Dynamic Cheap Talk

Mikhail Golosov, MIT & NBER Vasiliki Skreta, UCLA
Aleh Tsyvinski, UCLA & NBER

January 2005

1. Introduction

This paper studies strategic information transmission in a dynamic environment, where an expert and a decision maker interact for arbitrarily many, but finite, number of periods. Contrary to the seminal work on static information transmission by Crawford and Sobel (1982) we find that equilibria will not always be simple, in the sense that disconnected set of types of the experts may be inducing the same action at some point in time. In general we can have more information released compared to the static case, but it is impossible to have equilibria where the decision maker always learns the truth.

There are a large number of relationships where parties that own information will behave strategically in transmitting it. For instance, the relationship between manager and shareholders, workers-supervisors, doctors and patients and many more. A number of these relationships are dynamic. The literature on strategic information is either static, (Crawford and Sobel (1982), Battaglini (2002), (2003), Levy and Razin (2004)), or when it is dynamic it has a different focus than the one of the present study. Krishna and Morgan (2002) study the effect of adding a stage of face-to-face communication to a static information transmission game. There is a series of papers, (Krishna and Morgan (2000), Morris (2001), Ottaviani and Sorensen (2001, 2004a, 2004b)), that study strategic information transmission in dynamic environments when the expert cares about his/her reputation. Reputation issues can be important in such kinds of relationships, but it is important to know how information transmission differs in dynamic versus static environments when only strategic motives are present. This is the purpose of the present study.

In particular we are interested in answering the following questions. Can the decision maker obtain a more precise estimate about the truth compared to the static case? Will the decision maker eventually learn the truth?If not, what is the maximal amount of information that will be revealed overtime? Are equilibria still simple in structure? One may view this dynamic strategic information transmission game as not fundamentally that different from the static one, in the sense that in the dynamic version the expert chooses a vector of messages, instead of sending a single message, and this vector of messages determines a vector of actions, instead of a single action. Now the payoffs of the expert and the decision maker depend on the information of the expert, which remains constant overtime, and on the vector of actions chosen . We maintain the assumption that preferences satisfy the single crossing property and that the interests of the expert and the decision maker differ in the sense that the most preferred action for the expert is different from the one of the decision maker. Even though these problems may appear to be similar equilibria are qualitatively different. In contrast to the static case where all equilibria are equivalent to partition equilibria, there exist equilibria that are non-partition in an non-trivial sense, meaning that there is no partition equilibrium that induces same sequences of actions. Moreover the fact that complicated maybe complicated makes difficult to characterize exactly how much is the maximal amount of information released as a function of the number of stages in the game. This is again different from the static environment where the simple nature of equilibria allows us to say that there will be a natural number p that describes the maximal number of sub-intervals that the type space will be partitioned into. But even though the potential complexity of equilibria does not allow us to say exactly how much information is released we have two things to say. First compared to the static case there will be more information released, but there will be no full Information Revelation even if T is arbitrarily large, but finite. That is, we at every equilibrium we will eventually reach a point where learning stops.

2. The Environment

There are two players, an expert, (sender), and a decision maker, (receiver). The expert has a piece of information that is private information and distributed on $[0,1]$ according to F. This piece of information, which we denoted by m affects the payoffs of both the expert and the receiver. Their payoffs also depend on the actions chosen by the decision maker, $y \in \mathbb{R}$. The expert's preferences over m and y can be represented by a twice differentiable utility function $U^E(y,m)$. The decision maker's preferences also depend on an additional parameter $a \in \mathbb{R}$ which measures the differences in the objectives of the expert and the decision maker. They are represented by $U^D(y,m,a)$. We assume that U^E and U^D have for each (m,a) a unique maximum in y. We also assume a single crossing condition, that is $U^i_{12} > 0$. This condition ensures that the optimal value of y for the expert and the decision maker when they know m, is strictly increasing in m.

At the beginning of the game the expert learns the value of m, which remains constant over the whole relationship. The game lasts for $T < \infty$ stages. At each stage t, $t = 1,...,T$ the expert makes a report and the decision maker chooses an action y based on the report that he received.

The decision maker does *not* observe his payoff at each stage, otherwise the problem at hand is trivial because the decision maker can learn the expert's information by simply inverting his payoff.

Assessments

An *assessment* consists of a strategy profile and a belief system. A *strategy profile* $\sigma = (\sigma_i)_{i=S,B}$, specifies a strategy for each player. Let h_t denote denote a history that contains all the reports submitted by the expert and all the actions chosen by the decision maker up to stage t. The set of all feasible histories at t is denoted by H_t. A *behavioral* strategy of the expert, σ_E, consists of a sequence of signaling rules that map $V \times H_t$ to a probability distribution over reports. We use $q(n\,|m,h_t)$ to denote the probability that the expert reports message n at history h_t when his type is m. A strategy for the decision maker, σ_{DM}, is a sequence of maps from H_t to actions. We use $y(n\,|h_t) \in \mathbb{R}$ to denote the action that the decision maker chooses at h_t given a report n. A *belief system*, μ, maps H_t to the set of probability distributions over

[0, 1]. Let $F(m|h_t)$ denote the decision maker's beliefs about the experts's type after a history of moves h_t, $t = 1, ..., T$. A strategy profile σ and a belief system μ is an assessment.

Solution Concept

A *Perfect Bayesian Equilibrium, (PBE),* is a strategy profile, σ, and a belief system, μ, that satisfy:

1. For all $m \in [0, 1]$ and h_t we have that $\int_N q_t(n\,|m, h_t)dn$, where N stands for the Borel set that contains all feasible signals. If a message $\hat{n}$ is in the support of $q_t(n\,|m, h_t)$ it must be the case that $\hat{n} \in \arg\max_{n\in N} U_t^E(y(n\,|h_t), m)$
2. Given $F_t(m|h_t)$ and the expert's strategy, the decision maker chooses at each h_t an optimal action, that is $y_t(n\,|h_t)$ solves $\max_{y\in\mathbb{R}} \int_0^1 U^{DM}(y, m, a)F(m|h_t)dm$.
3. $F_t(m|h_t)$ is derived from F_{t-1} given h_t using Bayes' rule whenever possible.

Payoffs for the expert and the decision maker over the whole duration of the game are respectively given by summing up per period payoffs.

Notice that our environment is exactly the same as the one considered by Crawford and Sobel (1982) with the only difference that players interact overtime.

3. A Benchmark Case

We start our investigation by going over the of the problem and results of the literature on static strategic information transmission in a static setup. We do so in for the simple case where the expert's parameter is uniformly distributed on $[0, 1]$ and the per period preferences over m and y are represented by

$$U_t^E(y_t, m) = -(y_t - m)^2$$

$$U_t^{DM}(y_t, m, a) = -\Sigma_{t=1}^T (y_t - m - a)^2 .$$

These preferences satisfy the assumptions in Crawford-Sobel (1982). There is namely a unique maximizer in y for both the expert and the decision maker and U_t^E and U_t^{DM} satisfy a single crossing property. Crawford and Sobel (1982) establish that all equilibria are equivalent to partition equilibria. At a partition equilibrium the type space is divided in a finite number of subintervals described by a sequence of cutoffs $(m_1, ..., m_p)$. The expert uses a simple signaling where he for types in interval $[m_{i-1}, m_i]$ he sends message μ_i with probability one. The optimal action by the decision maker who observes message μ_i is then given by

$$y_i = \frac{m_{i-1} + m_i}{2} + a.$$

At an equilibrium the cutoffs have to be such that types in $[m_{i-1}, m_i]$ prefer y_i to the actions that are induced by sending another message. Given single crossing this is ensured if m_i, $i = 1, ..., p$ satisfies the following Indifference condition

$$\left(\frac{m_{i-1} + m_i}{2} - m_i + a\right)^2 = \left(\frac{m_i + m_{i+1}}{2} - m_i + a\right)^2,$$

which reduced to the following difference equation

$$m_{i+1} = 2m_i - m_{i-1} - 4a.$$

Its solution is given by

$$m_i = \left(\frac{1}{p} + 2ap\right) i - 2ai^2,$$

where p is the number of subintervals that the type space is divided into. This number must be clearly in $[0, 1]$. Using this observation we can obtain the largest number of subintervals that the type space can be divided into, which we denote by $p^{\max}$. This is the largest integer that satisfies

$$-2ap^2 + 2ap + 1 > 0$$

Equilibrium with $p = p^{\max}$ gives the maximal number of subintervals that the expert's type space can be divided into and for each $p = 1, ..., p^{\max}$ we get a different equilibrium. It follows that there are multiple equilibria in a static strategic information game, but equilibria are simple. We will later show that in the dynamic game this will not necessarily be the case by calculating a non-partition equilibrium. Still is will be instructive to characterize partition equilibria in our dynamic game. This is done in the section that follows.

4. Partition Equilibria in Dynamic Cheap Talk

2 periods

Type m does not change

Payoff is not observed until period 2

In this example we focus on partition equilibrium in both periods

$t = 1$: partitions $m_1, ..., m_{p_1}$

$t = 2$: for each interval i in period 1, partitions are $n^i_1, ..., n^i_{p(i)}$

Trade-off: how much info to reveal today versus tomorrow.

Information revealed by expert at $t = 1$ will be also used at $t = 2$.

Solution (simple dynamic case)

$t = 2$: For each interval $[m_i, m_{i+1}]$ – a static Crawford-Sobel.

$t = 1$: Indifference condition for type m_i

$$
\begin{aligned}
& \left(\frac{m_{i-1} - m_i}{2} + a\right)^2 + \left(\frac{n^i_{p(i)-1} - m_i}{2} + a\right)^2 \\
= & \left(\frac{m_{i+1} - m_i}{2} + a\right)^2 + \left(\frac{n^{i+1}_1 - m_i}{2} + a\right)^2
\end{aligned}
$$

Substitute from static Crawford-Sobel for $n^i_{p(i)-1}$ and n^{i+1}_1:

$$
\begin{aligned}
& \left(\frac{m_{i-1} - m_i}{2} + a\right)^2 - \left(\frac{m_{i+1} - m_i}{2} + a\right)^2 \\
= & \left(\frac{m_{i+1} - m_i}{2p(i+1)} + ap(i+1)\right)^2 - \left(\frac{m_{i-1} - m_i}{2p(i)} + ap(i)\right)^2
\end{aligned}
$$

Which can be rewritten as

$$\begin{aligned}&\left[\frac{m_{i-1}-m_{i+1}}{2}\right]\left[\frac{m_{i-1}+m_{i+1}-2m_i}{2}+2a\right]\\ =\ &\left[\frac{p\left(m_{i+1}-m_i\right)-\hat{p}\left(m_{i-1}-m_i\right)}{2p(i+1)p(i)}+a(p(i+1)-p(i))\right]\\ &\left[\frac{p\left(m_{i+1}-m_i\right)+\hat{p}\left(m_{i-1}-m_i\right)}{2p(i+1)p(i)}+a(p(i+1)+p(i))\right]\end{aligned}$$

The formula is not elegant and maybe not very informative....

Let us assume $p(i+1)=p(i)$:

$$m_{i+1}=2m_i-m_{i-1}-\frac{8ap^2}{p^2+1}$$

Compare to static case:

$$m_{i+1}=2m_i-m_{i-1}-4a$$

If $p=1$ (no info revealed at $t=2$) – dynamic case reduces to static case.

Define:

$$\hat{a}=\frac{2p^2a}{p^2+1},$$

Then dynamic solution:

$$m_{i+1}=2m_i-m_{i-1}-4\hat{a}$$

Same as static but with higher "difference" factor $\hat{a}>a$. Intuition: Multi period solution with the same number of subdivisions at $t=2$ is like a static problem with different "a"remember that this parameter indexes how different the preference of the sender and the receiver are.Can also show that $m_i^{dynamic}>m_i^{static}$

$(i<p)$

Observation: In partition equilibria of dynamic cheap talk games, the number of intervals that the type space can be divided into at $t=1$ is decreasing in the number of intervals that each subinterval is divided into at $t=2$, Let us look at the simple case where each interval at $t = 1$ is divided in the same number of subintervals at $t = 2$. Then from our previous analysis we have that the first stage cutoffs are described by the following indifference equation

$$m_{i+1} = 2m_i - m_{i-1} - 4\hat{a},$$

where

$$\hat{a} = \frac{2ap^2}{p^2 + 1},$$

where p is the number of partitions in the second period.

We use $[z_1, z_2]$ to denote a generic subinterval at $t = 2$. The solution is $m_i = \lambda i^2 + \mu i + v$ and from the difference equation we get $\lambda = -2\hat{a}$. Now the initial condition is $m_0 = z_1$, which implies that $v = z_1$. The final condition is $m_p = z_2$ which allows us to find $\mu : z_2 = -2\hat{a}p^2 + \mu p + z_1$ $\mu = \frac{1}{p}(z_2 - z_1) + 2\hat{a}p$ Following Salanie we can show that the following condition must be satisfied:

$$m_1 - m_0 - 4\hat{a}(p - 1) > 0 \tag{1}$$

We know that $m_0 = z_1$ and $m_1 = \lambda + \mu + v = -2\hat{a} + \frac{1}{p}(z_2 - z_1) + 2\hat{a}p + z_1$ The condition (1) then becomes

$$-2\hat{a} + \frac{1}{p}(z_2 - z_1) + 2\hat{a}p - 4\hat{a}(p - 1) > 0$$

which simplifies to

$$\frac{1}{p}(z_2 - z_1) - 2\hat{a}p + 2\hat{a} > 0$$

It equivalent to

$$-2\hat{a}p^2 + 2\hat{a}p + (z_2 - z_1) > 0.$$

Now at $t = 1$ the interval of types is $[0, 1]$. Then for $z_1 = 0$ and for $z_2 = 1$

$$\begin{aligned} -2\hat{a}p^2 + 2\hat{a}p + 1 &= 0 \\ 2\hat{a}p^2 - 2\hat{a}p - 1 &= 0 \end{aligned}$$

Solutions

$$p_1^{(1)} = \frac{1}{4\hat{a}}\left(2\hat{a} + 2\sqrt{(\hat{a}^2 + 2\hat{a})}\right)$$
$$p_1^{(2)} = \frac{1}{4\hat{a}}\left(2\hat{a} - 2\sqrt{(\hat{a}^2 + 2\hat{a})}\right)$$

after substituting in for

$$\hat{a} = \frac{2ap^2}{p^2+1},$$

root	$p_1^{(1)} = \frac{1}{2p}\frac{ap+\sqrt{a}\sqrt{(ap^2+p^2+1)}}{a}$	$p_1^{(2)} = \frac{1}{2p}\frac{ap-\sqrt{a}\sqrt{(ap^2+p^2+1)}}{a}$
derivative	$\frac{\partial p_1^{(1)}}{\partial p} = -\frac{1}{2}\frac{\sqrt{a}\sqrt{(ap^2+p^2+1)}p+1}{\sqrt{a}p^2\sqrt{(ap^2+p^2+1)}}$ $\downarrow$ in p	$\frac{\partial p_1^{(2)}}{\partial p} = -\frac{1}{2}\frac{\sqrt{a}\sqrt{(ap^2+p^2+1)}p-1}{\sqrt{a}p^2\sqrt{(ap^2+p^2+1)}}$ $\downarrow$ in p at least sometimes

We have studied how partition equilibria may look like in a multi stage game. For a particular class of equilibria where each interval is divided in the same number of sub-intervals in the second stage we show that the location of first period cutoffs will be the same as in a static game but with the preference parameter changed to $\hat{a} = \frac{2ap^2}{p^2+1}$ where p is the number of subintervals intervals will be divided in the second stage.

For the case of partition equilibria we obtain an upper bound on the maximal number of intervals that the original type space will be divided to using the condition

$$-2ap^2 + 2ap + (z_2 - z_1) > 0$$

it always holds for $p = 1-$ it will fail to hold $p = 2$ if

$$z_2 - z_1 \leq 4a.$$

Can we get an upper bound on the amount of information released in general (i.e. not just in partition equilibria)?

5. Preliminary Observations

It is possible to have non-partition equilibria

One of the most important results in the literature on static strategic information transmission is to establish that all equilibria are "essentially" simple in the sense that the expert uses uniform signaling rules where he sends the same messages with probability one if his types belongs in a certain interval. Put it differently the type space is partitions in a finite number of intervals, and types in each subinterval send the same report with probability one. This implies that experts of "similar" type tell the same advice. Our first observation is that this is no longer true in a dynamic environment. It is a rather surprising result. We illustrate this point by computing an equilibrium in a simple two-period example where a disconnected set of types send the same report at $t = 1$.

Example 1 *Possibility of Non-Partition. Consider the following two-stage example. The expert's type is uniformly distributed on* $[0, 1]$. *The period* t *payoffs of the expert and the decision maker are given by*

$$U^E(y, m) = -(y - m)^2$$

$$U^{DM}(y, m, a) = -(y - m - a)^2$$

and $a = 0.001$. *We will construct an assessment that is a* PBE *and where a disconnected set of types choose the same signal at* $t = 1$. *Suppose that the expert uses the following signaling rule. At stage 1, the expert for*

$[0, 0.253] \cup [0.597, 1]$ *sends a message* n_1^A *with prob 1 and*

$[0.253, 0.597]$ *sends a message* n_1^B *with prob 1*

Now at stage 2 the expert adopts the following signaling rule. For types in

$[0, 0.253]$ *sends a message* n_2^A *with prob 1 and*

$[0.253, 0.597]$ *sends a message* n_2^B *with prob 1*

$[0.597, 1]$ *sends a message* n_2^C *with prob 1.*

Given the strategy of the expert, the decision maker at stage 1 after observing message n_1^A *he things that the experts type lies somewhere in* $[0, 0.253] \cup [0.597, 1]$ *and chooses the action that solves*

$$y(n_1^A) \in \arg\max \int_0^{0.253} -(y-m-a)^2 dm + \int_{0.597}^{1} -(y-m-a)^2 dm$$

(where we ignore the constant that we have to multiply this expression in order for it to integrate to one), from which we get that

$$y(n_1^A) \simeq 0.463.$$

Now when the decision maker observes n_1^B *he thinks that the type of the expert must lie in* $[0.253, 0.597]$ *and chooses the action that solves*

$$\int_{0.253}^{0.597} -(y-m-a)^2 dm$$

so

$$y(n_1^B) \simeq 0.425.$$

In an similar fashion we can compute the optimal action that the decision maker chooses at stgae 2 as function of the signal that he receives. We get that

$$\begin{aligned} y(n_2^A) &\simeq 0.127 \\ y(n_2^B) &\simeq 0.425 \\ y(n_2^C) &\simeq 0.8, \end{aligned}$$

One can check that this is indeed an equilibrium by verifying that indeed type 0.253 *is indifferent between "inducing"* $y(n_1^A)$ *at* $t=1$ *and* $y(n_2^A)$ *at* $t=2$ *and inducing* $y(n_1^B)$ *at* $t=1$ *and* $y(n_2^B)$ *at* $t=2$*. Types below* 0.253 *prefer inducing* $y(n_1^A)$ *at* $t=1$ *and* $y(n_2^A)$ *at* $t=2$ *and types above* 0.253 *prefer inducing* $y(n_1^B)$ *at* $t=1$ *and* $y(n_2^B)$ *at* $t=2$*. One can also chech that type* 0.597 *is indifferent between "inducing"* $y(n_1^A)$ *at* $t=1$ *and* $y(n_2^C)$ *at* $t=2$ *and inducing* $y(n_1^B)$ *at* $t=1$ *and* $y(n_2^B)$ *at* $t=2$*. Types below* 0.597 *prefer inducing* $y(n_1^B)$ *at* $t=1$ *and* $y(n_2^B)$ *at* $t=2$ *and types above* 0.597 *prefer inducing* $y(n_1^A)$ *at* $t=1$ *and* $y(n_2^C)$ *at* $t=2$*. This example demonstrates that it is possible that the decision maker receives that same advice from quite different types of experts!*

There will be more information released

The previous example tells us that equilibria in the dynamic cheap talk game can be quite complicated. Still there is an number of important questions that we can answer without characterizing all possible equilibria. One of these questions is whether it is possible to have more information released compared to the static case. The example that follows shows that this is possible.

Example: More Information

As before, suppose that the expert's and the decision maker's preferences are represented by

$$U^S(y,m) = -(y-m)^2$$

$$U^R(y,m,a) = -(y-m-a)^2$$

Let us first characterize the equilibria in the static game. Suppose the support is in $[z_1, z_2]$. From Crawford and Sobel (1982) we know that all equilibria are essentially partition equilibria. The cutoffs of the subintervals are characterized by the difference equation

$$m_{i+1} = 2m_i - m_{i-1} - 4a.$$

The solution is $m_i = \lambda i^2 + \mu i + v$ and from the difference equation we get $\lambda = -2a$. Now the initial condition is $m_0 = z_1$, which implies that $v = z_1$. The final condition is $m_p = z_2$ which allows us to find $\mu :$ $z_2 = -2ap^2 + \mu p + z_1$ $\mu = \frac{1}{p}(z_2 - z_1) + 2ap$, where p is the number of subintervals that we divide the type space. The maximal number of subintervals that we can divide the type space is the larger p that satisfies

$$m_1 - m_0 - 4a(p-1) > 0 \tag{1}$$

We know that $m_0 = z_1$ and $m_1 = \lambda + \mu + v = -2a + \frac{1}{p}(z_2 - z_1) + 2ap + z_1$. The condition (1) then becomes

$$-2a + \frac{1}{p}(z_2 - z_1) + 2ap - 4a(p-1) > 0$$

which simplifies to

$$\frac{1}{p}(z_2 - z_1) - 2ap + 2a > 0$$

It equivalent to

$$-2ap^2 + 2ap + (z_2 - z_1) > 0. \tag{2}$$

Let's $p^{\max}$ be the maximal p such that the inequality above holds. This $p^{\max}$ depends only on the difference $(z_2 - z_1)$ and this dependence is monotonic. This, however, does not imply that number of partitions will be the same as in the static case.

Suppose $a = 1/12$. The interval at which agent's types could be are $[0,1]$. **Static economy:** There are at most two intervals. Check (2). If $p = 2$ then $-2 * \frac{1}{12} * 4 + 2 * \frac{1}{12} * 2 + 1 = \frac{8}{12} > 0$. We can not have 3 partitions since $-2\frac{1}{12}9 + 2\frac{1}{12}3 + 1 = 0$. Using the formulas we get $\lambda = -\frac{1}{6}$ and $\mu = \frac{5}{6}$. The break point m is $m_1 = \frac{4}{6}$, so that there are two intervals: $[0, 4/6]$ and $[4/6, 1]$. We can show that in the 2 period economy more information is revealed than in the static case. **Dynamic economy:** First, note that the interval $[0, 4/6]$ can also be partitioned into two. In particular, for the difference equation we can $v = 0$, $\lambda = -\frac{1}{6}$ and $\mu = \frac{1}{2} * \frac{4}{6} + 2 * 2 * \frac{1}{12} = \frac{4}{6}$ and the interval will be partitioned into $[0, 3/6]$ and $[3/6, 4/6]$. Now let's try to find the equilibria. Suppose in the first period the interval is separated into $[0, m_1]$ and $[m_1, 1]$. The first interval is partitioned in the second period into $[0, n_1^1]$ and $[n_1^1, m_1]$. The indifference condition is

$$\begin{aligned} & \left(\frac{-m_1}{2} + a\right)^2 + \left(\frac{n_1^1 - m_1}{2} + a\right)^2 \\ = \; & 2\left(\frac{1 - m_1}{2} + a\right)^2 \end{aligned}$$

From the formulas we can find that $n_1^1 = \lambda + \mu + v = -2a + \left(\frac{1}{p}(z_2 - z_1) + 2ap\right) + 0 = -\frac{2}{12} + \left(\frac{m_1}{2} + \frac{4}{12}\right) = \frac{m_1}{2} + \frac{2}{12}$. Then the condition above can be re-written as

$$\left(\frac{-m_1}{2} + a\right)^2 + \left(\frac{1/6 - m_1/2}{2} + a\right)^2 = 2\left(\frac{1 - m_1}{2} + a\right)^2$$

or

$$\left(-m_1 + \frac{1}{6}\right)^2 + \left(\frac{2}{6} - \frac{1}{2}m_1\right)^2 = 2\left(\frac{7}{6} - m_1\right)^2$$

Using Matlab, we can see that $m_1 \approx 0.75$ and $n_1^1 \approx 0.54$. Thus one possible equilibrium is the interval

$[0; 0.75]$ and $[0.75; 1]$ in the first period and further division $[0; 0.54]$ and $[0.54; 0.75]$ in the second period. So in the second period there are three intervals, while in the static case there are only 2 intervals.

This example demonstrates in a dynamic setup that we can have more information released in equilibrium compared to the static. This raises the question of how much more information will be released? Is the expert's type for all types of the expert? We now move one establish that the answer to this question is no.

6. No-Full Information Revelation

We will now establish that there does not exist an equilibrium where the decision maker learns the truth at some stage in the game. In other words, no matter for how many periods the expert and the decision maker interact, there will be a stage where learning stops.

First let us describe formally what we mean when we say that the decision maker has learnt the true type of the expert. Is there a full revealing equilibrium? That is, there an equilibrium where the decision maker learns the truth about the expert's type. It means that there exists a stage t in the game where the decision maker knows m no matter where the value of m lies. This implies that at the previous stage all types have chosen revealing signaling rules that is chose a different message for each type.

We will show that there cannot exist an equilibrium where at some stage t we have full revelation. We will argue by contradiction. Consider a PBE of our dynamic information transmission game and suppose that there exists a stage $t-1$ where all types of the expert choose revealing actions. We will show that this cannot be an equilibrium. First we argue that since no full information revelation has taken place up to $t-1$, it must be the case that at $t-1$ after some history the support of the posterior will contain an open interval of types. Now we will argue that it is impossible at stage $t-1$ all types choose revealing actions

Observation: The main point is that we cannot have full separation of types at a point where the posterior contains an open set. The reason is that the expert's and the decision makers interests differ. Now can we have an equilibrium where at some stage the posterior of the decision maker never contains an open set? If this is the case then there must be an equilibrium where the expert employs complicated signaling rules which have to be of the following form for types different sets of countable types $m_i \neq m_j$ the expert chooses the same message. We will argue that this is impossible. Suppose that there exists some stage where for all possible histories up to that stage the posterior has support a countable set of types. To fix ideas suppose that after some history the posterior at stage $t-1$ assigns positive probability to three types m_A, m_B and m_C. Suppose that given this posterior action y_t is chosen at t. From Bester and Strausz (2001) it follows that for the particular continuation of the game it is enough that there can be up to three messages that the expert can send, it is therefore without any loss to think that at $t+1$ at this continuation game there can be at most three actions induced, let us call them $y(n_1), y(n_2)$, $y(n_3)$. If there is only a single action induced we can take all $y(n_1), y(n_2)$, $y(n_3)$ to be equal to each other.

Now if this is an equilibrium it must be the case type m_A prefers the corresponding messages to the one that $m-\varepsilon$ chooses:

$$\begin{aligned}
-(y_t - m_A)^2 - (y(n_1) - m_A)^2 &\geq -(y_t(m_A-\varepsilon) - m_A)^2 - (y(m_A-\varepsilon) - m_A)^2 \\
-(y_t - m_A)^2 - (y(n_1) - m_A)^2 &\geq -(y_t(m_A+\varepsilon) - m_A)^2 - (y(m_A+\varepsilon) - m_A)^2
\end{aligned}$$

$$\begin{aligned}
-(y_t(m_A-\varepsilon) - (m_A-\varepsilon))^2 - (y(m_A-\varepsilon) - (m_A-\varepsilon))^2 &\geq -(y_t - (m_A-\varepsilon))^2 - (y(n_1) - (m_A-\varepsilon))^2 \\
-(y_t(m_A+\varepsilon) - (m_A+\varepsilon))^2 - (y(m_A+\varepsilon) - (m_A+\varepsilon))^2 &\geq -(y_t - (m_A+\varepsilon))^2 - (y(n_1) - (m_A+\varepsilon))^2
\end{aligned}$$

and then letting ε go to zero we get that

$$\begin{aligned}
-(y_t - m_A)^2 - (y(n_1) - m_A)^2 &= -(y_t(m_A-\varepsilon) - m_A)^2 - (y(m_A-\varepsilon) - m_A)^2 \\
-(y_t - m_A)^2 - (y(n_1) - m_A)^2 &= -(y_t(m_A+\varepsilon) - m_A)^2 - (y(m_A+\varepsilon) - m_A)^2
\end{aligned}$$

(incomplete)

Impossible to have a fully separating equilibrium

(I will write arguments for two stages- but generalize to more than two stages)

If there is separation then at stage one each type of the sender chooses a different action. The responder chooses $y^R(m)$ and the same for stage 2 but since for all m we have that $y^R(m) \neq y^S(m,b)$ there exists at least one m that would like to choose the message of another type - say $\hat{m}-$ at stage one- for instance one can easily show that at least for one m there exists $\hat{m}$ such that $-(y^R(\hat{m})-m) \geq -(y^R(m)-m)$ because $y^R(\hat{m})$ is "closer" to $y^S(m,b)$.

7. Appendix

Algebra for Partition Equilibria

from Mike's draft

$$m_1 = \lambda + \mu + v = -2a + \frac{1}{p}(z_2 - z_1) + 2ap + z_1$$

CF for $[0,1]$

$$m_i = \left(\frac{1}{p} + 2ap\right) i - 2ai^2$$

Hence the general formula is

$$n_i = (-2a)\, i^2 + \left(\frac{1}{p}(z_2 - z_1) + 2ap\right) i + z_1$$

where z_1 and z_2 are the beginning and the end of the interval.

Let's find the general expression for n_1 and n_{p-1}

$$\begin{aligned} n_1 &= (-2a) + \frac{1}{p}(z_2 - z_1) + 2ap + z_1 \\ &= \frac{z_2}{p} + \frac{z_1\,(p-1)}{p} + 2a\,(p-1)\,.\ \text{same} \end{aligned}$$

and

$$\begin{aligned} n_{p-1} &= (-2a)\,(p-1)^2 + \left(\frac{1}{p}(z_2 - z_1) + 2ap\right)(p-1) + z_1 \\ &= 2\alpha(p-1)(1-p+p) + \frac{(p-p+1)\,z_1}{p} + \frac{(p-1)z_2}{p} \\ &= 2\alpha(p-1) + \frac{z_1}{p} + \frac{(p-1)z_2}{p} \\ oleg &= \frac{z_1}{p} + \frac{p-1}{p}z_2 + 2a\,(p-1) \end{aligned}$$

A general indifference equation

$$\left(\frac{m_{i-1}-m_i}{2}+a\right)^2+\left(\frac{n^i_{p(i)-1}-m_i}{2}+a\right)^2 \tag{2}$$
$$=\left(\frac{m_{i+1}-m_i}{2}+a\right)^2+\left(\frac{n^{i+1}_1-m_i}{2}+a\right)^2$$

Now the first cut-off of the $i+1^{th}$ partition at $t=2$ is n_1^{i+1} from the formula above this is given by

$$n_1^{i+1}=\frac{z_2}{p}+\frac{z_1\,(p-1)}{p}+2a\,(p-1)$$

where $z_1=m_i$ and $z_2=m_{i+1}$ and $p(i+1)$ this is the number of subintervals that the $i+1$ interval is going to be divided in the second period. We therefore get

$$n_1^{i+1}=\frac{m_{i+1}}{p(i+1)}+\frac{m_i\,(p(i+1)-1)}{p(i+1)}+2a\,(p(i+1)-1)$$

Now subtracting m_i from both sides we get that

$$\begin{aligned} n_1^{i+1}-m_i &= \frac{m_{i+1}}{p(i+1)}+\frac{m_i\,(p(i+1)-1)}{p(i+1)}+2a\,(p(i+1)-1)-m_i \\ &= \frac{m_{i+1}}{p(i+1)}+\frac{m_i\,(p(i+1)-1-p(i+1))}{p(i+1)}+2a\,(p(i+1)-1) \\ &= \frac{m_{i+1}-m_i}{p(i+1)}+2a\,(p(i+1)-1)\ \text{ vasiliki} \end{aligned}$$

$$\begin{aligned} n_1^{i+1}-m_i &= \frac{m_{i+1}}{p}+\frac{m_i\,(p(i+1)-1)}{p(i+1)}+ap(i+1)-m_i \\ &= \frac{m_{i+1}-m_{i-1}}{p(i)}+2a\,(p\,(i+1)-1)-\ \text{ oleg} \end{aligned}$$

Now let us check the expression for $n^i_{p(i)-1}-m_i$. $n^i_{p(i)-1}$ is the penultimate cut-off at the second period of the i^{th} interval. From the general formula above we have

$$n_i=(-2a)\,i^2+\left(\frac{1}{p}(z_2-z_1)+2ap\right)i+z_1$$

which for $i = p(i) - 1$; $m_{i-1} = z_1$ and $m_i = z_2$ we get that

$$\begin{aligned} n^i_{p(i)-1} &= (-2a)\,(p(i)-1)^2 + \left(\frac{1}{p(i)}(m_i - m_{i-1}) + 2ap(i)\right)(p(i)-1) + m_{i-1} \\ &= 2a\,(p(i)-1)\,(-p(i)+1+p(i)) + \frac{m_i\,(p(i)-1)}{p(i)} + \frac{m_{i-1}\,(p(i)-p(i)+1)}{p(i)} \\ &= 2a\,(p(i)-1) + \frac{m_i\,(p(i)-1)}{p(i)} + \frac{m_{i-1}}{p(i)} \end{aligned}$$

Subtracting m_i from both sides

$$\begin{aligned} n^i_{p(i)-1} - m_i &= 2a\,(p(i)-1) + \frac{m_i\,(p(i)-1)}{p(i)} + \frac{m_{i-1}}{p(i)} - m_i \\ &= 2a\,(p(i)-1) + \frac{m_{i-1} - m_i}{p(i)} - \text{ vasiliki} \end{aligned}$$

Summarizing - before subtractions: (V)

$$\begin{aligned} n^{i+1}_1 &= \frac{m_{i+1}}{p(i+1)} + \frac{m_i\,(p(i+1)-1)}{p(i+1)} + 2a\,(p(i+1)-1) \\ n^i_{p(i)-1} &= \frac{m_i\,(p(i)-1)}{p(i)} + \frac{m_{i-1}}{p(i)} + 2a\,(p(i)-1) \end{aligned}$$

after subtracting m_i from both sides: (V)

$$\begin{aligned} n^{i+1}_1 - m_i &= \frac{m_{i+1} - m_i}{p(i+1)} + 2a\,(p(i+1)-1) \\ n^i_{p(i)-1} - m_i &= 2a\,(p(i)-1) + \frac{m_{i-1} - m_i}{p(i)}, \end{aligned}$$

now we substitute these expressions into the indifference equation

$$\begin{aligned} & \left(\frac{m_{i-1} - m_i}{2} + a\right)^2 + \left(\frac{n^i_{p(i)-1} - m_i}{2} + a\right)^2 \\ = & \left(\frac{m_{i+1} - m_i}{2} + a\right)^2 + \left(\frac{n^{i+1}_1 - m_i}{2} + a\right)^2 \end{aligned}$$

and get that

$$\left(\frac{m_{i-1}-m_i}{2}+a\right)^2+\left(\frac{2a\left(p(i)-1\right)+\frac{m_{i-1}-m_i}{p(i)}}{2}+a\right)^2$$
$$=\left(\frac{m_{i+1}-m_i}{2}+a\right)^2+\left(\frac{\frac{m_{i+1}-m_i}{p(i+1)}+2a\left(p(i+1)-1\right)}{2}+a\right)^2$$

$$\left(\frac{m_{i-1}-m_i}{2}+a\right)^2+\left(\frac{m_{i-1}-m_i}{2p(i)}+ap(i)\right)^2$$
$$=\left(\frac{m_{i+1}-m_i}{2}+a\right)^2+\left(\frac{m_{i+1}-m_i}{2p(i+1)}+ap(i+1)\right)^2$$

Now let $p(i)=p(i+1)$ and rearrange we get

$$\left(\frac{m_{i-1}-m_i}{2}+a\right)^2-\left(\frac{m_{i+1}-m_i}{2}+a\right)^2$$
$$=\left(\frac{m_{i+1}-m_i}{2p}+ap\right)^2-\left(\frac{m_{i-1}-m_i}{2p}+ap\right)^2$$

Left hand side

$$\left[\left(\frac{m_{i-1}-m_i}{2}+a\right)-\left(\frac{m_{i+1}-m_i}{2}+a\right)\right]\left[\left(\frac{m_{i-1}-m_i}{2}+a\right)+\left(\frac{m_{i+1}-m_i}{2}+a\right)\right]$$
$$=\left[\frac{m_{i-1}-m_{i+1}}{2}\right]\left[\frac{m_{i-1}+m_{i+1}-2m_i}{2}+2a\right]$$

Right Hand side

$$\left[\left(\frac{m_{i+1}-m_i}{2p}+ap\right)-\left(\frac{m_{i-1}-m_i}{2p}+ap\right)\right]\left[\left(\frac{m_{i+1}-m_i}{2p}+ap\right)+\left(\frac{m_{i-1}-m_i}{2p}+ap\right)\right]$$
$$=\left[\frac{m_{i+1}-m_{i-1}}{2p}\right]\left[\frac{m_{i+1}+m_{i-1}-2m_i}{2p}+2ap\right] \qquad (*)$$

Combining the left and the right hand side we get that

$$\left[\frac{m_{i-1}-m_{i+1}}{2}\right]\left[\frac{m_{i-1}+m_{i+1}-2m_i}{2}+2a\right]$$
$$=\left[\frac{m_{i+1}-m_{i-1}}{2p}\right]\left[\frac{m_{i+1}+m_{i-1}-2m_i}{2p}+2ap\right]$$

$$
\begin{aligned}
-p\left[\frac{m_{i-1}+m_{i+1}-2m_i}{2}+2a\right] &= \left[\frac{m_{i+1}+m_{i-1}-2m_i}{2p}+2ap\right] \\
(m_{i-1}+m_{i+1}-2m_i)\left[\frac{1}{2p}+\frac{p}{2}\right] &= -2ap-2ap \\
(m_{i-1}+m_{i+1}-2m_i)\frac{1+p^2}{2p} &= -4ap \\
m_{i+1} &= 2m_i-m_{i-1}-\frac{8ap^2}{1+p^2}
\end{aligned}
$$

Let us try to see what happens for different subdivisions in the second period:

$$
\begin{aligned}
& \left(\frac{m_{i-1}-m_i}{2}+a\right)^2+\left(\frac{m_{i-1}-m_i}{2p(i)}+ap(i)\right)^2 \\
= & \left(\frac{m_{i+1}-m_i}{2}+a\right)^2+\left(\frac{m_{i+1}-m_i}{2p(i+1)}+ap(i+1)\right)^2
\end{aligned}
$$

Rearrange

$$
\begin{aligned}
& \left(\frac{m_{i-1}-m_i}{2}+a\right)^2-\left(\frac{m_{i+1}-m_i}{2}+a\right)^2 \\
= & \left(\frac{m_{i+1}-m_i}{2p(i+1)}+ap(i+1)\right)^2-\left(\frac{m_{i-1}-m_i}{2p(i)}+ap(i)\right)^2
\end{aligned}
$$

Left hand side

$$
\begin{aligned}
& \left[\left(\frac{m_{i-1}-m_i}{2}+a\right)-\left(\frac{m_{i+1}-m_i}{2}+a\right)\right]\left[\left(\frac{m_{i-1}-m_i}{2}+a\right)+\left(\frac{m_{i+1}-m_i}{2}+a\right)\right] \\
= & \left[\frac{m_{i-1}-m_{i+1}}{2}\right]\left[\frac{m_{i-1}+m_{i+1}-2m_i}{2}+2a\right] \qquad (**) \\
= & \; a\left[2m_i-m_{i-1}-m_{i+1}\right]+\left[\frac{2m_i-m_{i-1}-m_{i+1}}{2}\right]\left[\frac{m_{i+1}-m_{i-1}}{2}\right]
\end{aligned}
$$

Right Hand side

$$
\begin{aligned}
& \left[\left(\frac{m_{i+1}-m_i}{2\hat{p}}+a\hat{p}\right)-\left(\frac{m_{i-1}-m_i}{2p}+ap\right)\right]\left[\left(\frac{m_{i+1}-m_i}{2\hat{p}}+a\hat{p}\right)+\left(\frac{m_{i-1}-m_i}{2p}+ap\right)\right] \\
= & \left[\frac{m_{i+1}-m_i}{2\hat{p}}-\frac{m_{i-1}-m_i}{2p}+a(\hat{p}-p)\right]\left[\frac{m_{i+1}-m_i}{2\hat{p}}+\frac{m_{i-1}-m_i}{2p}+a(\hat{p}+p)\right] \qquad (*) \\
= & \left[\frac{2p\left(m_{i+1}-m_i\right)-2\hat{p}\left(m_{i-1}-m_i\right)}{4\hat{p}p}+a(\hat{p}-p)\right]\left[\frac{2p\left(m_{i+1}-m_i\right)+2\hat{p}\left(m_{i-1}-m_i\right)}{4\hat{p}p}+a(\hat{p}+p)\right]
\end{aligned}
$$

Combining the left and the right hand side we get that

$$\begin{aligned} & \left[\frac{m_{i-1}-m_{i+1}}{2}\right]\left[\frac{m_{i-1}+m_{i+1}-2m_i}{2}+2a\right] \\ = & \left[\frac{p\left(m_{i+1}-m_i\right)-\hat{p}\left(m_{i-1}-m_i\right)}{2\hat{p}p}+a(\hat{p}-p)\right]\left[\frac{p\left(m_{i+1}-m_i\right)+\hat{p}\left(m_{i-1}-m_i\right)}{2\hat{p}p}+a(\hat{p}+p)\right] \end{aligned}$$

Algebra for Crazy Equilibria

Example

An Example of Non-Partition equilibria

Suppose that the types in Y

$$Y=[\underline{m},m_L]\cup[m_H,\bar{m}]$$

stage 1:

all types in Y choose the same action call it b

the responder the chooses $y(b)$

at stage 2 **types in** $[\underline{m},m_L]$ choose b_2^L - the responder the chooses $y(b_2^L)$

and types in $[m_H,\bar{m}]$ choose b_2^H - the responder the chooses $y(b_2^H)$

Types in (m_L,m_H)

at stage 1 choose $\hat{b}-$ the responder the chooses $y(\hat{b})$

at stage 2 choose $\hat{b}-$ the responder the chooses $y(\hat{b})$

Payoffs over in the game are given by

$$U^S(y(b),y(b_2^L),m_L)=-(y(b)-m_L)^2-(y(b_2^L)-m_L)^2$$

$$U^S(y(b),y(b_2^H),m_H)=-(y(b)-m_H)^2-(y(b_2^H)-m_H)^2.$$

For a type $\hat{m}\in(m_L,m_H)$ payoff at this PBE is given by

$$U^S(y(\hat{b}),y(\hat{b}_2),\hat{m})=-(y(\hat{b})-\hat{m})^2-(y(\hat{b}_2)-\hat{m})^2.$$

then since this is a PBE and the sender's strategy is a best response it must be the case that

$$-(y(\hat{b})-\hat{m})^2-(y(\hat{b}_2)-\hat{m})^2 \geq -(y(b)-\hat{m})^2-(y(b_2^L)-\hat{m})^2$$

and

$$-(y(\hat{b})-\hat{m})^2-(y(\hat{b}_2)-\hat{m})^2 \geq -(y(b)-\hat{m})^2-(y(b_2^H)-\hat{m})^2$$

Moreover it must be the case that

$$-(y(b)-m_L)^2-(y(b_2^L)-m_L)^2 \geq -(y(\hat{b})-m_L)^2-(y(\hat{b}_2)-m_L)^2$$

by continuity there must exists a type that is indifferent between these sequences of actions. In order that the scenario under consideration is an equilibrium it must be the case that this type is m_L

$$-(y(b)-m_L)^2-(y(b_2)-m_L)^2 = -(y(\hat{b})-m_L)^2-(y(\hat{b}_2)-m_L)^2$$

or

$$(y(\hat{b})-m_L)^2-(y(b)-m_L)^2 = (y(b_2)-m_L)^2-(y(\hat{b}_2)-m_L)^2$$

$$\begin{aligned} & \Big[(y(\hat{b})-m_L)-(y(b)-m_L)\Big]\Big[(y(\hat{b})-m_L)+(y(b)-m_L)\Big] \\ = & \Big[(y(b_2)-m_L)-(y(\hat{b}_2)-m_L)\Big]\Big[(y(b_2)-m_L)+(y(\hat{b}_2)-m_L)\Big] \end{aligned}$$

$$\Big[(y(\hat{b})-y(b)\Big]\Big[(y(\hat{b})+y(b)-2m_L\Big] = \Big[(y(b_2)-y(\hat{b}_2)\Big]\Big[(y(b_2)+y(\hat{b}_2)-2m_L\Big]$$

$$\begin{aligned} & \Big[(y(\hat{b})-y(b)\Big]\Big[(y(\hat{b})+y(b)\Big]-2m_L\Big[(y(\hat{b})-y(b)\Big] \\ = & \Big[(y(b_2)-y(\hat{b}_2)\Big]\Big[(y(b_2)+y(\hat{b}_2)\Big]-2m_L\Big[(y(b_2)-y(\hat{b}_2)\Big] \\ & -2m_L\Big[\Big[(y(\hat{b})-y(b)\Big]-\Big[(y(b_2)-y(\hat{b}_2)\Big]\Big] \\ = & \Big[(y(b_2)-y(\hat{b}_2)\Big]\Big[(y(b_2)+y(\hat{b}_2)\Big]-\Big[(y(\hat{b})-y(b)\Big]\Big[(y(\hat{b})+y(b)\Big] \end{aligned}$$

or

$$\begin{aligned} & -2m_L\Big[\Big[(y(\hat{b})-y(b)\Big]-\Big[(y(b_2)-y(\hat{b}_2)\Big]\Big] \\ = & \Big[(y(b_2)-y(\hat{b}_2)\Big]\Big[(y(b_2)+y(\hat{b}_2)\Big]-\Big[(y(\hat{b})-y(b)\Big]\Big[(y(\hat{b})+y(b)\Big] \end{aligned} \tag{3}$$

Similarly by continuity there exists a type that is indifferent between $y(b)$ and $y(b_2^H)$ and $y(\hat{b})$ and $y(\hat{b}_2)$. In order that the scenario under consideration is an equilibrium it must be the case that this type is m_H

$$-(y(b)-m_H)^2-(y(b_2^H)-m_H)^2 \geq -(y(\hat{b})-m_H)^2-(y(\hat{b}_2)-m_H)^2$$
$$-(y(b)-m_H)^2-(y(b_2^H)-m_H)^2 = -(y(\hat{b})-m_H)^2-(y(\hat{b}_2)-m_H)^2$$

or

$$\begin{aligned} & -2m_H\left[\left[(y(\hat{b})-y(b)\right]-\left[(y(b_2^H)-y(\hat{b}_2)\right]\right] \\ = & \left[(y(b_2^H)-y(\hat{b}_2)\right]\left[(y(b_2^H)+y(\hat{b}_2)\right]-\left[(y(\hat{b})-y(b)\right]\left[(y(\hat{b})+y(b)\right] \end{aligned} \tag{4}$$

Recap:

type m_L is choosing $y(b)$ and $y(b_2^L)$

type $\hat{m}$ is choosing $y(\hat{b})$ and $y(\hat{b}_2)$

type m_H is choosing $y(b)$ and $y(b_2^H)$

type $\tilde{m}^L$ is indifferent between $y(b)$ and $y(b_2^L)$ and $y(\hat{b})$ and $y(\hat{b}_2)$

type $\tilde{m}^H$ is indifferent between $y(b)$ and $y(b_2^H)$ and $y(\hat{b})$ and $y(\hat{b}_2)$

Let us examine the details closer

a) if $\left[\left[(y(\hat{b})-y(b)\right]-\left[(y(b_2^L)-y(\hat{b}_2)\right]\right]>0$ then all types below m_L strictly prefer $y(b)$ and $y(b_2^L)$ to $y(\hat{b})$ and $y(\hat{b}_2)$ and all types above m_L strictly prefer $y(\hat{b})$ and $y(\hat{b}_2)$ to $y(b)$ and $y(b_2^L)$.

b) if $\left[\left[(y(\hat{b})-y(b)\right]-\left[(y(b_2^L)-y(\hat{b}_2)\right]\right]<0$ then all types above m_L strictly prefer $y(b)$ and $y(b_2^L)$ to $y(\hat{b})$ and $y(\hat{b}_2)$ and all types below m_L strictly prefer $y(\hat{b})$ and $y(\hat{b}_2)$ to $y(b)$ and $y(b_2^L)$

c) if $\left[\left[(y(\hat{b})-y(b)\right]-\left[(y(b_2^H)-y(\hat{b}_2)\right]\right]>0$ then all types below m_H strictly prefer $y(b)$ and $y(b_2^H)$ to $y(\hat{b})$ and $y(\hat{b}_2)$ and all types above m_H strictly prefer $y(\hat{b})$ and $y(\hat{b}_2)$ to $y(b)$ and $y(b_2^H)$.

d) if $\left[\left[(y(\hat{b})-y(b)\right]-\left[(y(b_2^H)-y(\hat{b}_2)\right]\right]<0$ then all types above m_H strictly prefer $y(b)$ and $y(b_2^H)$ to $y(\hat{b})$ and $y(\hat{b}_2)$ and all types below m_H strictly prefer $y(\hat{b})$ and $y(\hat{b}_2)$ to $y(b)$ and $y(b_2^H)$.

Now let us examine which of the above combinations are equilibrium feasible, Implications

a) implies that $m_L < \hat{m}$

b) implies $\hat{m} < m_L$ impossible

c) implies $m_H < \hat{m}$ impossible

d) implies $\hat{m} < m_H$

- $y(b_2^L)$ solves a problem of the form

$$\int_{\underline{m}}^{m_L} -(y-m-a)^2 dm$$

so

$$y(b_2^L) = \frac{\underline{m} + m_L}{2} + a$$

- $y(b_2^H)$ solves a problem of the form

$$\int_{m_H}^{\bar{m}} -(y-m-a)^2 dm$$

so

$$y(b_2^H) = \frac{\bar{m} + m_H}{2} + a$$

- $y(\hat{b})$ solves a problem of the form

$$\int_{m_L}^{m_H} -(y-m-a)^2 dm$$

so

$$y(\hat{b}) = \frac{m_L + m_H}{2} + a$$

since $\underline{m} \le m_L$ and $m_H \le \bar{m}$ we have that

$$y(b_2^L) \le y(\hat{b}) \le y(b_2^H)$$

Also

$$y(b) \in \arg\max \int_{\underline{m}}^{m_L} -(y-m-a)^2 dm + \int_{m_H}^{\bar{m}} -(y-m-a)^2 dm$$

FOC's

$$
\begin{aligned}
\int_{\underline{m}}^{m_L} -2(y-m-a)dm + \int_{m_H}^{\bar{m}} -2(y-m-a)dm &= 0 \\
-2(ym - \frac{m^2}{2} - am)\Big|_{\underline{m}}^{m_L} + -2(ym - \frac{m^2}{2} - am)\Big|_{m_H}^{\bar{m}} &= 0 \\
(y - \frac{m}{2} - a)\Big|_{\underline{m}}^{m_L} + (y - \frac{m}{2} - a)\Big|_{m_H}^{\bar{m}} &= 0 \text{ assuming } m \neq 0
\end{aligned}
$$

$$2y = \frac{\underline{m} + m_L}{2} + \frac{\bar{m} + m_H}{2} + 2a$$

or

$$y(b) = \frac{\underline{m} + m_L + \bar{m} + m_H}{4} + a.$$

One more observation: because $\hat{b}$ is chosen by types in (m_L, m_H) it is always the case that

$$y(\hat{b}_2) \leq y(b_2^H).$$

Recall that $y(b) = \frac{\underline{m}+m_L+\bar{m}+m_H}{4} + a$ and $y(\hat{b}) = \frac{m_L+m_H}{2} + a$; depending on the level of $\underline{m}$ and $\bar{m}$ it can be the case that either $y(b) \leq y(\hat{b})$ or $y(b) \geq y(\hat{b})$.

In order that the just described scenario be a PBE we have noted that it must be the case that

$$-(y(b) - m_L)^2 - (y(b_2^L) - m_L)^2 = -(y(\hat{b}) - m_L)^2 - (y(\hat{b}_2) - m_L)^2$$

$$-(y(b) - m_H)^2 - (y(b_2^H) - m_H)^2 = -(y(\hat{b}) - m_H)^2 - (y(\hat{b}_2) - m_H)^2$$

Substituting the expressions for $y(b)$, $y(\hat{b})$, $y(\hat{b}_2)$ and $y(b_2^L)$ and $y(b_2^H)$, we get that

$$-2(\frac{m_L + m_H}{2} + a - \hat{m})^2 \geq -(\frac{\underline{m} + m_L + \bar{m} + m_H}{4} + a - \hat{m})^2 - (\frac{m_0^L + m_L}{2} + a - \hat{m})^2$$

and

$$-(\frac{\underline{m} + m_L + \bar{m} + m_H}{4} + a - m_L)^2 - (\frac{m_0^L + m_L}{2} + a - m_L)^2 = -2(\frac{m_L + m_H}{2} + a - m_L)^2$$

$$-(\frac{\underline{m} + m_L + \bar{m} + m_H}{4} + a - m_H)^2 - (\frac{m_I^H + m_H}{2} + a - m_H)^2 = -2(\frac{m_L + m_H}{2} + a - m_H)^2$$

Actually m_L and m_H must be indifferent, hence following by now familiar arguments we obtain:

for type m_L

$$-2m_L \left[\left[(y(\hat{b}) - y(b)\right] - \left[(y(b_2) - y(\hat{b}_2)\right]\right] = \left[(y(b_2) - y(\hat{b}_2)\right]\left[(y(b_2) + y(\hat{b}_2)\right] - \left[(y(\hat{b}) - y(b)\right]\left[(y(\hat{b}) + y(b)\right] \tag{5}$$

$$\begin{aligned} & -2m_L \left[\frac{m_L + m_H}{2} - \frac{\underline{m} + m_L + \bar{m} + m_H}{4} - \left[\frac{\underline{m} + m_L}{2} - \frac{m_L + m_H}{2}\right]\right] \\ = & \left[\frac{\underline{m} + m_L}{2} - \frac{m_L + m_H}{2}\right]\left[\frac{\underline{m} + m_L}{2} + a + \frac{m_L + m_H}{2} + a\right] \\ & - \left[\frac{m_L + m_H}{2} - \frac{\underline{m} + m_L + \bar{m} + m_H}{4}\right]\left[(\frac{m_L + m_H}{2} + a + \frac{\underline{m} + m_L + \bar{m} + m_H}{4} + a\right] \end{aligned}$$

simplifying

$$\begin{aligned} & -2m_L \left[m_L + m_H - \frac{\underline{m} + m_L + \bar{m} + m_H}{4} - \frac{2\underline{m}}{4} - \frac{2m_L}{4}\right] \\ = & \left[\frac{\underline{m} - m_H}{2}\right]\left[\frac{\underline{m} + 2m_L + m_H}{2} + 2a\right] \\ & - \left[\frac{2m_L + 2m_H}{4} - \frac{\underline{m} + m_L + \bar{m} + m_H}{4}\right]\left[(\frac{2m_L + 2m_H}{4} + \frac{\underline{m} + m_L + \bar{m} + m_H}{4} + 2a\right] \end{aligned}$$

$$\begin{aligned} & -2m_L \left[m_L + m_H - \frac{(3\underline{m} + 3m_L + \bar{m} + m_H)}{4}\right] \\ = & \left[\frac{\underline{m} - m_H}{2}\right]\left[\frac{\underline{m} + 2m_L + m_H}{2} + 2a\right] \\ & - \left[\frac{m_L + m_H - \underline{m} - \bar{m}}{4}\right]\left[\frac{3m_L + 3m_H + \underline{m} + \bar{m}}{4} + 2a\right] \end{aligned}$$

for type m_H

$$-2m_H \left[\left[(y(\hat{b}) - y(b)\right] - \left[(y(b_2^H) - y(\hat{b}_2)\right]\right] = \left[(y(b_2^H) - y(\hat{b}_2)\right]\left[(y(b_2^H) + y(\hat{b}_2)\right] - \left[(y(\hat{b}) - y(b)\right]\left[(y(\hat{b}) + y(b)\right] \tag{6}$$

which by substituting $y(\hat{b})$, $y(b)$, $y(b_2^H)$ and $y(\hat{b}_2)$

$$
\begin{aligned}
& -2m_H\left[\left[\frac{m_L+m_H}{2}-\frac{\underline{m}+m_L+\bar{m}+m_H}{4}\right]-\left[\frac{\bar{m}+m_H}{2}-\frac{m_L+m_H}{2}\right]\right] \\
= \quad & \left[\frac{\bar{m}+m_H}{2}-\frac{m_L+m_H}{2}\right]\left[\frac{\bar{m}+m_H}{2}+a+\frac{m_L+m_H}{2}+a\right] \\
& -\left[(\frac{m_L+m_H}{2}-\frac{\underline{m}+m_L+\bar{m}+m_H}{4}\right]\left[(\frac{m_L+m_H}{2}+a+\frac{\underline{m}+m_L+\bar{m}+m_H}{4}+a\right]
\end{aligned}
\tag{7}
$$

simplifying

$$
\begin{aligned}
& -2m_H\left[m_L+m_H-\frac{\underline{m}+m_L+\bar{m}+m_H}{4}-\frac{2\bar{m}}{4}-\frac{2m_H}{4}\right] \\
= \quad & \left[\frac{\bar{m}-m_L}{2}\right]\left[\frac{\bar{m}+m_L+2m_H}{2}+2a\right] \\
& -\left[\frac{2m_L+2m_H}{4}-\frac{\underline{m}+m_L+\bar{m}+m_H}{4}\right]\left[\frac{2m_L+2m_H}{4}+\frac{\underline{m}+m_L+\bar{m}+m_H}{4}+2a\right]
\end{aligned}
\tag{8}
$$

or

$$
\begin{aligned}
& -2m_H\left[m_L+m_H-\frac{(\underline{m}+m_L+3\bar{m}+3m_H)}{4}\right] \\
= \quad & \left[\frac{\bar{m}-m_L}{2}\right]\left[\frac{\bar{m}+m_L+2m_H}{2}+2a\right] \\
& -\left[\frac{m_L+m_H-\underline{m}-\bar{m}}{4}\right]\left[\frac{3m_L+3m_H+\underline{m}+\bar{m}}{4}+2a\right]
\end{aligned}
\tag{9}
$$

Summary of formulas:

$$y(b_2^L)=\frac{\underline{m}+m_L}{2}+a$$

$$y(b_2^H)=\frac{\bar{m}+m_H}{2}+a$$

$$y(b)=\frac{\underline{m}+m_L+\bar{m}+m_H}{4}+a.$$

$$y(\hat{b})=y(\hat{b}_2)=\frac{m_L+m_H}{2}+a$$

$$
\begin{aligned}
& -2m_L\left[m_L+m_H-\frac{(3\underline{m}+3m_L+\bar{m}+m_H)}{4}\right] \\
= & \left[\frac{\underline{m}-m_H}{2}\right]\left[\frac{\underline{m}+2m_L+m_H}{2}+2a\right] \\
& -\left[\frac{m_L+m_H-\underline{m}-\bar{m}}{4}\right]\left[\frac{3m_L+3m_H+\underline{m}+\bar{m}}{4}+2a\right]
\end{aligned}
$$

$$
\begin{aligned}
& -2m_H\left[m_L+m_H-\frac{(\underline{m}+m_L+3\bar{m}+3m_H)}{4}\right] \\
= & \left[\frac{\bar{m}-m_L}{2}\right]\left[\frac{\bar{m}+m_L+2m_H}{2}+2a\right] \\
& -\left[\frac{m_L+m_H-\underline{m}-\bar{m}}{4}\right]\left[\frac{3m_L+3m_H+\underline{m}+\bar{m}}{4}+2a\right]
\end{aligned} \tag{10}
$$

second stage best response implies that the following should be satisfied

$$
\begin{aligned}
-(y(b_2^L)-m_L)^2 &\geq -(y(b_2^H)-m_L)^2 \\
(y(b_2^H)-m_L)^2-(y(b_2^L)-m_L)^2 &\geq 0 \\
\left(y(b_2^H)-m_L-y(b_2^L)+m_L\right)\left(y(b_2^H)-m_L+y(b_2^L)-m_L\right) &\geq 0 \\
\left(y(b_2^H)-y(b_2^L)\right)\left(y(b_2^H)+y(b_2^L)-2m_L\right) &\geq 0
\end{aligned}
$$

or

$$
\left(\frac{\bar{m}+m_H}{2}-\frac{\underline{m}+m_L}{2}\right)\left(\frac{\bar{m}+m_H}{2}+\frac{\underline{m}+m_L}{2}+2a-2m_L\right)\geq 0
$$

and

$$
\begin{aligned}
-(y(b_2^H)-m_H)^2 &\geq -(y(b_2^L)-m_H)^2 \\
\left(y(b_2^L)-y(b_2^H)\right)\left(y(b_2^L)+y(b_2^H)-2m_H\right) &\geq 0 \\
\left(\frac{\underline{m}+m_L}{2}-\frac{\bar{m}+m_H}{2}\right)\left(\frac{\underline{m}+m_L}{2}+\frac{\bar{m}+m_H}{2}-2m_H\right) &\geq 0
\end{aligned}
$$

References

[1] Battaglini M. (2002): "Multiple Referrals and Multi-dimensional Cheap Talk." *Econometrica* 70, 1379-1401.

[2] Crawford, V. and J. Sobel (1982): "Strategic Information Transmission," *Econometrica*, 50, 1431-1451.

[3] Levy, G. and R. Razin (2004): "Multidimensional Cheap Talk," mimeo.

[4] Krishna V. and J. Morgan (2002): "The Art of Conversation," mimeo.

[5] Morris, Stephen (2001): "Political Correctness," *Journal of Political Economy*, 109, 2, 231-265.

[6] Sobel, J. (1985): "The Theory of Credibility," *The Review of Economic Studies*," LII, 557-573.

Taxation

[7] Auerbach A. and J. Hines (2002): "Taxation and Economic Efficiency," *Handbook of Public Economics*, Volume 3, 1347-1421.

[8] Battaglini, M and S. Coate (2004): "Pareto Efficient Income Taxation with Stochastic Abilities," mimeo.

[9] Battaglini, M. (2004): "Optimality and Renegotiation in dynamic Contracting,' mimeo.

[10] Battaglini, M. (2004): "Long-Term Contracting with Markovian Consumers," mimeo.

[11] Benhabib J. and A. Rustuchini (1997): "Optimal Taxes Without Commitment," *Journal of Economic Theory*, 77, 231-259.

[12] Benhabib J. and A. Rustuchini and A. Velasco, (1998): "Public Spending and Optimal taxes without Commitment," mimeo.

[13] Berliant, M. and J. Ledyard, (2003): "Optimal Dynamic Nonlinear Income Taxes with No Commitment," mimeo.

[14] Brito, D., J. Hamilton, S. Slutsky and J. Stiglitz (1991): "Dynamic Optimal Income Taxation with Government Commitment," *Journal of Public Economics*, 44, 15-35.

[15] Brito, D., J. Hamilton, M. Intiligator, E. Sheshinski and S. Slutsky (2003): "Private Information, Coasian Bargaining, and the Second Welfare Theorem," mimeo.

[16] Diamond P. and J. Mirrlees, ? "Optimal Taxation and Public Production I: Production Efficiency," *American Economic Review.*

[17] Diamond P. and J. Mirrlees, ? "Optimal Taxation and Public Production II: Tax Rules," *American Economic Review.*

[18] Dillen, M. and M. Lundholm, (1996): "Dynamic Income Taxation, Redistribution and the Rathchet Effect," *Journal of Public Economics*, 59, 69-93.

[19] Hamilton, J. and S. Slutsky (2003): "Optimal Nonlinear Taxation with a Finite Population," mimeo.

[20] Hamilton, J. and S. Slutsky (2004): "Nonlinear Price Discrimination with a Finite Number of consumers and Constrained Recontracting," mimeo.

[21] Laffont, J.J. and J. Tirole, (1987): "Comparative Statics and the Optimal Incentive Contract," *European Economic Review*, 31, 901-926.

[22] Mirrlees, J. (1971): "An Exploration in the Theory of Omptimum Income Taxation," *The Review of Economic Studies*, 38,2, 175-208.

[23] Roberts, K, (1984): "The Theoretical Limits to Redistribution," *The Review of Economic Studies*, LI, 177-195.

[24] Roberts, K.. (2000): "A Reconsideration of the Optimal Income Tax," mimeo Oxford University.

[25] Saez, E. (2001): "Using Elasticities to Derive Optimal Income Tax rates," *The Review of Economic Studies*, 68, 205-229.