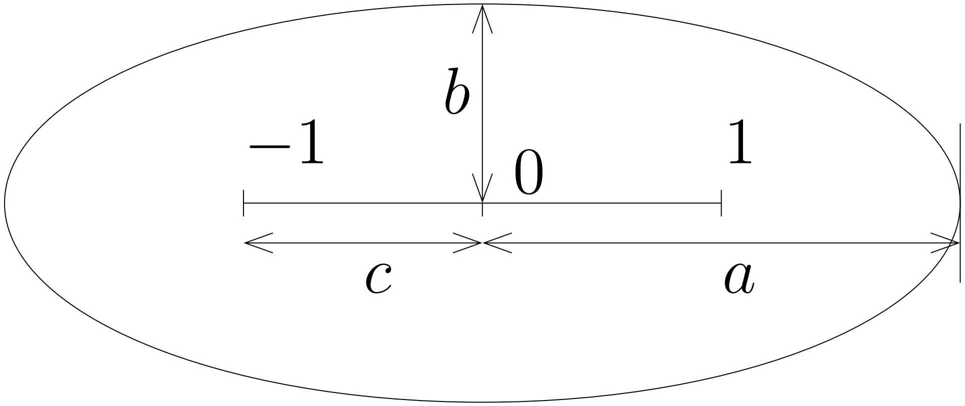

Let u be analytic inside $\hat{\mathcal{E}}_\rho$, but not inside $\hat{\mathcal{E}}_{\rho'}$ with $\rho' > \rho$; then

$$\frac{1}{\rho} = \overline{\lim}_{i \to \infty} |a_i|^{1/i}, \qquad \rho > 1 \,. \qquad \text{(Davis 1963)}$$

Domain of analyticity

The quantity

$$\theta = \frac{1}{\rho}$$

is the measure of the size of the domain of analyticity of u relative to $(-1, 1)$.

Domain of analyticity

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$$\theta = \frac{1}{\rho}$$

is the measure of the size of the domain of analyticity of u relative to $(-1, 1)$.

- $\theta = 0$: entire analytic function
- $\theta = 1$: function with singularity support in $[-1, 1]$
  (finite Sobolev regularity in I).

Domain of analyticity

The quantity

$$\theta = \frac{1}{\rho}$$

is the measure of the size of the domain of analyticity of u relative to $(-1, 1)$.

Adaptive Algorithm

- Select $0 < \theta_{\max} < 1$, say $\theta_{\max} = 1/2$.

Domain of analyticity

The quantity

$$\theta = \frac{1}{\rho}$$

is the measure of the size of the domain of analyticity of u relative to $(-1, 1)$.

Adaptive Algorithm

- Select $0 < \theta_{\max} < 1$, say $\theta_{\max} = 1/2$.
- Given an each element κ in the mesh $\mathcal{T}_h$:
  - If $\theta > \theta_{\max}$, then $u|_\kappa$ is smooth;
  - otherwise $u|_\kappa$ has finite Sobolev regularity.

H., Senior & Süli 2003, H. & Süli 2005.

Melenk 2002, Eibner & Melenk 2004

Burgers' Equation

$$u_t + \left((1/2)\,u^2\right)_x = 0,$$

subject to the initial condition

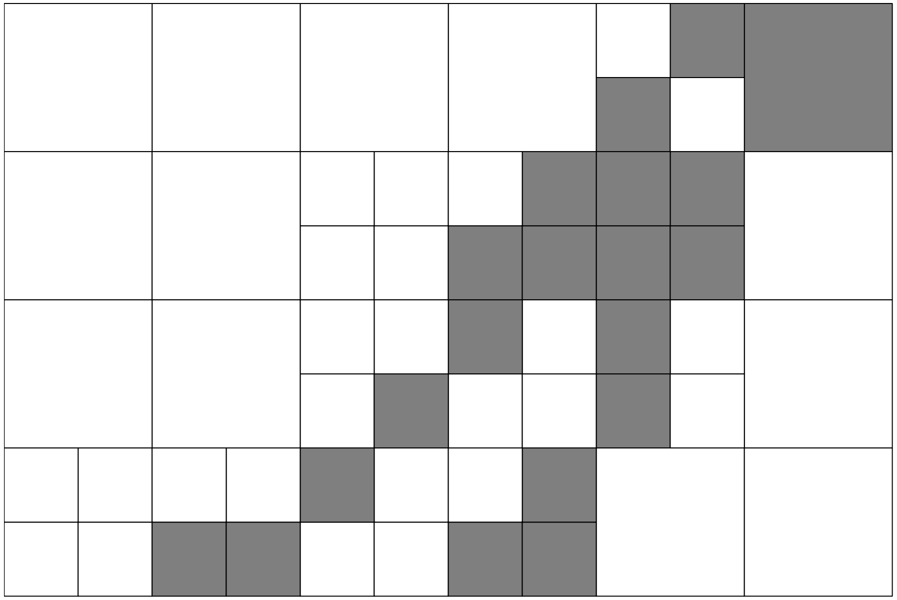
$$u(x,0) = \begin{cases} 2\,\sin^2(\pi x) & 0 \le x \le 1, \\ \sin^2(\pi x) & 1 \le x \le 2, \\ 0 & 2 \le x \le 3. \end{cases}$$

$$J(u) \equiv u(2.3, 1.5) = 0.66444240397525467O\ .$$

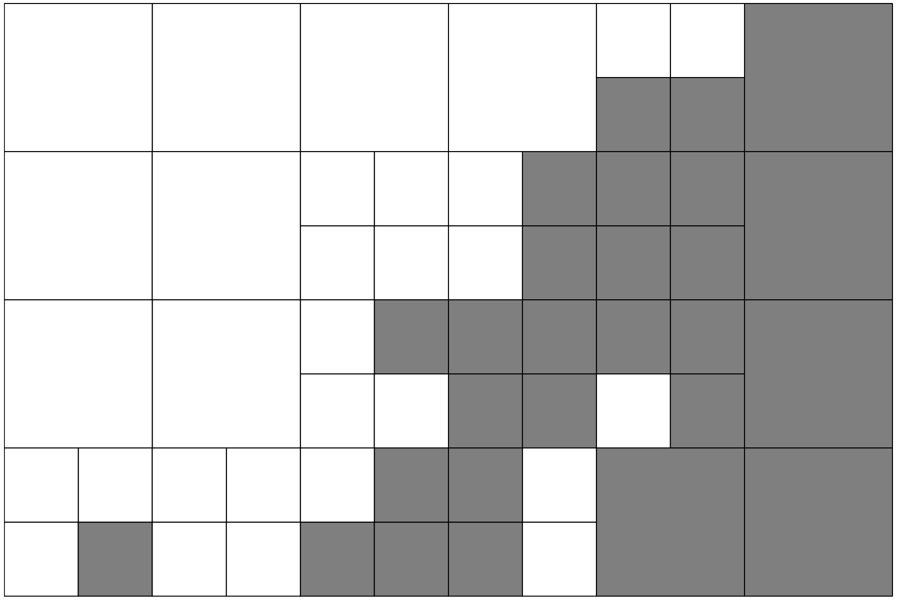
Burgers' Equation

DOF	$J(u - u_{h,p})$	$\sum_\kappa \eta_\kappa$	θ_1	$\sum_\kappa \lvert\eta_\kappa\rvert$	θ_2
4279	-3.17e-4	-5.24e-4	1.65	3.10e-3	9.77
6227	-6.51e-5	-4.74e-5	0.73	8.03e-4	12.32
9008	1.03e-5	8.52e-6	0.83	1.23e-4	11.96
13045	-5.40e-7	-1.25e-6	2.31	3.72e-5	68.82
18070	-5.41e-7	-5.70e-7	1.05	1.10e-5	20.23
26020	1.01e-7	5.86e-8	0.58	1.11e-6	11.01
37181	1.46e-9	1.09e-9	0.75	8.65e-8	59.49
57850	-6.20e-11	-4.91e-11	0.79	2.84e-9	45.86
94649	3.16e-12	2.82e-12	0.89	9.51e-11	30.05

Burgers' Equation

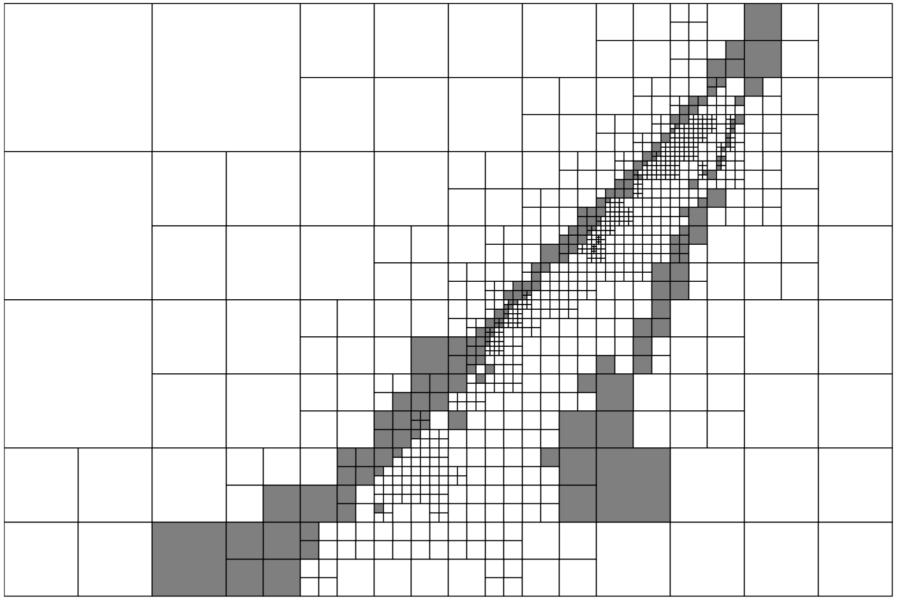


Primal Regularity

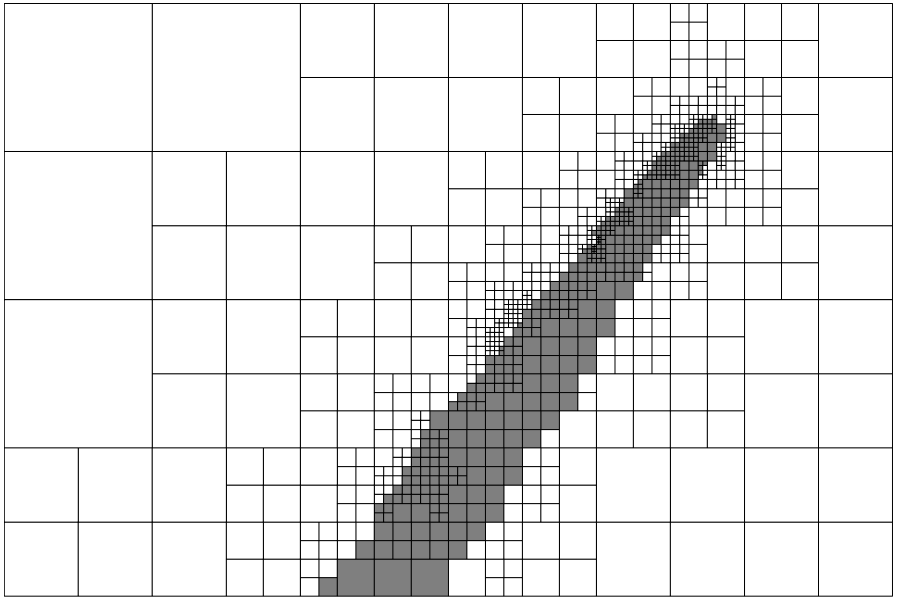


Dual Regularity

Burgers' Equation



Primal Regularity

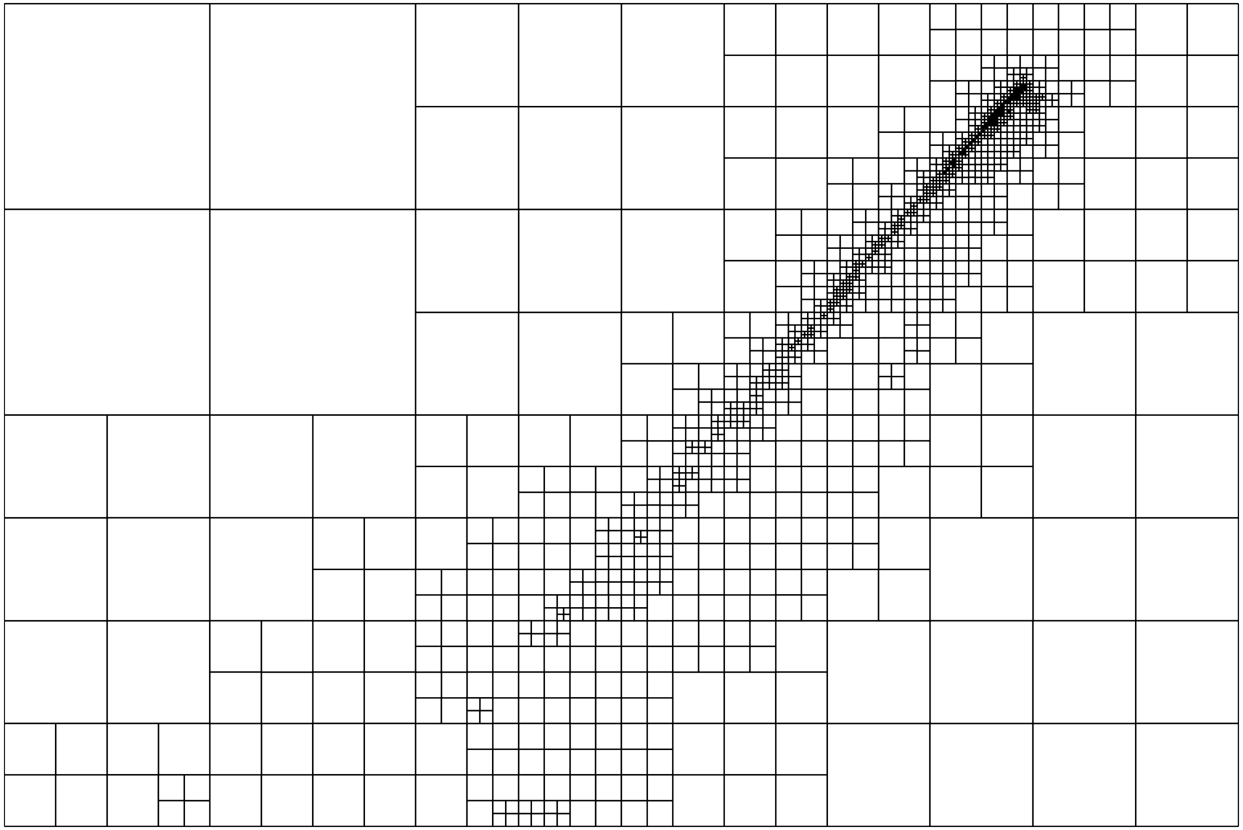


Dual Regularity

Burgers' Equation

1626 elements and 40175 degrees of freedom.

$$|J(u) - J(u_{h,p})| = 1.5615 \times 10^{-8}.$$



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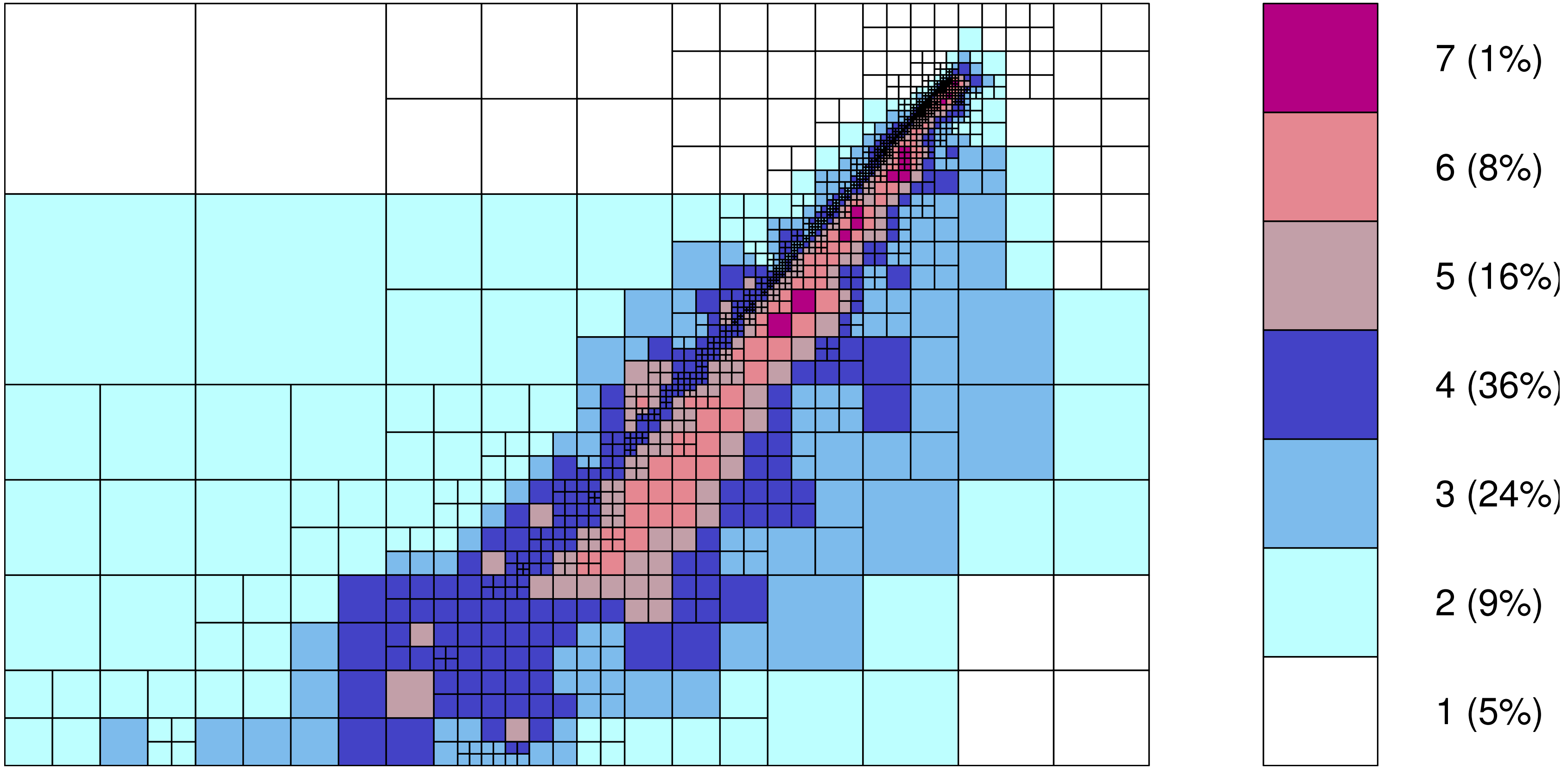

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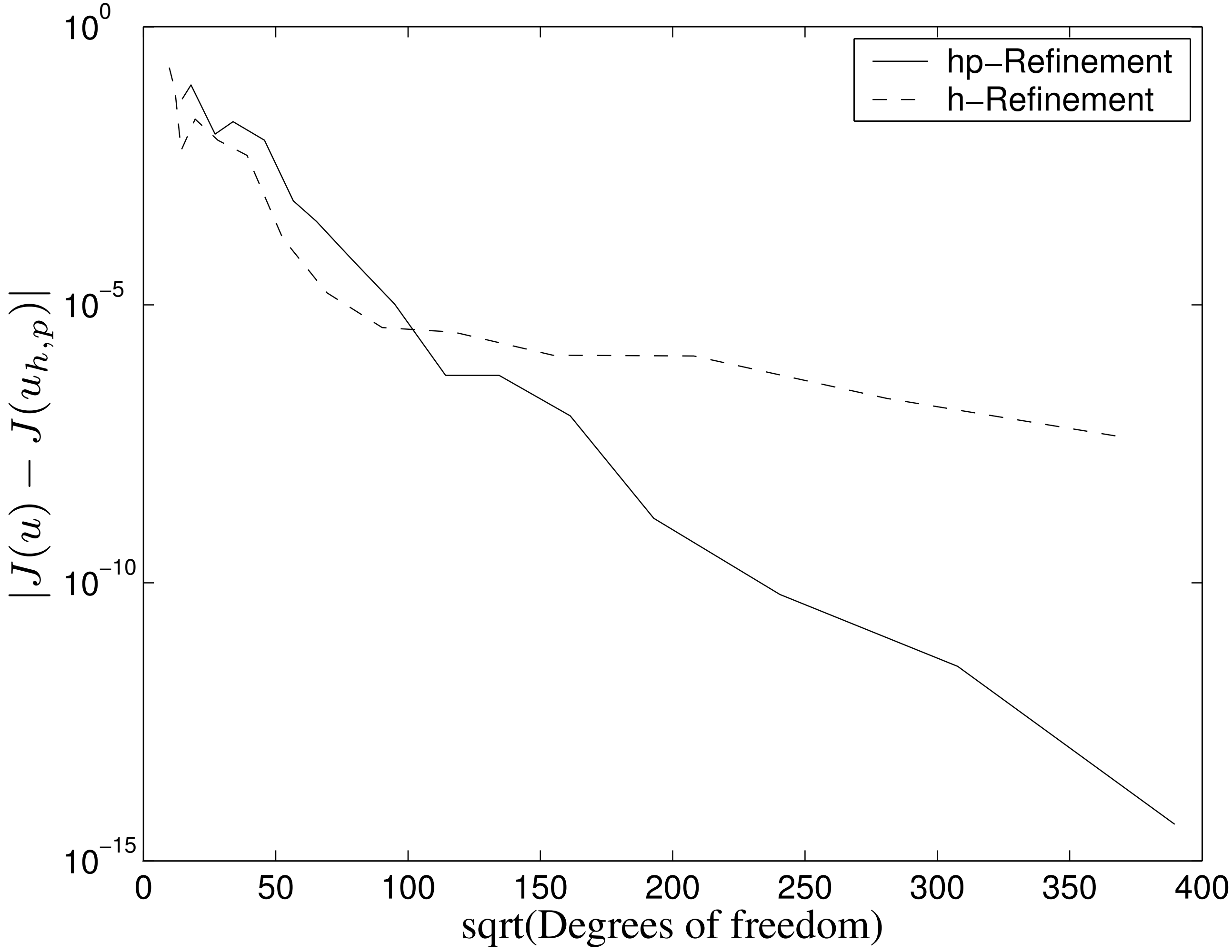

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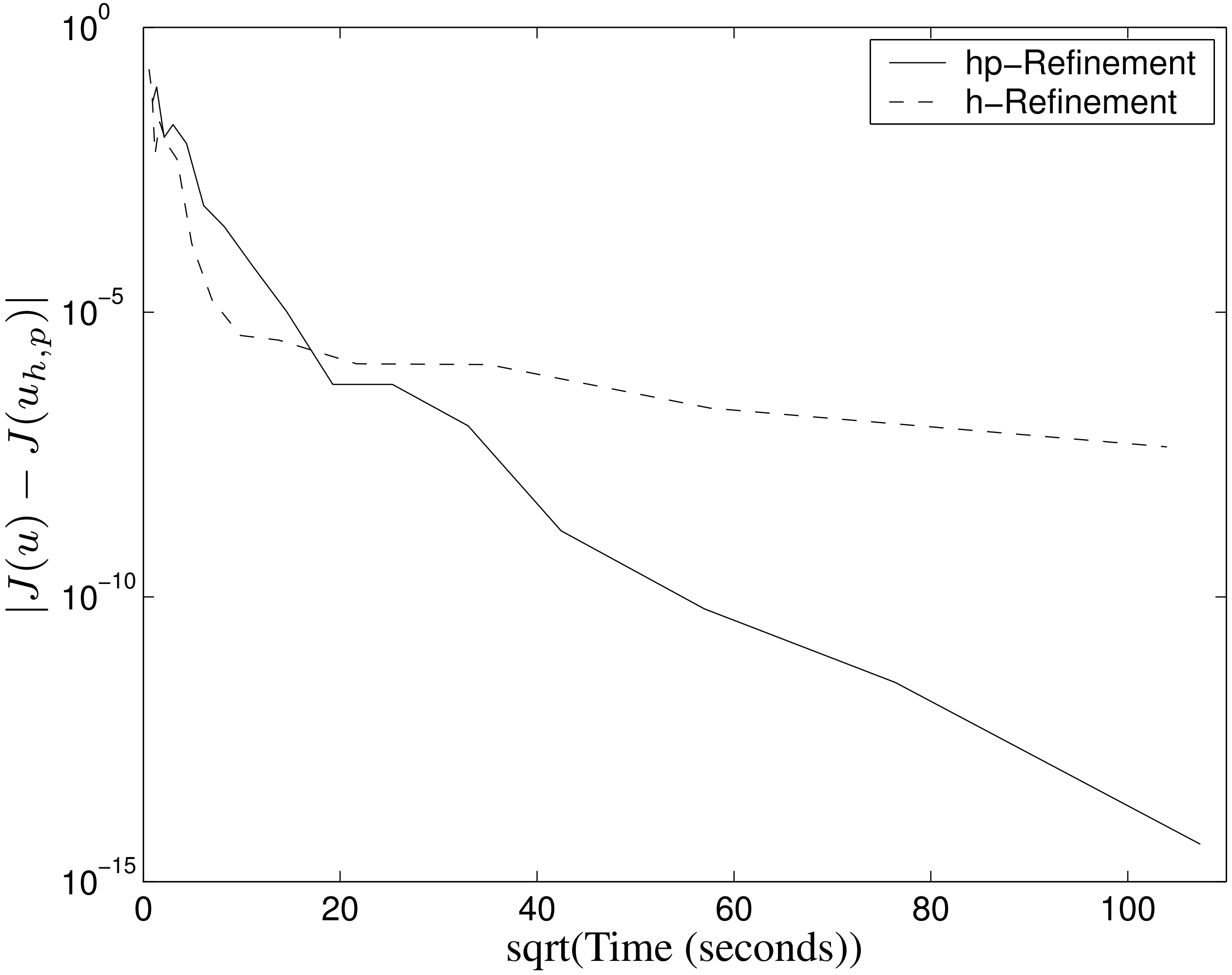

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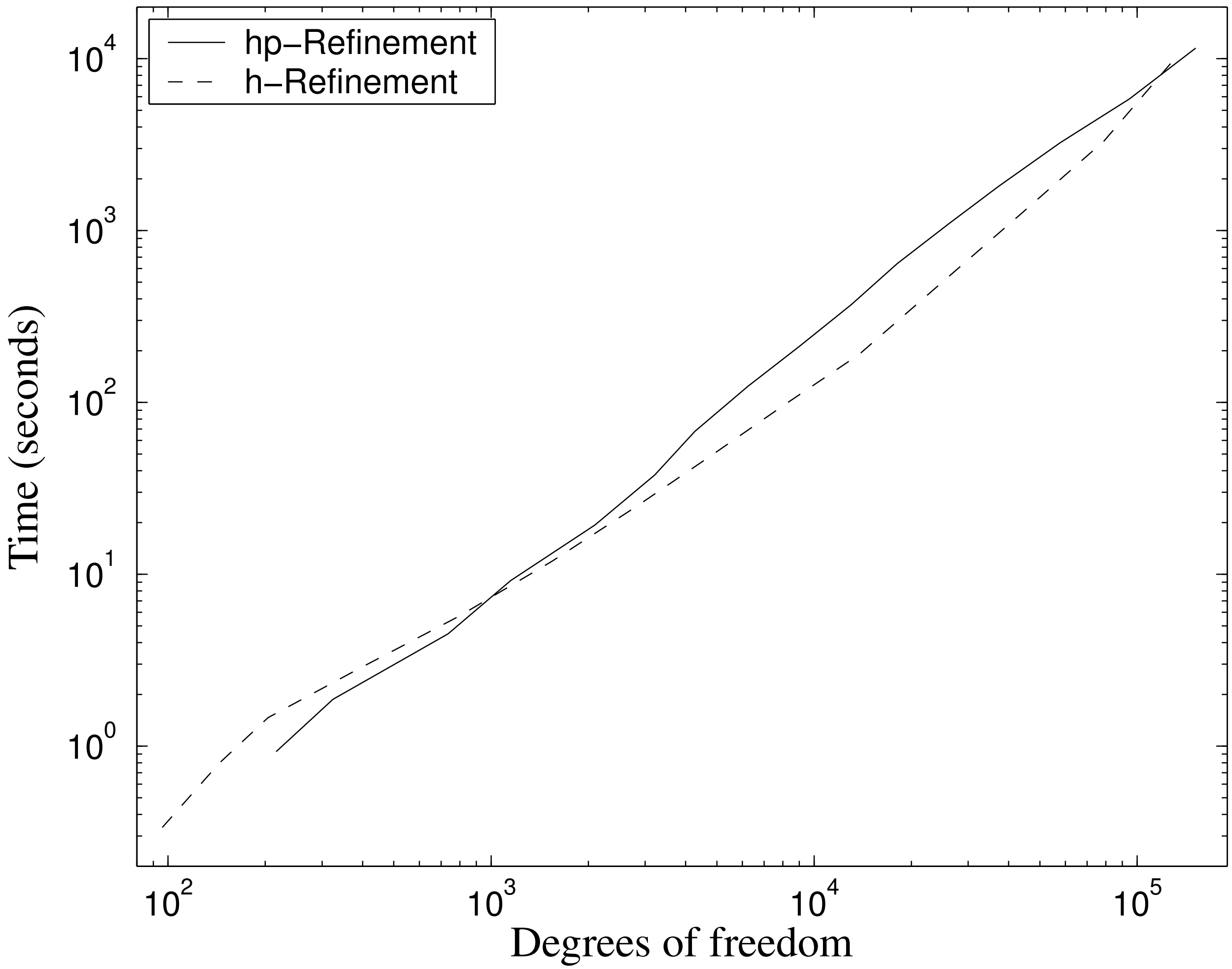
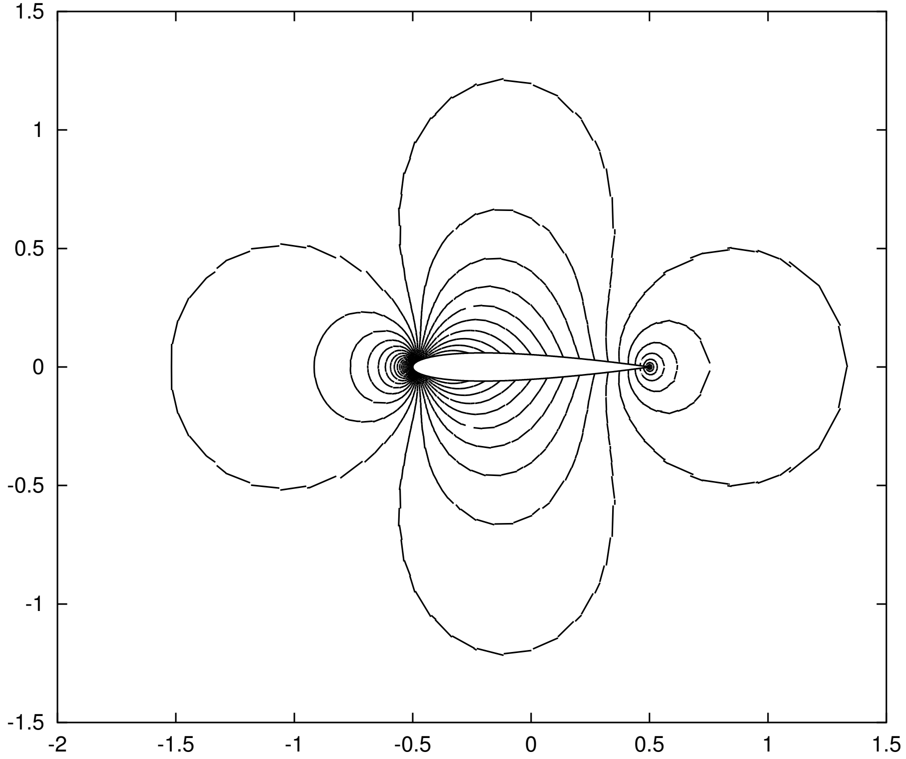
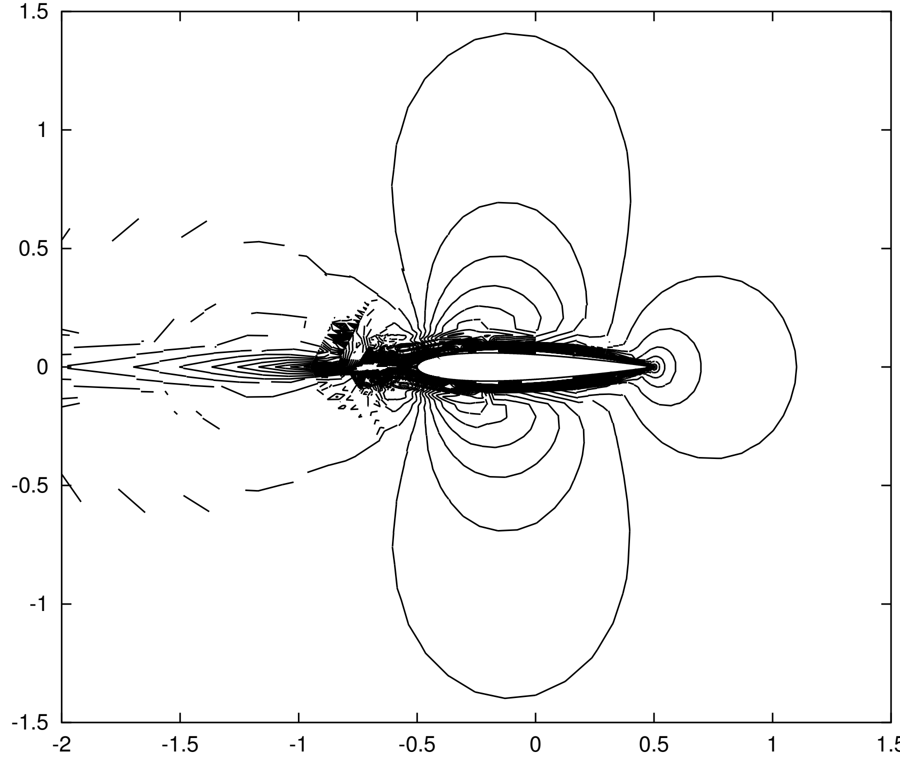


NACA0012 Airfoil (Euler)

$\mathrm{Ma} = 0.5$ and $\alpha = 0°$.



Primal Solution



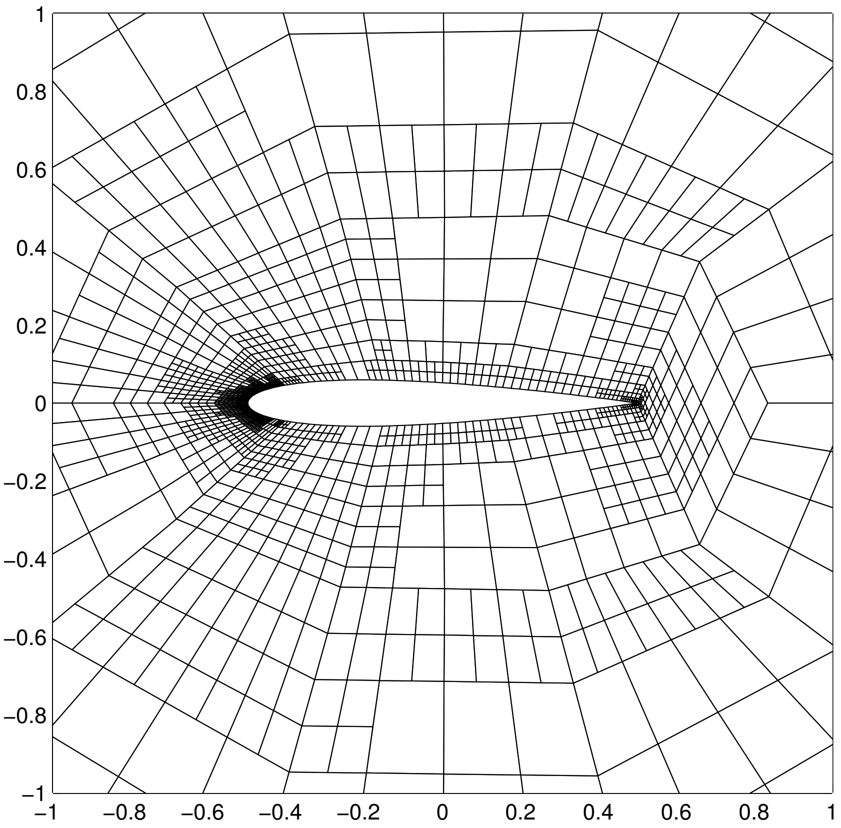
Dual Solution

We select $J(\cdot) \equiv J_{c_{\mathrm{dp}}}(\cdot)$; here, $J_{c_{\mathrm{dp}}}(\boldsymbol{u}) = 0$.

NACA0012 Airfoil (Euler)

6126 elements and 98016 degrees of freedom.

$$|J_{c_{\mathrm{dp}}}(\boldsymbol{u}) - J_{c_{\mathrm{dp}}}(\boldsymbol{u}_{h,p})| = 4.051 \times 10^{-5}.$$

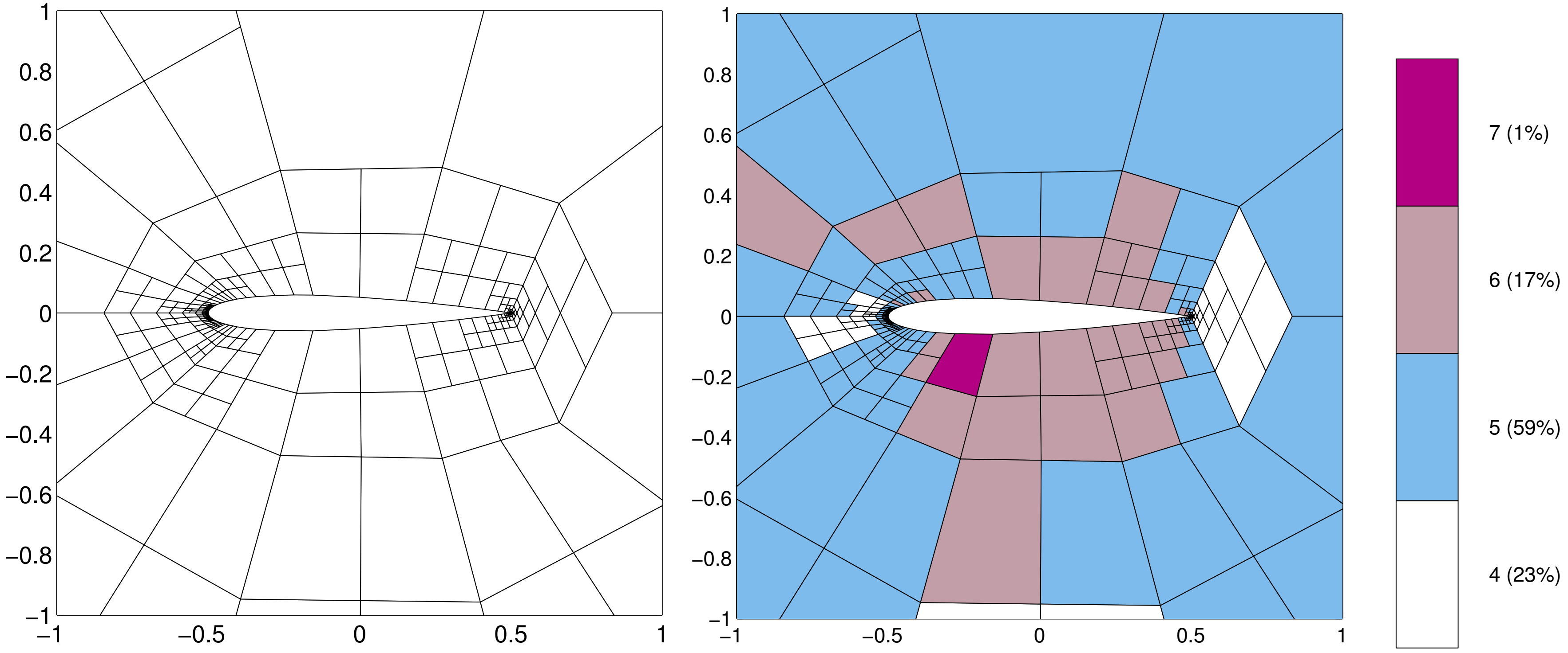


h–refinement

NACA0012 Airfoil (Euler)

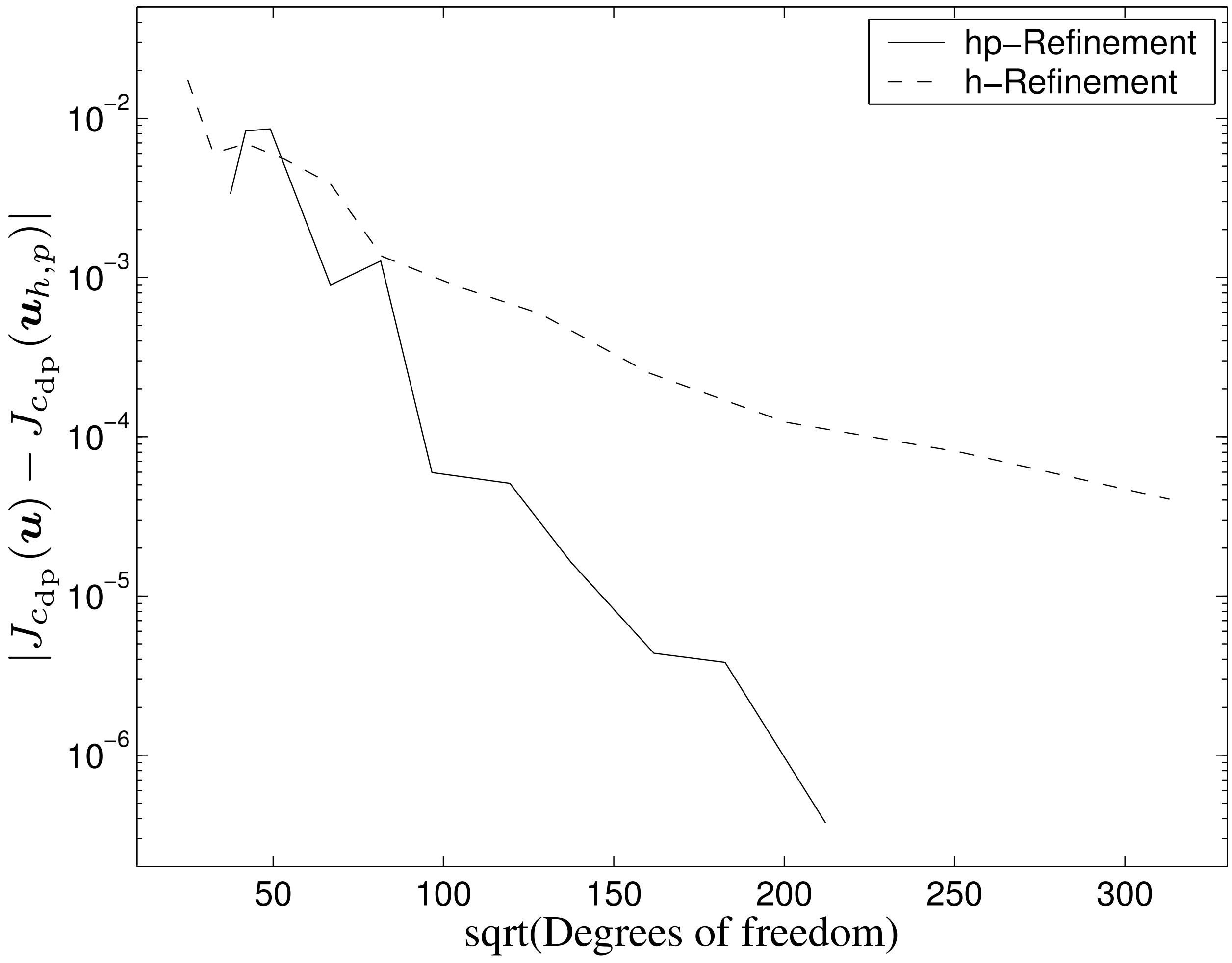
325 elements and 45008 degrees of freedom.

$$|J_{c_{\rm dp}}(\boldsymbol{u}) - J_{c_{\rm dp}}(\boldsymbol{u}_{h,p})| = 3.756 \times 10^{-7}.$$



hp–refinement

NACA0012 Airfoil (Euler)



NACA0012 Airfoil (Comp. NS)

- $\mathrm{Ma} = 0.5$, $\alpha = 0^\circ$ and $Re = 5000$.
- We select $J(\cdot) \equiv J_{c_{\mathrm{dp}}}(\cdot) \approx 0.0222875$.

NACA0012 Airfoil (Comp. NS)

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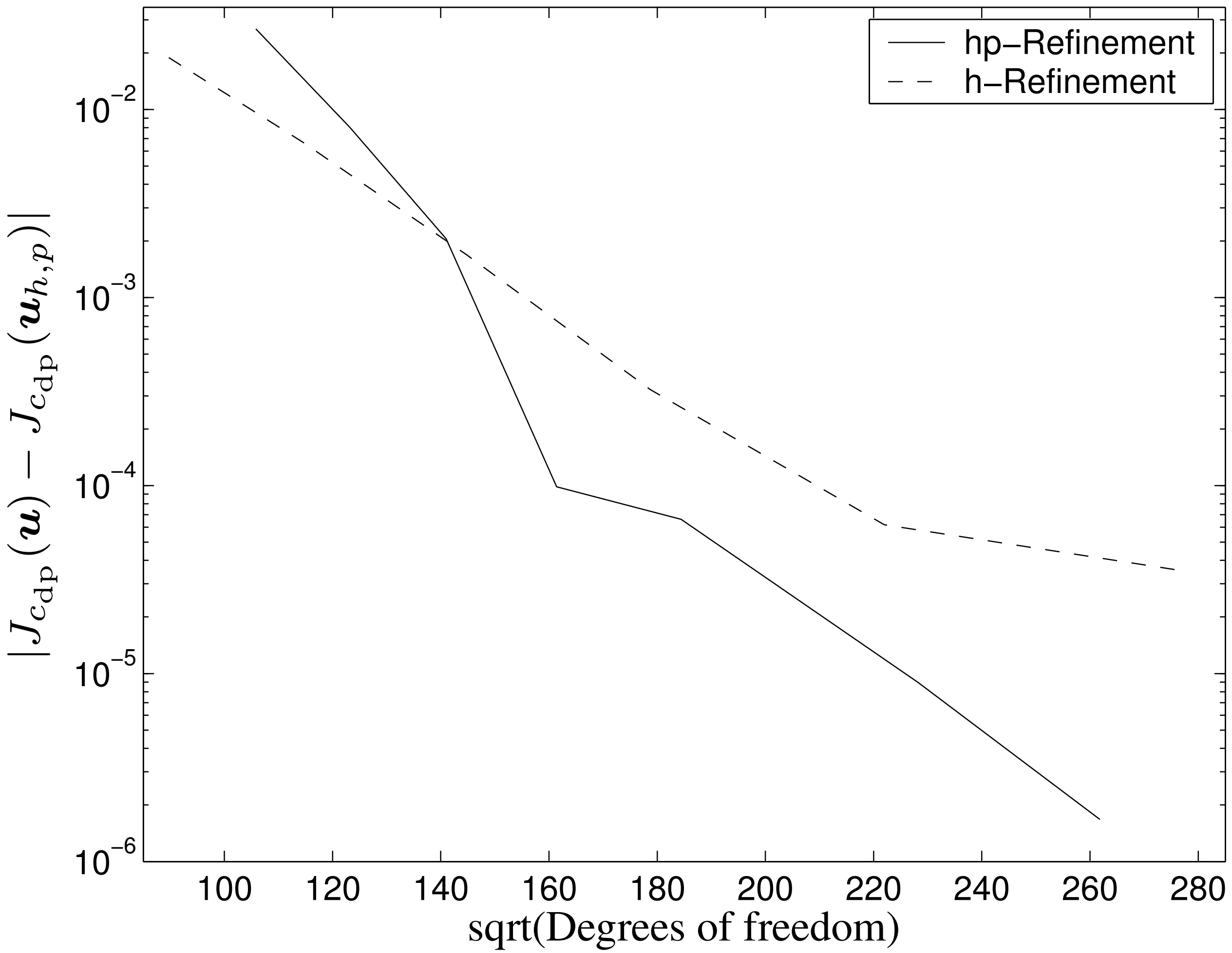
- We select $J(\cdot) \equiv J_{c_{\mathrm{dp}}}(\cdot) \approx 0.0222875$.



6. Summary and Outlook

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- Anisotropic mesh adaptation.

Future Challenges

- Anisotropic mesh adaptation

  Mesh movement, mesh regeneration, general refinement of elements, anisotropic p–adaptation

- Unsteady problems

  Reliable error control v. economical mesh design, grid and solution storage/check pointing

- Modelling error control

  Coupling models of different complexity and different (spatial) dimension.

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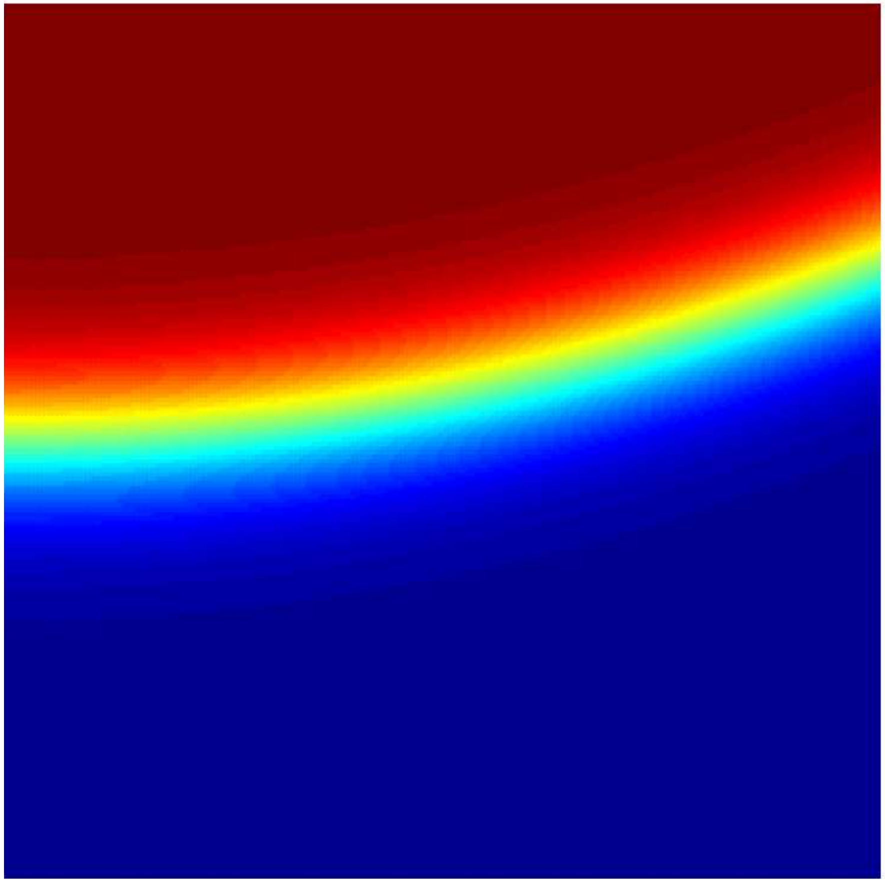
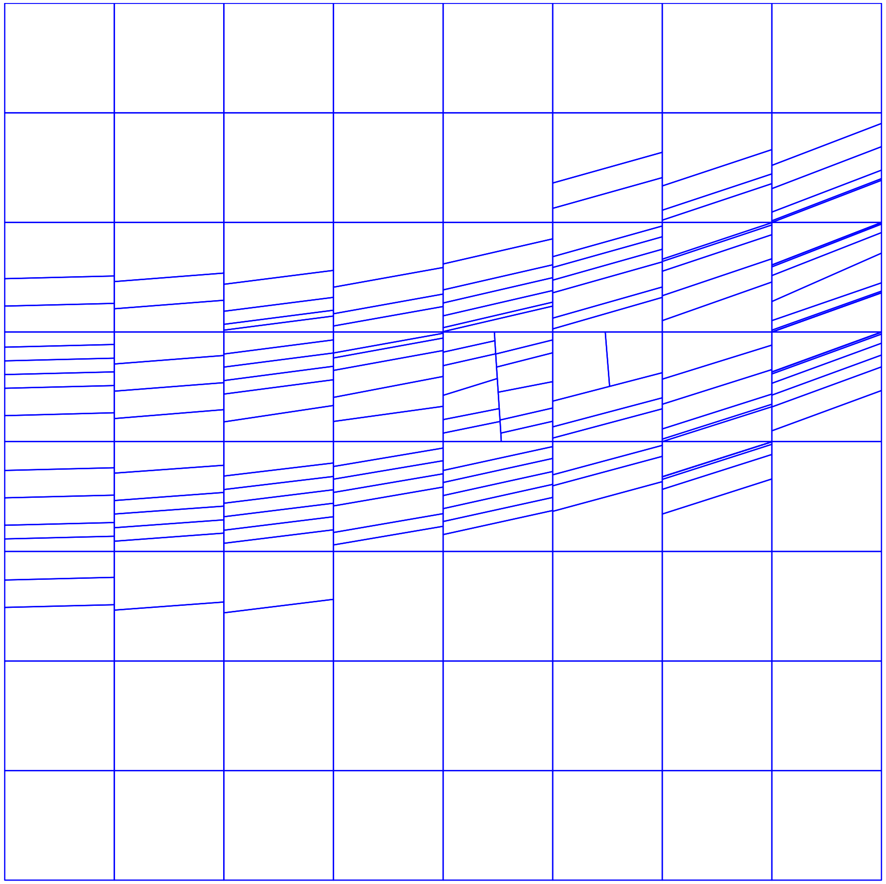
Mesh Adaptation





Joint work with Edward Hall.