# Impossibility of Collusion under Imperfect Monitoring with Flexible Production. †

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**ABSTRACT.** We show that it is impossible to achieve collusion in a duopoly when (1) goods are homogenous and firms compete in quantities, (2) new, imperfect information arrives continuously, without sudden events and (3) firms are able to respond to this new information quickly. The result holds even if we allow for asymmetric equilibria or monetary transfers. The intuition is that the flexibility to respond to new information quickly unravels any collusive scheme and that signals about the aggregate behavior only cannot be used effectively to provide individual incentives via transfers. Our result applies both to a simple stationary model and a more complicated one with prices following a mean-reverting Markov process.

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# 1. Introduction

Collusion is a major problem in many markets and has been an important topic in both applied and theoretical economics. As a result, we know how numerous real-life cartels have organized their cooperation and what kinds of agreements (implicit or explicit) are likely to be successful in obtaining collusive profits.[1] At least since Stigler (1964) economists have recognized that imperfect monitoring can destabilize cartels. In their seminal paper, Green and Porter (1984) proposed a class of equilibria that despite imperfect monitoring improve upon the static Nash equilibrium. In those equilibria, to provide incentives, price wars necessarily break out on the equilibrium path reducing the maximal payoffs achievable by a cartel. The monitoring imperfections reduce collusive payoffs but some collusion is still possible.

We study the scope of collusion in a quantity-setting duopoly with homogenous goods, constant marginal costs and flexible production, that is, if firms can change output flow frequently.[2] Like in Green and Porter (1984), in our model firms cannot observe each others' production decisions directly. They only observe noisy market prices/signals that depend on the total market supply. There are many markets that fit the general features of our model: the OPEC cartel's individual quantities are imperfectly monitored while the resulting world price of oil is affected by the total available supply and is commonly observable. One can also think about firms producing a chemical and selling both at a spot market and to clients, with client deals being private and affecting the spot market.

Our main result is that collusion is impossible to achieve if

(1) new, imperfect information arrives continuously,

[1] There are many papers describing explicit and tacit collusion among firms; for comprehensive studies see for example Hay and Kelley (1974) or Levenstein and Suslow (2004).

[2] The term flexible production is usually understood as low costs of changing flow and variety of output. In this paper we use it in a narrower sense, restricting the firms to produce only one type of output. The flexibility on the variety dimension would have separate effects on the scope of collusion: increased product differentiation may improve collusion, but increased complexity of monitoring may destabilize cartels.

(2) the firms have flexible production technology to respond to this new information quickly and

(3) public signals depend on total market supply only, and not individual decisions.

The result holds for symmetric and asymmetric equilibria (public perfect equilibria in pure strategies), even if we allow for monetary transfers and bounded money burning. It holds in a simple stationary world, analogous to Green and Porter (1984), and in a world where prices change slowly over time, according to a mean-reverting process.

A first guess may be that fast arrival of information and the flexibility to respond to it facilitate collusion, as firms can punish potential deviators more quickly. However, as shown first in a classic paper by Abreu, Milgrom and Pearce (1991) (hereafter AMP), although this intuition is true in games with perfect monitoring, in games with imperfect monitoring there is a tradeoff: more frequent moves make it difficult to discriminate between noise and deviations and hence can lead to a reduction of collusive profits. In this paper we extend the insights of AMP to a new environment and show that with information arriving continuously, high frequency of moves eliminates any possibility of collusion.

To illustrate the result, let us discuss an example, based on the mean-reverting model of Section 7. Suppose that the price is perturbed by random shocks, but conditional on the total production rate $Q$, the price tends towards the mean $1 - Q$. Assuming zero marginal costs, Figure 1 shows two sample paths of the price level when the firms produce Cournot Nash quantities and when they split the monopoly quantity.[3] Just by looking at the price level at any moment of time, it is obvious whether the firms are colluding or not. Yet, our results show that collusion is impossible when firms can act sufficiently frequently.

[3] In this non-stationary setting the absence of collusion is characterized by a Markov Perfect Equilibrium and the first-best collusion is similarly a state-dependent strategy. However, the Nash equilibrium and monopoly quantity of a one-shot analogue of this game (with constant demand $1 - Q$) serve well for illustration.

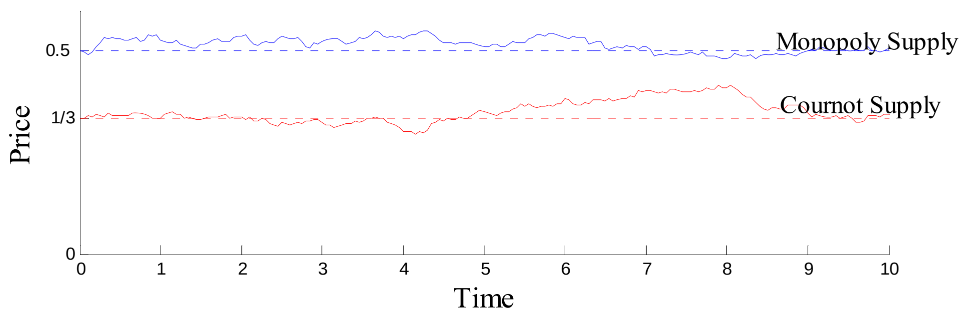


**Figure 1**

Our impossibility result has two main reasons. First, as firms act more frequently, statistical tests that detect deviations become very inefficient due to disaggregation of information. Second, it is difficult to use a signal that depends only on the sum of quantities to discriminate between deviators and non-deviators.

The first reason is best illustrated in *symmetric* equilibria. In a symmetric collusive scheme firms behave identically after all histories. Following Abreu, Pearce and Stacchetti (1986) (hereafter APS'86), an optimal symmetric equilibrium has two regimes: collusive and price war/punishment regimes. In the collusive regime, firms produce less than the static Nash equilibrium quantities. If a price drops below a critical level, this arouses enough suspicion of cheating to cause a price war. In the price war regime, firms strongly overproduce, because the intensity of the price war (i.e. low prices) makes the return to the collusive regime more likely.

As the frequency of moves increases, two forces influence the scope of collusion. First, as mentioned above, the quality of per-period information deteriorates. For a statistical test of cheating with a given likelihood difference (between deviation and no-deviation) the probability of a false positive (i.e. of observing a signal that indicates cheating even though no deviation took place) increases with the frequency of moves and makes the firms spend more time in price wars. Second, the single-period return to a deviation becomes smaller (compared to the continuation profits) and less powerful tests are needed to prevent deviations (and hence the probability of false-positives can be smaller). As we show, with information arriving continuously the first effect is much stronger. The reason lies in the aggregation of information. When firms can change actions twice in a time interval of length $\Delta$, they must test each price innovation individually against a cut-off level. Figure 2 illustrates the critical regions for these tests. If firms could change actions only once per time interval $\Delta$, a more effective joint test (illustrated by the dashed diagonal line) that aggregates information would be available.

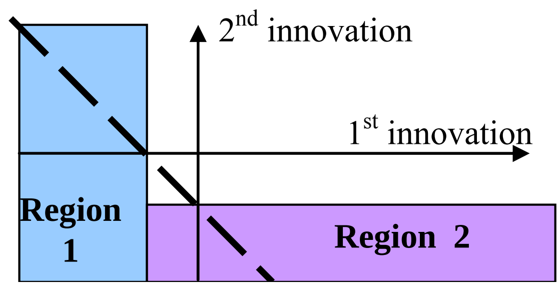


**Figure 2**

The second reason collusion is not possible pertains to *asymmetric* equilibria. In many games players can enforce collusive schemes without price wars using asymmetric continuation equilibria (or monetary transfers). Namely, rather than destroying value, they can transfer payoffs among themselves, keeping the average profits high. As we show, such collusive scheme does not work in our environment either. The intuition is that with a one-dimensional signal, transfers used to provide incentives for one player interfere with the incentives of the other player. As a result, collusion cannot be sustained by transfers alone. This intuition is related to the concept of identifiability from Fudenberg, Levine and Maskin (1994). The provision of incentives necessarily involves the destruction of value.

We carry most of our analysis on a simple stationary model. In this model, like in Green and Porter (1984), keeping the quantities fixed, prices are independently and identically distributed in all time periods. We allow firms to change actions more frequently while keeping the amount of information that they learn per unit of time constant.[4] Denoting

[4] As firms act more frequently, this model becomes less realistic than our non-stationary model with correlated prices presented in Section 7. However, the analysis is more transparent with the familiar techniques of APS so we present the stationary model first.

by 1/Δ the frequency of moves, Figure 3 shows a stylized relationship between the frequency of moves and the scope of collusion in that model (see Figure 6 in Section 5 for a detailed example). As we see it is not monotonic and the highest collusive payoffs are achievable for an interior value of Δ. Moreover, as Δ becomes small, the highest equilibrium profits decrease to the stage-game Nash equilibrium profits.

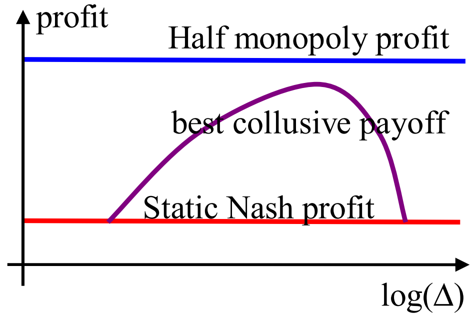


**Figure 3**

There is a large and growing theoretical literature trying to asses the impact of information on the scope of collusion in an environment with imperfect monitoring. Green and Porter (1984) were the first to propose a *symmetric* collusive equilibrium in which price wars are used on the equilibrium path to prevent deviations. Abreu, Pearce and Stacchetti (1986) characterize optimal symmetric equilibria in this setting and show that they involve two extreme regimes. The general analysis of games with hidden actions has been extended to *asymmetric* equilibria by Abreu, Pearce and Stacchetti (1990), Fudenberg, Levine and Maskin (1994) and Sannikov (2004) among others.

Many papers look at the role of asymmetric equilibria and monetary transfers in achieving collusion. In a setting with private information, McAfee and McMillan (1994) show in a static model the necessity of transfers for any collusion utilizing the private information. Athey and Bagwell (2001), Aoyagi (2003), Blume and Heidhues (2003) and Skrzypacz and Hopenhayn (2004) among others extend this intuition by studying collusion in repeated games without monetary transfers and emphasizing the need for transfers of continuation payoffs. Harrington and Skrzypacz (2004) study repeated competition with hidden prices and observed stochastic market shares, and show that symmetric equilibria (with price wars after skewed market shares) do not improve upon competitive outcomes but asymmetric equilibria do. Similarly, Athey and Bagwell (2001) show that asymmetric strategies greatly improve the scope of collusion, (which is shown to be quite limited in the symmetric equilibria of Athey, Bagwell and Sanchirico (2004)). In contrast, our impossibility result holds even for asymmetric equilibria and any monetary transfers.

As mentioned before, the idea that the frequency of actions can reduce the scope of collusion in a repeated game with imperfect monitoring was first shown in AMP (Abreu, Milgrom and Pearce 1991). They study a repeated Prisoner's Dilemma, in which noisy signals about players' actions arrive according to a Poisson process. More frequent moves imply a smaller chance of a signal arriving in a given period. They establish that depending on the signal structure collusion becomes more difficult or impossible with frequent moves.

We enrich the results of AMP in a number of new dimensions. First, we find that this effect is particularly strong in a setting where information arrives continuously through a Brownian-type of process, rather than via a Poisson arrival process. Unlike in AMP, where some collusion may be possible in the limit as the frequency of actions increases, in our setting collusion becomes completely impossible. Second, we analyze a different game: the Green and Porter duopoly game with a continuum of action choices per period, which allows stronger punishments than Nash reversion. Third, our result holds even if one considers asymmetric equilibria and the possibility of monetary transfers, while AMP focus on symmetric equilibria in a repeated Prisoner's Dilemma. Fourth, we show that collusion becomes impossible even if one considers a non-stationary setting and compares collusive payoffs with those in Markov perfect equilibria.

Analysis of asymmetric equilibria is especially important, as they allow the provision of incentives via transfers of continuation payoffs, without the need of destroying value in price wars. In fact, in the AMP model even when no collusion is possible in symmetric equilibria, it is feasible in asymmetric ones for a range of parameter values. That is not the case in our environment.[5]

Although we make many simplifying assumptions to illustrate our result, not all of them are important. It is essential that deviations of different firms cannot be statistically distinguished, and that information arrives continuously and without delay. If players could distinguish deviations of different players, then collusion would be possible (e.g. see Sannikov (2004)). Also, collusion may be possible if information arrives discontinuously or with delay (as shown in AMP). However, our results extend to many other settings; for example, with more than two firms, with multiple signals, and for a wide range of cost functions.

The paper proves the impossibility of collusion if firms act very frequently in a number of settings. Section 2 presents a simple model of a repeated game with stationary prices. Section 3 proves the main result for symmetric public perfect equilibria. Section 4 proves the result for asymmetric equilibria with and without monetary transfers. Section 5 presents a numerical example. Section 6 explores the possibility of nonlinear costs and capacity constraints. Section 7 proves our impossibility result in a more complicated and realistic model in which market prices are correlated over time. Section 8 discusses various modifications of the model. Section 9 concludes.

# 2. The Stationary Model

Two firms compete in a stationary market with homogenous products. Time horizon is infinite and firms discount the future profits with a common rate $r$. The firms set quantities every period and the resulting prices depend on quantities and noise. We study how the scope of potential collusion depends on the firms' flexibility of production. We describe the flexibility of production in terms of $\Delta$, the duration of time for which the production decisions are made.

[5] The general characterization of the class of games in which collusion becomes impossible as frequency of moves increases is still incomplete. The continuous information arrival per se is not a cause of such impossibility, as can be seen by comparing the current paper with Sannikov (2004).

In a repeated game $\Gamma^{\Delta}$ firms play the stage game at time points $t = 0,\ \Delta,\ 2\Delta, \ldots$. The stage game is as follows: at time $t = n\Delta$ the firms choose (privately) supply rates $q_{it} \in [0, \overline{q}]$ for a time interval $[t,\ t+\Delta)$ (where $\overline{q}$ is a large, exogenous capacity constraint). Denote by $Q_t = q_{1t} + q_{2t}$ the total supply rate of the two firms. During this time interval $[t,\ t+\Delta]$ firms supply $\Delta Q_t$ of the product. Both firms have the same constant marginal cost, which we normalize to zero.[6] The profit of each firm is just the revenue, which depends on the price and supply in a given period. Prices are publicly observable and depend on total supply and random shocks

$$p_t = P(Q_t) + \varepsilon_t$$

where the inverse demand function $P : [0, 2\overline{q}] \to \Re$ is strictly decreasing, twice continuously differentiable and $P(0) > 0$. Firm $i$'s revenue in time interval $[t,\ t+\Delta]$ is

$$\Delta q_{it}(P(Q_t) + \varepsilon_t)$$

To facilitate comparisons of equilibrium outcomes for different $\Delta$, assume that random shocks come from a standard Brownian motion $\{Z_t; t \geq 0\}$. Shocks $\varepsilon_t$ are formed in a way that makes total revenue of each firm depend not on $\Delta$, but on the supply rates only. Specifically, the revenue that firm $i$ gets from time 0 to $t$ is

$$\sum_{s=0,\Delta,2\Delta\ldots t-\Delta} \Delta q_{is}\left(P(q_{1s} + q_{2s}) + \varepsilon_s\right) = \int_0^t q_{is}\left(P(q_{1s} + q_{2s})ds + \sigma dZ_s\right),$$

where $q_{is}$ with $s \in [n\Delta,\ n\Delta+\Delta)$ is the supply rate that firm $i$ chose at time $n\Delta$; and $\sigma^2$ is the variance of the noise. Then $\varepsilon_t$, the average price shock over time interval $[t,\ t+\Delta)$, is simply

$$\varepsilon_t = \frac{\sigma}{\Delta}(Z_{t+\Delta} - Z_t).$$

Therefore, $\varepsilon_t \sim N(0, \sigma^2/\Delta)$. Note that random shocks have greater variance over small time intervals, so that the information that firms learn by observing prices is proportional

[6] See Section 6 for the extension to increasing marginal cost.

to time interval. We can interpret the noise structure also in the following way. Suppose that market prices are quoted every second and that these prices have a normal distribution with mean $P(Q)$ and variance $\sigma^2$. If firms can change their supply rates every Δ seconds, then the average price over that interval has mean $P(Q)$ and variance $\sigma^2/\Delta$ (and as firms are risk neutral, these two interpretations of $p_t$ are equivalent).

The fact that average prices can become unboundedly negative due to very high variance over short time periods is unattractive. However, it allows for a simple model in which the argument behind our main result is particularly clean.[7] With the simple model we develop intuition that translates easily to more complex situations. See Section 7 for a more realistic non-stationary model, with bounded instantaneous prices. Also, Section 8 explains how our results extend to a stationary model with distribution of per-period prices not necessarily normal. In particular, prices can be a nonnegative function of total supply and random shock – for example could have a lognormal distribution.

A crucial assumption is that like in Green and Porter (1984) the price depends on the total supply only, so that the distribution of prices changes in the same way whether firm 1 or firm 2 increases its production by a unit, no matter what production levels they start with. That is, we assume that the products are homogenous. As demonstrated in Sections 7 and 8, the assumption that the random price shocks are distributed normally and independently of past prices is not essential for the results.

In the repeated game firms choose supply rates after every history to maximize their expected discounted profit. Firm *i*'s (normalized) expected payoff is:

$$E\left[\left(1-e^{-r\Delta}\right)\sum_{t=0,\Delta,2\Delta\ldots} e^{-rt}q_{it}(P(Q_t)+\varepsilon_t)\right],$$

where the expectation takes into account the dependence of future supply rates on past prices.

[7] It is important to point out that our results do not follow simply from the per-period variance increasing to infinity as Δ→ 0 but rather from the rate at which it does.

A history of firm $i$ at time $t$ is the sequence of price realizations and own supply decisions up to time $t$. A public history contains only the realizations of the past prices.[8] We analyze pure strategy public perfect equilibria (PPE) of the game. A strategy of a firm is public if it depends only on public history of the game. Two public strategies form a PPE if after any public history continuation strategies form a Nash equilibrium. When we consider pure strategies, it is not restrictive to focus on public strategies. Indeed, for every private strategy, one can find a public strategy that induces the same probability measure over private histories.

Throughout the paper we use the term PPE in the sense of pure strategy perfect public equilibria. In Section 8 we discuss the extension of the results to mixed-strategy PPE.

**Remark 1.** In the Abstract and Introduction we have used extensively an informal phrase 'information arrives continuously'. We can now explain the term formally. If prices are i.i.d. and the average price from time 0 to time $t$ is a continuous function of $t$, then prices must take the form specified in our model (in particular, the noise has to be generated by a Brownian motion). Formally speaking, if $\overline{P}_t$, the average price between times 0 and $t$ is continuous, then $t\overline{P}_t$ is a Lévy process without jumps. Such a process can be represented as a sum of drift and diffusion terms, so average prices in each period are normally distributed.

**Remark 2**. It is important to note that the flexibility $\Delta$ plays two roles in our model: it affects variance of the distribution of prices in a given period and the per-period discount rate $\delta \equiv e^{-r\Delta}$. The first effect makes collusion more difficult for smaller $\Delta$ as it makes statistical inference less precise and hence deviations more difficult to detect. The second (standard in repeated games) makes it easier to collude because the single-period benefits to deviation become small relative to the continuation payoffs.

## 2.1. Structure of the Stage Game

We make the following two assumptions about the inverse demand function:

[8] One can also allow for public randomizations, so that before each period the players observe a realization of a public random variable (which becomes a part of the public history). It does not change any of the results - as prices have full support, any randomization can be supported using only them.

- **A1:** The marginal revenue (of total demand) is decreasing: $\frac{\partial^2}{\partial Q^2}\left(QP(Q)\right) < 0$.

- **A2:** The static best response to any $q \in \left[0, \overline{q}\right]$ is less than $\overline{q}$, i.e. the marginal revenue of residual demand is negative at $\overline{q}$ : $\left.\frac{\partial(q'P(q'+q))}{\partial q'}\right|_{q'=\overline{q}} < 0$ .

Assumptions A1 and A2 are sufficient to guarantee that best response in the stage game, $q^*(q)$, is unique and that the Nash equilibrium is symmetric and unique. Denote the Nash equilibrium of the stage game by $(q_N, q_N)$. We refer to it also as the 'competitive equilibrium' or the 'static equilibrium' especially when we consider the dynamic game and talk about the repetition of $(q_N, q_N)$ in each stage game. Define $v_N = P(2q_N)\, q_N$.

> **LEMMA 1.** Assume A1 and A2. The static best response $q^*(q)$ is unique and less than $\overline{q}$ . Also, static Nash equilibrium is symmetric and unique.

The proof is standard and can be found in the appendix.

## 3. Impossibility of Collusion: Symmetric Equilibria

In this section, we prove that collusion becomes impossible as $\Delta \to 0$ in symmetric PPE. From APS'86 we know that optimal symmetric PPE involve two regimes; a collusive regime and a price war regime. In each period, the decision whether to remain in the same regime or to switch is guided by the outcome of the price in that period alone. We show that as $\Delta \to 0$, transitions from collusion to the price war regime must happen very frequently to provide the players with incentives. Because of that, price wars destroy all collusive gains, and payoffs above static Nash become impossible in the limit. The methods we develop in this section are important for asymmetric PPE (possibly with monetary transfers) as well, but the analysis of those will require further insights. Denote by $[\underline{v}(\Delta), \overline{v}(\Delta)] \subset \Re$ the set of payoffs achievable in symmetric PPE in the game $\Gamma^{\Delta}$. We would like to show that as $\Delta \to 0$, $\overline{v}(\Delta)$ converges to the 'competitive payoff'
$v_N = P(2q_N)q_N$.

First, let us informally present the core of our argument. Following APS'86, in the collusive regime players expect total payoffs $\overline{v}(\Delta)$ and choose some supply rates $(q, q) < (q_N, q_N)$ in the current period. In the next period, depending on the realized price, players either stay in the collusive regime with continuation payoff $\overline{v}(\Delta)$ or go to a price war with a continuation payoff $\underline{v}(\Delta)$. The lower $\underline{v}(\Delta)$, the better are collusive payoffs $\overline{v}(\Delta)$: harsher punishments make it easier to provide incentives. We will show that $\overline{v}(\Delta)$ converges to $v_N$ even if players could use punishment 0 (clearly $\underline{v}(\Delta) \geq 0$ because 0 is the minmax payoff – a firm can guarantee itself that payoff by producing 0). As $\Delta \to 0$, collusion becomes impossible because a statistical test to prevent deviations sends players to a price war with a disproportionately high probability.

To see this in greater detail, consider a deviation from $q$ to a static best response $q^*(q) > q$ that reduces the mean of the observed average price from $\mu = P(2q)$ to $\mu' = P(q^*(q)+q) < \mu$. Lemma **3** below shows that the best statistical test to prevent this deviation is a tail test, which triggers a punishment when the price falls within a critical region $(-\infty, c]$. Given such a test, a deviation increases the probability of punishment by

$$\text{likelihood difference } = G'(c) - G(c),$$

where $G'$ and $G$ are cumulative distribution functions of normal distributions with variance $\sigma^2/\Delta$ and means $\mu'$ and $\mu$ respectively. The probability of type I error, i.e. triggering punishment when no deviation has occurred, is given by the size of the test:

$$\text{size} = G(c)$$

Because the gain from a deviation in one period is on the order of $\Delta$, we explore tail tests with a likelihood difference on the order of $\Delta$. Lemma **2** shows that with such tests as $\Delta \to 0$, the probability of making type I error in each period blows up in comparison with $\Delta$. Therefore, $\overline{v}(\Delta)$ cannot be sustained as illustrated in Figure 4. The total expected payoff is a weighted sum of the current period payoff and the expected continuation payoff with weights equal to $(1-\delta)$ and $\delta$. The continuation payoff is a weighted average of $\overline{v}(\Delta)$ and 0 (recall that we allow for punishments that are probably harsher than the equilibrium ones, which relaxes the problem) with weights $1-G(c)$ and $G(c)$:

$$\overline{v}(\Delta) = (1-\delta)qP(2q) + \delta\big(\overline{v}(\Delta)(1-G(c)) + 0\cdot G(c)\big)$$

That is represented graphically as:

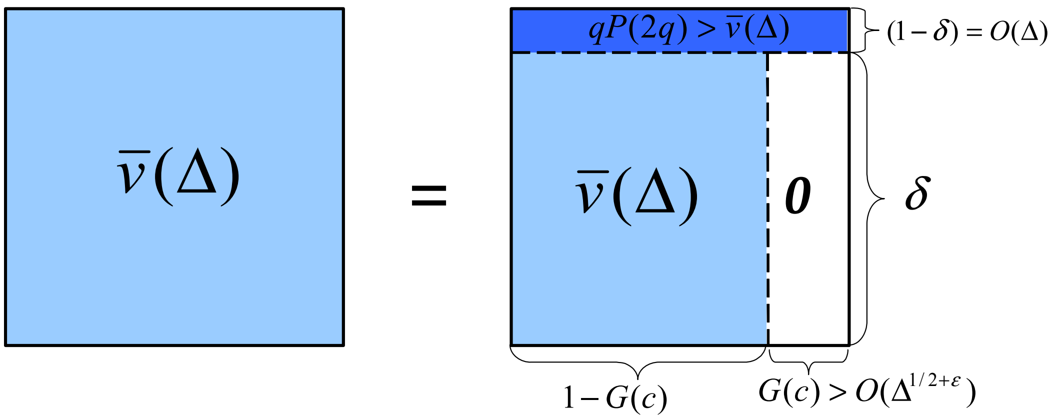


**Figure 4**

Now, as $\Delta \to 0$, we have two effects: the gain from colluding in the current period, $(1-\delta)\big(qP(2q) - q_N P(2q_N)\big)$ and the payoff loss due to punishment become small. However, the payoff loss of $\delta\,\overline{v}(\Delta)G(c)$ due to punishment becomes disproportionately large compared to the current-period payoff gain in the limit. As a result, any payoff above Nash cannot be sustained even if we

- worry about only one deviation to a static best response and
- allow 0 as the harshest punishment, even though typically $\underline{v}(\Delta) > 0$.

When comparing games with different Δ one may be tempted to employ the following reasoning: consider two games with Δ and Δ′=Δ/2. Suppose that with Δ we can construct a profitable collusive equilibrium. Now, moving to Δ′, instead of employing the factorization techniques of APS (that it is sufficient to factorize the game into current period and continuation payoffs) consider the following strategies as a candidate for equilibrium. In the odd periods firms are recommended to follow the same strategies as in the game with Δ, using the average price over the last two periods to decide on continuation play. In the even periods the recommendation is to ignore the current prices and keep the quantities fixed from last period.

Clearly, if firms follow these recommendations, they will obtain the same payoffs as in the game with $\Delta$. Unfortunately, these recommendations are not incentive compatible for several reasons. First, even if firm 2 follows that recommendation, firm 1 after seeing a high first period price has incentives to increase quantity and seeing a low price to decrease it. That 'option value' will also make firm 1 increase its quantity in the first period. Second, by that reasoning firm 1 will expect firm 2 to react in the second period to the observed first-period price. That makes firm 1 increase its first-period quantity even more: it gets all the benefit, while the cost of reducing quantities in the second period if prices turn out to be low will be shared between the two firms.[9]

## 3.1. Formal Argument

First, we prove that as $\Delta \to 0$, the probability of type I error under a tail test to prevent any deviation becomes disproportionately high.

> **LEMMA 2.** Fix $C_1 > 0$ and $\mu - \mu' > 0$. If $\Delta > 0$ is sufficiently small, then a tail test of a deviation with a likelihood difference of $C_1\Delta$ has a probability of type I error greater than $O(\Delta^{1/2+\varepsilon})$ for any $\varepsilon > 0$.

**PROOF:** See Appendix.

Next, we want to show that a tail test is an optimal way to deter a given deviation. Consider a deviation with instantaneous gain $D\Delta$, which reduces the mean of observed prices from $\mu$ to $\mu'$. The optimal test maximizes the expected continuation payoff subject to providing incentives against this deviation. Lemma **3** shows that if we are worried about only one deviation, a tail test with a bang-bang property is best.

> **LEMMA 3.** Suppose $D > 0$. Consider the problem

$$\max_{v(x)} \int_{-\infty}^{\infty} v(x)\, g(x)\, dx$$

$$s.t$$

$$D\Delta \leq \delta \int_{-\infty}^{\infty} v(x)\big(g(x) - g'(x)\big)\, dx \quad \text{and} \quad \forall x \in \Re \;\; v(x) \in [\underline{v}, \overline{v}]$$

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[9] The second of these two effects is the main reason cooperation is impossible for small $\Delta$, but clearly the two effects are related.

where $g$ and $g'$ are the densities of normal distributions with variance $\sigma^2/\Delta$ and means $\mu$ and $\mu' < \mu$ respectively.

If this problem has a solution, then it takes the form

$$v(x) = \begin{cases} \overline{v} & \text{if } x > c \\ \underline{v} & \text{if } x \leq c \end{cases} \tag{1}$$

for some $c \in (-\infty, (\mu + \mu')/2]$.

**PROOF:** See Appendix.

With the help of Lemmas **2** and **3** we are ready to formulate our main result. Proposition I shows that as $\Delta \to 0$, the maximal payoff achievable in symmetric PPE converges to static Nash.

**PROPOSITION I.** We have $\overline{v}(\Delta) \to v_N$ as $\Delta \to 0$.

Some of the intuition behind the result has been first presented in AMP: in games with imperfect monitoring more frequent moves lead to weaker statistical tests and hence can make collusion more difficult to sustain. The proof shows that in our setup this deterioration is so extreme, that no collusion is possible at all as $\Delta \to 0$. We present the proof in the main body of the text, as it illustrates well the methods we use (and we believe these methods are applicable to other models). For methodological reasons we divided the proof into very simple steps. That makes it longer, but hopefully easier to read.

**PROOF:** We can characterize the best symmetric equilibrium (using the methods from APS'86) as a solution to the following problem:

$$\begin{aligned} &\overline{v}(\Delta) = \max_{q, v(x)} \left(1 - e^{-r\Delta}\right) qP(2q) + e^{-r\Delta} \int_{-\infty}^{\infty} v(x) g(x) dx \\ &\text{s.t.} \\ &\forall q' \in [0, \overline{q}]: \quad \left(1 - e^{-r\Delta}\right)\left(q'P(q+q') - qP(2q)\right) \leq e^{-r\Delta} \int_{-\infty}^{\infty} v(x)\left(g(x) - g_{q'}(x)\right) dx \\ &v(x) \in \left[\underline{v}(\Delta), \overline{v}(\Delta)\right] \end{aligned} \tag{2}$$

where $(q,q)$ is the quantity pair chosen in the current period, $g$ denotes the density of prices for total supply $2q$, $g_{q'}$ denotes the density for total supply $q+q'$, and $v(x)$

denotes the continuation payoff when the realized price is $x$. The first constraint is a standard one-period incentive compatibility constraint and the second constraint states continuation payoffs must be achievable in equilibrium.

Let us relax the problem. First, consider only one incentive constraint against a deviation to a static best response. Second, allow a wider range of continuation payoffs. Third, weaken slightly the incentive constraint by noting that $r\Delta < \left(1-e^{-r\Delta}\right)/e^{-r\Delta}$. In this way we obtain problem:

$$\begin{aligned}
&\overline{v}_R(\Delta) = \max_{q,v(x)} \left(1-e^{-r\Delta}\right) qP(2q) + e^{-r\Delta}\int_{-\infty}^{\infty} v(x)\, g(x)\, dx \\
&\text{s.t.} \\
&q' = q^*(q) \;:\quad r\Delta\left(q'P(q+q') - qP(2q)\right) \le \int_{-\infty}^{\infty} v(x)\left(g(x)-g'(x)\right) dx \\
&v(x) \in \left[0, \overline{v}_R(\Delta)\right]
\end{aligned} \tag{3}$$

where $g'$ denotes the density of prices for total supply $q+q^*(q)$. Clearly, $\overline{v}_R(\Delta) \ge \overline{v}(\Delta) \ge v_N$ so it will be sufficient to show that $\overline{v}_R(\Delta) \to v_N$ as $\Delta \to 0$. Consider any $\varepsilon > 0$. Let us show that if $\Delta$ is sufficiently small, then no supply rates $q \in \left[0, q_N - \varepsilon\right]$ can achieve $\overline{v}_R(\Delta)$. We will follow the reasoning presented in Figure 4.

Suppose the solution to (3) is $q \in \left[0, q_N - \varepsilon\right]$. Let $\varepsilon_\pi > 0$ be a lower bound on static profit from deviation $q^*(q)$ for $q \in \left[0, q_N - \varepsilon\right]$, and $M$ be an upper bound on drop in prices from such deviations (see Lemma **4** in the Appendix). Denote the static gain from $q^*(q)$ by

$$D = P\left(q^*(q)+q\right)q^*(q) - P(2q)\,q \ge \varepsilon_\pi .$$

This deviation makes the expected price drop by $P(2q) - P\left(q^*(q)+q\right) \le M$ .

Keeping $q$ that solves (3) fixed, $v(x)$ solves:

$$\begin{aligned}&\max_{v(x)} \int_{-\infty}^{\infty} v(x)\, g(x)\, dx \\ &\text{s.t.} \\ &q' = q^*(q) \;\; : \qquad r\Delta D \le \int_{-\infty}^{\infty} v(x)\big(g(x) - g'(x)\big)\, dx \\ &v(x) \in \big[0, \overline{v}_R(\Delta)\big]\end{aligned} \tag{4}$$

By Lemma **3**, $v(x)$ takes the form:

$$v(x) = \begin{cases} \overline{v}_R(\Delta) & \text{if } x > c \\ 0 & \text{if } x \le c \end{cases} \tag{5}$$

This form of $v$ is associated with a tail test with critical region (-∞, $c$]. The likelihood difference in this test is

$$\text{likelihood difference } = \int_{-\infty}^{c} (g'(x) - g(x))dx$$

Denote by $v_M \ge \overline{v}_R(\Delta)$ half the monopoly profit. From the incentive compatibility constraint in (4), we have

$$r\Delta D \le \int_{c}^{\infty} \overline{v}_R(\Delta)\big(g(x) - g'(x)\big)dx \le v_M \int_{c}^{\infty} \big(g(x) - g'(x)\big)dx = v_M \underbrace{\int_{-\infty}^{c} \big(g'(x) - g(x)\big)dx}_{\text{likelihood difference}} \tag{6}$$

That implies:

$$\text{likelihood difference } \ge r\Delta D / v_M \ge r\Delta \varepsilon_\pi / v_M \tag{7}$$

so it is of order O(Δ).

Referring to Lemma **2**, let $\Delta^* > 0$ be such that for all $\Delta \le \Delta^*$, a tail test for a difference in mean of $M$ with likelihood difference $\varepsilon_\pi \, \Delta r / v_M$ has a probability of type I error greater than $C\Delta^\alpha$ for some $C > 0$ and $\alpha < 1$. Then any test for a difference in mean of less than $M$ with the same likelihood difference has an even greater probability of type I error.

Consider $\Delta < \Delta^*$. The probability of making a type I error is

$$\int_{-\infty}^{c} g(x)\, dx > C\Delta^{\alpha} \tag{8}$$

This implies:

$$\int_{-\infty}^{\infty} v(x)\,g(x)\,dx = \int_{c}^{\infty} \overline{v}_R(\Delta)\,g(x)\,dx < \overline{v}_R(\Delta)\left(1 - C\Delta^{\alpha}\right) \tag{9}$$

Finally, returning to (3):

$$\underbrace{\left(1-e^{-r\Delta}\right)P(2q)\,q}_{O(\Delta)} + \underbrace{e^{-r\Delta}}_{1-O(\Delta)} \underbrace{\int_{-\infty}^{\infty} v(x)\,g(x)\,dx}_{<\overline{v}_R(\Delta)-O(\Delta^{\alpha})} < \overline{v}_R(\Delta) \tag{10}$$

for sufficiently small Δ. This leads to a contradiction for any $\alpha < 1$, and from Lemma **2** we know that we can pick any $\alpha > 1/2$. Therefore if $q$ is the current-period supply rate that achieves value $\overline{v}_R(\Delta)$, then $q > q_N - \varepsilon$. But then

$$\overline{v}_R(\Delta) \le \left(1-e^{-r\Delta}\right)P(2q)\,q + e^{-r\Delta}\overline{v}_R(\Delta) \quad \Rightarrow \quad \overline{v}_R(\Delta) \le P(2q)\,q$$

Note that $P(2q)q$ is continuous and (by **A1**) decreasing for $q$ greater than a half of the monopoly quantity. Letting $\varepsilon \to 0$, we conclude that $\overline{v}_R(\Delta) \to v_N$ as $\Delta \to 0$. ◆

# 4. Asymmetric Equilibria and Monetary Transfers

As we discussed in the Introduction, in many games asymmetric equilibria achieve much higher payoffs than symmetric ones. The reason is that in symmetric equilibria incentives are provided via price wars, hence they require destruction of value. In asymmetric equilibria incentives can often be provided by transferring payoffs between the agents, keeping the sum of profits high. Using asymmetric strategies, can some collusion be sustained in our setting? We claim that not.

To see intuition behind this claim, let us informally attempt to construct a natural asymmetric equilibrium and show why this attempt fails. Consider a hypothetical collusive scheme in which one of the players always produces the monopoly quantity while the other produces nothing, and they change roles over time. One may hope that this collusive scheme can be made successful, because only one player who produces nothing is tempted to deviate in any period. We can provide incentives against this deviation by letting the opponent, who is in the role of a monopolist, stay in that role longer if prices become too low. However, this arrangement fails because it interferes with the incentives of the monopolist: he becomes tempted to overproduce to stay in the role of the monopolist longer. Therefore, the scheme fails to keep the total supply low. The reason for such failure, which is formalized in the proof of Proposition II, is that individual incentives created by transfers of future payoffs interfere with each other and cause the players to jointly produce the Cournot Nash sum of quantities.

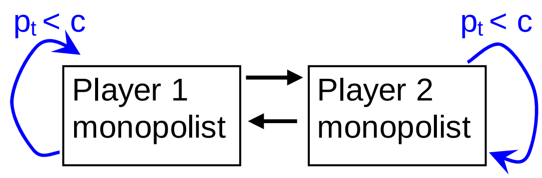


**Figure 5**

The proof that collusion is impossible in asymmetric equilibria depends crucially on the assumption that deviations of different players cannot be statistically distinguished. The intuition is as follows. Consider a pair of quantities ($q_1$, $q_2$) with $Q = q_1+q_2 \leq 2q_N$ and an upward deviation by $\varepsilon$ separately by each player. Both such deviations change the density of prices from $g$ to $g'$ and therefore are observationally equivalent. The incentive constraints against these deviations are

$$\begin{aligned} r\Delta\big((q_1+\varepsilon)P(Q+\varepsilon)-q_1P(Q)\big) &\leq \int_{-\infty}^{\infty} v_1(x)\big(g(x)-g'(x)\big)dx \\ r\Delta\big((q_2+\varepsilon)P(Q+\varepsilon)-q_2P(Q)\big) &\leq \int_{-\infty}^{\infty} v_2(x)\big(g(x)-g'(x)\big)dx \end{aligned}, \qquad (11)$$

where $v_1(x)$ and $v_2(x)$ are next-period continuation values as a function of price $x$. Adding up the two constraints and dividing by 2, we get a *joint constraint*

$$r\Delta\left(\left(\frac{Q}{2}+\varepsilon\right)P(Q+\varepsilon)-\frac{Q}{2}P(Q)\right) \leq \int_{-\infty}^{\infty} \frac{v_1(x)+v_2(x)}{2}\big(g(x)-g'(x)\big)dx \qquad (12)$$

This joint constraint is identical to an incentive constraint in a symmetric equilibrium with quantity pair $(Q/2, Q/2)$, against an upward deviation by ε. From the previous section, we know that the provision of such incentives destroys too much value, which is measured by the sum of the players' payoffs here. Therefore, collusion is impossible as $\Delta \to 0$ when deviations of different players are statistically indistinguishable.

Proposition II formalizes the preceding argument. Let $\overline{v}_a(\Delta)$ be the highest sum of payoffs that the players can achieve in equilibrium.

**PROPOSITION II.** We have $\overline{v}_a(\Delta) \to 2v_N$ as $\Delta \to 0$.

**PROOF:** Denote by $E_a(\Delta)$ the set of payoff pairs achievable in asymmetric PPE of game $\Gamma^\Delta$. Denoting $Q = q_1 + q_2$, $\overline{v}_a(\Delta)$ solves

$$\overline{v}_a(\Delta) = \max_{q_1, q_2, v_1(x), v_2(x)} \left(1 - e^{-r\Delta}\right) QP(Q) + e^{-r\Delta} \int_{-\infty}^{\infty} \left(v_1(x) + v_2(x)\right) g(x)\, dx \tag{13}$$

subject to the players' incentive constraints, and the constraint that future continuation values $(v_1(x), v_2(x))$ must be in the set $E_a(\Delta)$, i.e. $(v_1(x), v_2(x)) \in E_a(\Delta)$. We relax the problem in several steps. First, as in the proof of Proposition I, we relax the problem by keeping only the incentive constraints against upward deviations by $\varepsilon$ by each firm, where $\varepsilon$ will be specified later. Each deviation has the same effect on the distribution of prices, decreasing the mean of the observed price to $P(Q + \varepsilon)$. The incentive constraints against these deviations are given above by (11). We further relax the problem by replacing constraints (11) with their sum (see (12)) and by replacing the constraint $(v_1(x), v_2(x)) \in E_a(\Delta)$ with a weaker constraint $v_1(x) + v_2(x) \in [0, \overline{v}_a(\Delta)]$. As a result, we arrive at the problem

$$\begin{aligned}
&\overline{v}_{a,R}(\Delta) = \max_{q_1, q_2, v_1(x), v_2(x)} \left(1 - e^{-r\Delta}\right) QP(Q) + e^{-r\Delta} \int_{-\infty}^{\infty} (v_1(x) + v_2(x)) g(x)\, dx \\
&\text{s.t.} \\
&r\Delta\left(\left(\frac{Q}{2} + \varepsilon\right) P(Q + \varepsilon) - \frac{Q}{2} P(Q)\right) \le \int_{-\infty}^{\infty} \frac{v_1(x) + v_2(x)}{2} \left(g(x) - g'(x)\right) dx \\
&v_1(x) + v_2(x) \in \left[0, \overline{v}_{a,R}(\Delta)\right]
\end{aligned} \tag{14}$$

Clearly, $\overline{v}_{a,R}(\Delta) \geq \overline{v}_a(\Delta) \geq 2v_N$ . Problem (14) is the same as problem (3) multiplied by 2 (here $v_1(x)+v_2(x)$ plays the same role as $2v(x)$ in problem (3)) so $\overline{v}_{a,R}(\Delta) = 2\overline{v}_R(\Delta)$. It was shown in the proof of Proposition I that $\overline{v}_R(\Delta) \to v_N$ as $\Delta \to 0$. Therefore, $\overline{v}_{a,R}(\Delta) \to 2v_N$ and $\overline{v}_a(\Delta) \to 2v_N$. ◆

**Monetary transfers**

So far we have assumed that the players cannot use any monetary transfers. This is a realistic assumption for some cartels (especially tacit ones). However, if we apply our analysis to other environments (for example, partnerships) in which transfers are legal and/or often used, we should relax that assumption. It turns out that the proof of Proposition II easily extends to show that collusion becomes impossible as $\Delta \to 0$ even if firms can use monetary transfers.

To allow for the transfers, we assume that the players can write binding contracts that specify a transfer of money conditional on the observed realization of prices. Such a contract specifies an amount of money $t_i(x)$ that each firm is supposed to pay (or receive if $t_i$ is negative) depending on the price realization. The case of *balanced transfers*, such that $t_1(x)+t_2(x)=0$, is particularly simple. The incentive constraints against an upward deviation by $\varepsilon$ separately by each player are now

$$r\Delta\big((q_1+\varepsilon)P(Q+\varepsilon)-q_1P(Q)\big) \leq \int_{-\infty}^{\infty}\big(v_1(x)-t_1(x)\big)\big(g(x)-g'(x)\big)dx$$
$$r\Delta\big((q_2+\varepsilon)P(Q+\varepsilon)-q_2P(Q)\big) \leq \int_{-\infty}^{\infty}\big(v_1(x)-t_2(x)\big)\big(g(x)-g'(x)\big)dx$$

Adding these two constraints and dividing by 2, we obtain the same joint constraint as (12):

$$r\Delta\left(\left(\frac{Q}{2}+\varepsilon\right)P(Q+\varepsilon)-\frac{Q}{2}P(Q)\right) \leq \int_{-\infty}^{\infty}\frac{v_1(x)+v_2(x)}{2}\big(g(x)-g'(x)\big)dx$$

From this point, the proof of Proposition II applies to balanced transfers without any change. The intuition behind this result is the same: the provision of joint incentives destroys too much value, as measured by the sum of the players' payoffs.

If we allow *unbalanced transfers* ($t_1(x)$, $t_2(x)$), we assume that they satisfy $t_1(x)+t_2(x)=b(x)\geq 0$, where $b(x)$ is the amount of money burning (e.g. payments made to a third party). Unbalanced transfers can improve somewhat the scope of collusion for a fixed Δ, as they allow for punishments stronger than the previous minmax payoff. However, our limit results still hold as long as the amount of money burning per period stays bounded. The reason is that as long as punishment is bounded, the likelihood difference for a statistical test to prevent a particular deviation must be still on the order of Δ. Such tests trigger punishment with a disproportionately high probability on the equilibrium path.

We summarize our findings in the following corollary.

> **Corollary.** We have $\overline{v}_a(\Delta)\rightarrow 2v_N$ as Δ→0 even if monetary transfers are allowed, as long as the amount of money burning is bounded.

**PROOF:**. As in the proof of Proposition II, we find that the sum of the best collusive profits is bounded above by a solution to the maximization problem

$$\overline{v}_{a,R'}(\Delta)=\max_{q_1,q_2,v_1(x),v_2(x)}\left(1-e^{-r\Delta}\right)QP(Q)+e^{-r\Delta}\int_{-\infty}^{\infty}(v_1(x)+v_2(x)-b(x))g(x)\,dx$$

s.t.

$$r\Delta\left(\left(\frac{Q}{2}+\varepsilon\right)P(Q+\varepsilon)-\frac{Q}{2}P(Q)\right)\leq\int_{-\infty}^{\infty}\frac{v_1(x)+v_2(x)-b(x)}{2}\left(g(x)-g'(x)\right)dx$$

$$v_1(x)+v_2(x)-b(x)\in\left[-\overline{b},\overline{v}_{a,R'}(\Delta)\right]$$

where $\overline{b}$ is an upper bound on money burning per period. As in the proof of Proposition I a solution to this problem is based on a tail test with a likelihood difference on the order of Δ. We know that such tests destroy too much value as Δ→0. ◆

We conclude this section with two observations related to unbounded money burning.

**REMARK.** Suppose that the players can sign binding contracts that specify a transfer $t_i(x)$ from player $i$ to a third party when the realization of the signal is $x$, with $t_1(x)+t_2(x)\geq 0$. Then:

i) If there is no bound on the transfers then for any $\Delta > 0$ and $\varepsilon > 0$ there exists equilibrium in which $\bar{v}_a(\Delta) \geq v_M - \varepsilon$, where $v_M$ is the monopoly profit.

ii) If the amount of money burning is bounded by the current wealth and the players start the game with finite wealth, then $\bar{v}_a(\Delta) \to 2v_N$ as $\Delta \to 0$ where $\bar{v}_a(\Delta)$ is calculated at the beginning of the game.

**PROOF:** See Appendix.

# 5. A Numerical Example

We now provide a numerical example that illustrates the effects of the changes in Δ on $\bar{v}(\Delta)$. The inverse demand is $P(Q) = 1 - Q$. It corresponds to a monopoly supply $Q = ½$ and expected price $P(Q) = ½$. In the static Nash (Cournot) equilibrium quantities are $q_N = 1/3$, average price is $P(2q_N) = 1/3$ and payoffs are $v_N = 1/9$.

We take $r$ = 10% to be the annual interest year, so we interpret Δ=1 as the firms making supply rate decisions every year. We take $\sigma$ = 0.03, so that the daily standard deviation of average price is $\sigma/\sqrt{\Delta} = 0.03/\sqrt{1/365} \approx 0.57$, which is on the order of average daily monopoly price.

We did not compute the actual best collusive payoff, because it depends on whether we consider symmetric or asymmetric equilibria and various details of the model (e.g. the production capacity $\bar{q}$, which is important for the worst equilibria). Instead, we computed a robust lower bound by finding the best symmetric equilibrium with Nash reversion as a punishment, and a robust upper bound using punishment by the minmax payoff of 0. Both bounds are valid for both symmetric and asymmetric equilibria[10], and collusion enforced by balanced monetary transfers. Figure 6 shows the two bounds. The horizontal axis is $\log_{10}(\Delta)$. The value of 0 corresponds to a move once a year, the value of -2, 100 times a year.

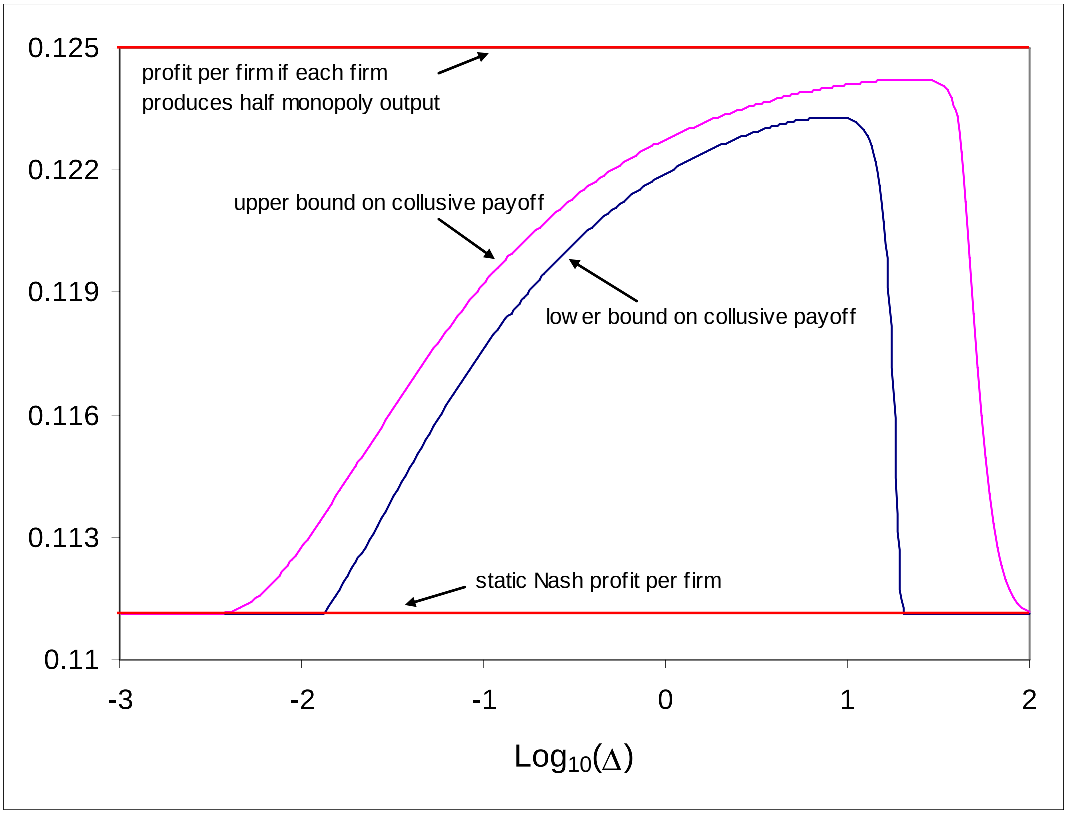


**Figure 6**

The behavior on the right hand side of the graph is standard: as the moves get less frequent the benefits from one-period deviation are large compared to continuation payoffs and collusion is difficult to sustain. The effect we focus on is the non-monotonicity of $\overline{v}(\Delta)$ in Δ: as we move to frequencies higher than once a year (we move to the left on the graph), $\overline{v}(\Delta)$ decreases. In fact if the players move 100 times a year then $\overline{v}(\Delta) \leq 0.113$ , at most a 1.5% improvement over the static Nash payoffs, while half monopoly profit would represent a 12.5% improvement.

## 6. Convex Costs and Capacity Constraints.

In this section we extend our results to the case of increasing marginal costs. Recall that in our basic model, we assumed constant marginal costs and normalized them to zero.

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[10] According to the proof of Proposition II, asymmetric equilibria cannot improve upon symmetric equilibria if punishment 0 is allowed.

Suppose that the costs of production accrue at rate $c(q)$ when the output flow is $q$, so that profits per period are

$$\Delta\left(q_{it}(P(Q_t)+\varepsilon_t)-c(q_{it})\right)$$

Assume that costs are increasing and convex:

- **A3**: $c(0)=0,\ c'(q)\geq 0,\ c''(q)\geq 0$

We show that under A3 still no collusion can be sustained with symmetric equilibria. The scope of collusion in asymmetric equilibria depends on the convexity of marginal costs (i.e. on the third derivative of $c$). If they are not too convex, then even asymmetric equilibria with monetary transfers do not improve upon static Nash payoffs. Otherwise, as we show in the extreme case of capacity constraints, collusion may be sustainable with transfers. The analysis of this section is brief as it directly applies the methodology of Sections 3 and 4.

## 6.1. Symmetric Equilibria.

Let us summarize the argument why collusion is impossible in a symmetric equilibrium as $\Delta\to 0$. To improve upon static Nash equilibrium payoffs, firms must use a pair of quantities $(q, q)$ less than Nash. At any such pair, both firms can improve static payoffs by an upward deviation. A statistical test to prevent such deviations must have likelihood difference on the order $O(\Delta)$. As $\Delta\to 0$, such a test gives false positives with a disproportionately high probability, $O(\Delta^{1/2+\varepsilon})$, and destroys all collusion benefits.

We now argue that this argument still applies with increasing marginal costs. The symmetric static Nash equilibrium is unique. It is given by a first-order condition

$$P'(2q_N)q_N+P(q_N)-c'(q_N)=0, \tag{15}$$

and note that by A1 and A3 the LHS of (15) is decreasing. When the firms produce $(q, q)$ the sum of profits is given by

$$\Pi(2q)=P(2q)q-2c(q)$$

By A1 and A3 this is a strictly concave function. As $\Pi'(2q_N)<0$, firms must choose quantities less than $q_N$ to improve upon static Nash equilibrium profits. Because the derivative of each firm's static profit with respect its quantity $P'(2q)q+P(q)-c'(q)$ is positive when $q < q_N$, each firm is tempted to overproduce. Moreover, the gain $D$ from the most profitable static deviation is uniformly bounded away from 0 by some number $\varepsilon_\pi > 0$ for all $q < q_N - \varepsilon$. Now the argument in the proof of Proposition I applies directly. Quantities $q < q_N - \varepsilon$ cannot be used for collusion due to false positives as $\Delta \to 0$. Letting $\varepsilon \to 0$ shows that collusion becomes impossible. Therefore even with convex costs symmetric equilibria cannot improve upon static Nash payoffs:

**PROPOSITION III** Suppose total costs satisfy A3. We have $\overline{v}(\Delta) \to v_N$ as $\Delta \to 0$.

## 6.2. Asymmetric Equilibria with Monetary Transfers.

Can asymmetric equilibria help collusion? With increasing marginal costs asymmetric quantities are inefficient, and may help collusion only if asymmetries significantly facilitate the provision of incentives. In most cases, the provision of asymmetric incentives is just as difficult as it is with constant marginal costs, and the provision of joint incentives destroys too much joint value. In a limited set of cases, e.g. with capacity constraints of subsection 6.3, collusion may be possible.

Let us go through the details of our argument. First, note that if firms move from a symmetric quantity pair ($Q/2$, $Q/2$) to an asymmetric pair ($q_1$, $q_2$) with $Q = q_1+q_2$, their total profits are hurt due to convex total costs (A3), i.e

$$\Pi(q_1,q_2) = QP(Q) - c(q_1) - c(q_2) \leq P(Q) - 2c(Q/2)$$

Therefore, as in the case of symmetric quantity pairs, the sum of profits from ($q_1$, $q_2$) can be higher than twice the static Nash payoffs only if $Q = q_1+q_2 < 2q_N$. Following our reasoning in Proposition II, we will consider the sum of IC constraints of the firms when each considers increasing its quantity by some $\varepsilon$. The sum of derivatives of single firms' profits at ($q_1$,$q_2$) is:

$$\frac{\partial \pi_1(q_1,q_2)}{\partial q_1} + \frac{\partial \pi_2(q_1,q_2)}{\partial q_2} = P'(Q)Q + 2P(Q) - c'(q_1) - c'(q_2) \tag{16}$$

If for any ($q_1$, $q_2$) such that $\Pi(q_1,q_2)-\Pi(q_N,q_N)>0$ this sum is strictly positive, then asymmetric quantities do not facilitate incentive provision. The intuition is that because signals depend only on aggregate quantity, balanced transfers (of continuation values or money) between the firms can provide incentives for one of the firms only at the cost of interfering with incentives of the other firm. Indeed, adding up two IC constraints for an upward deviation of each firm by ε, we obtain a *joint constraint* analogous to (12), where balanced transfers cancel out. As a result, like in the proof of Proposition II, providing incentives jointly requires a disproportionately large destruction of value. Therefore if the value of (16) is positive whenever $\Pi(q_1,q_2)-\Pi(q_N,q_N)>0$, collusion is impossible as $\Delta\to 0$.

A sufficient condition for this to be the case is:

- **A4**: $c'''(q)\le 0$ i.e. marginal costs are weakly concave.

A4 guarantees that moving to asymmetric equilibria the loss in total profits is faster than the gain in incentives provision. If A4 holds, then keeping the total quantity fixed at $Q = q_1+q_2$, the sum of derivatives at asymmetric quantities is higher than the sum at symmetric quantities:

$$P'(Q)Q+2P(Q)-c'(q_1)-c'(q_2)\ge P'(Q)Q+2P(Q)-2c'(Q/2)$$

As we argued in the discussion of symmetric equilibria, the RHS is strictly positive for all $Q < 2q_N$. We summarize our conclusion in a proposition.

**PROPOSITION IV** Suppose total costs are convex and satisfy A3 and A4. Then $\overline{v}_a(\Delta)\to 2v_N$ as $\Delta\to 0$.

## 6.3. Capacity Constraints.

We finish by showing that if A4 is violated then sometimes collusion is possible with asymmetric equilibria that use balanced monetary transfers. We illustrate this fact in an extreme case of capacity constraints. In this case the concavity assumption A4 is violated because marginal costs jump to infinity at capacity. We show that although the result for symmetric equilibria is very robust, for asymmetric equilibria it is somewhat delicate. Technically, once one of the firms is at capacity, the sum in equation (16) is minus infinity and therefore it is possible to provide incentives via transfers to one of the firms without disturbing incentives for the firm producing at capacity.

For example, suppose that both firms have zero marginal costs and capacity constraints at the monopoly level of production. Then it is still not possible to sustain profitable collusion with symmetric equilibria. In contrast, asymmetric equilibria with monetary transfers solve the monitoring problem. The way to build such equilibria is to allow the firms alternate between being a monopolist and producing nothing. When a firm produces nothing, its incentives come from punishments when the price is low and rewards when the price is high. Unlike in Section 4, these incentives can be provided only with monetary transfers between the firms, without money burning. Because the firm that acts as a monopolist is at capacity, it cannot overproduce to receive transfers by driving prices down. By this mechanism, capacity constraints help us resolve the problem of separating deviators from non-deviators.

To see in greater detail how a collusive equilibrium may look like, consider the following strategies. In odd periods firm 1 is recommended to produce at capacity ($q_M$) and firm 2 to produce nothing. In even periods the roles are reversed. After every period the firm that is supposed to produce nothing receives a transfer of $\Delta q_M(P(q_M)+\varepsilon_t)$, i.e. the realized profit (assuming no deviation). In such a scheme the current monopolist clearly has no incentives to deviate – lower quantity means lower profits and higher transfers. The other firm, which maximizes $\Delta\left(q_M+q_{it}\right)(P(q_M+q_{it})+\varepsilon_t)$, has no incentives to deviate either. Therefore the proposed strategies form an equilibrium (assuming the transfers can be enforced). That scheme achieves first-best.

This scheme requires the transfer to be very large after some extreme realizations and hence if transfers cannot be contractually enforced, players will have incentives to deviate. This can be resolved by truncating the realized price at 10 standard deviations away from the mean and slightly increasing the transfer within this range. This preserves incentives and keeps transfers on the order $O(\sqrt{\Delta})$. Note that firms do not need to sign a binding contract to enforce such transfers: they can be enforced by a threat of Nash reversion. For small enough $\Delta$ this threat is credible and prevents deviations of players refusing to pay the transfer: since the equilibrium achieves first best, the expected continuation payoffs are almost half the monopoly profits forever. Not paying the transfer saves much less than the benefit of continued cooperation.

**Remark**

This setup has three ingredients that allowed us to sustain collusion: monetary transfers, asymmetric equilibria and capacity constraints. We know that without any of the last two, collusion is not possible for small $\Delta$. It can also be shown that the monetary transfers are necessary as well, i.e. using only transfers of continuation payoffs (say, promises to be a monopolist longer in the future) is not sufficient to provide incentives (for small $\Delta$). This may be surprising, as in repeated games when players are sufficiently patient, often transfers of continuation payoffs are asymptotically as good as monetary transfers.

The intuition behind the result is as follows. First, in order to sustain the highest sum of payoffs, in the current period one of the players must be at capacity and the other player must choose strictly less than his static best response. In order to provide incentives for this player not to overproduce, the transfers have to be at least on the order of $O(\sqrt{\Delta})$. The second part of reasoning is about the largest difference in continuation payoffs that can be achieved in equilibrium for small $\Delta$. Using arguments similar to the ones we used so far, one can show that this upper bound on the continuation payoff transfer decreases to zero much faster than at rate $O(\sqrt{\Delta})$.[11] Putting it differently, the whole set of continuation payoffs rapidly collapses towards the 45 degree line as $\Delta$ gets small. Combining the two steps, for very flexible production the range of feasible transfers via continuation payoffs alone is too small to provide incentives to reduce quantity below the static best response.

[11] We subtract the local incentive compatibility constraints to arrive at a relaxed problem. This relaxed problem has a solution given by a cut-off rule. Then, bounding the type-one errors, we can prove that the maximal difference in equilibrium continuation payoffs drops to 0 at a rate much faster than $O(\sqrt{\Delta})$.

# 7. Correlated Demand

We have presented a simplest duopoly model with homogenous goods to illustrate the idea that collusion becomes impossible if firms can act frequently. This standard model with i.i.d. prices was motivated by Green and Porter (1984). As firms act frequently, a few unrealistic features arise in this model if we keep it simple. For example, prices are unbounded (positive and negative) and the variance of per-period average prices grows to infinity as we take $\Delta \to 0$. Here we illustrate how the intuition that we developed extends to a more complicated and realistic model.

In particular, assume that prices are a mean-reverting process with mean that depends on the total supply:

$$p_{t+\Delta} = p_t + \Delta f(Q, p_t) + \varepsilon_{t+\Delta},$$

where $\varepsilon_{t+\Delta} \sim N(0, \Delta\sigma^2(p_t))$.[12] This price process is different from the previous model, in particular, it is non-stationary and the conditional variance of per-period prices decreases to zero as $\Delta \to 0$. However, the impact on prices of a one-period deviation is on the order of $\Delta$ and the standard deviation of noise is on the order of $\sqrt{\Delta}$. As the ratio of these two orders is the same as in the stationary model, similar issues arise: it is more difficult to detect deviations over smaller periods of time. Assume that $f_1 < 0,\ f_{11} < 0,\ f_2 < 0$ and that $f(Q,\cdot)$ has root $P(Q)$, the mean of prices when the total supply is $Q$.[13]

The timing is like before: at periods $t \in \{0, \Delta, 2\Delta ...\}$ firms observe the current price $p_t$ and decide on supply rates to be held constant for $\Delta$ units of time. The first price $p_0$ is given exogenously. Per-period profits are $(1-\delta) q_t p_{t+\Delta}$.[14]

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[12] To guarantee that prices are bounded, we could truncate the distribution of $\varepsilon_t$ in far tails. This would not affect results.

[13] For example, we can have that $p_{t+\Delta} = (1-\alpha\Delta) p_t + \alpha\Delta\phi(Q) + \varepsilon_{t+\Delta}$ which yields $f(Q, p_t) \equiv \alpha(\phi(Q) - p_t)$.

[14] We can model correlated demand in many other ways. For example, we can imagine that the noise is correlated over time. Such a model is much less tractable, because a deviating firm would have private

To understand the potential for collusion in this model, denote by $F_M(p_t)$ the monopoly profit. A monopolist's Bellman equation when the current price is $p_t$ is

$$F_M(p_t) = \max_Q E\left[(1-\delta)Qp_{t+\Delta} + \delta F_M(p_{t+\Delta})\right],$$

and the first-order condition is $E\left[(1-\delta)p_{t+\Delta} + \Delta f(Q, p_t)((1-\delta)Q + \delta F_M^{'}(p_{t+\Delta}))\right] = 0.$

Monopoly profit cannot be achieved in an equilibrium with two firms. Indeed, if a firm anticipated half the monopoly profit as its continuation value then its Bellman equation would be

$$F_M(p_t) = \max_q E\left[(1-\delta)qp_{t+\Delta} + \delta \frac{1}{2} F_M(p_{t+\Delta})\right].$$

Taking the first-order condition at half the monopoly quantity, we get

$$E\left[(1-\delta)p_{t+\Delta} + \Delta \underbrace{f_1(Q, p_t)}_{<0}((1-\delta)Q + \frac{1}{2}\delta \underbrace{F_M{}'(p_{t+\Delta}))}_{>0}\right] > 0.$$

Therefore, each firm would be tempted to overproduce.

We argue that collusion is impossible in this setting as $\Delta \to 0$. First, we will discuss Markov Perfect Equilibria (MPE), which are analogous to Cournot competition in the simple model. Second, we will prove that it is impossible to achieve payoffs greater than those of an MPE.

Markov Perfect Equilibria

A (symmetric) MPE is a sequential equilibrium in which current supply rates $(\tilde{q}(p_t), \tilde{q}(p_t))$ depend only on current price. Denote by $F(p_t)$ the expected discounted payoff of each firm in a MPE. We will assume that $F(p_t)$ is a smooth function. Then $F(p_t)$ satisfies the following Bellman equation

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information about future distribution of prices. That introduces the problem of private monitoring, making analysis very difficult.

$$F(p_t) = E\left[(1-\delta)qp_{t+\Delta} + \delta F(p_{t+\Delta})\right] =$$

$$(1-\delta)qp_t + \delta\left( F(p_t) + \Delta f(2q, p_t)F'(p_t) + \frac{\Delta\sigma^2(p_t)}{2} F''(p_t) \right) + O(\Delta^2),$$

where we used a Taylor expansion to generate the second line, $1-\delta = 1-e^{-r\Delta} = O(\Delta)$ and $q = \tilde{q}(p_t)$ satisfies first-order condition

$$(1-\delta)p_t + \delta\Delta f_1(2q, p_t)F'(p_t) + O(\Delta^2) = 0.$$

We are not concerned with existence, uniqueness or further characterization of MPE. Assuming that at least one MPE exists, let us show that firms cannot achieve better payoffs in any PPE than in the best MPE as $\Delta \to 0$.

Impossibility of Collusion

Denote by $\overline{F}(p_t)$ the expected discounted payoff from the best symmetric PPE. We will assume that like $F(p_t)$ (which now denotes the payoffs in the best MPE) it is smooth and that there is an interior $p_t$ at which the difference between these two functions is maximized.

**PROPOSITION V.** $\overline{F} \to F$ as $\Delta \to 0$.

For a sketch of a proof see the Appendix. The logic behind this proof is the same as in the simple model that we analyzed in detail. In order to enforce collusion, firms must execute a statistical test of a normally distributed random variable that sends them to punishment if there is evidence of cheating. The probability of punishment becomes disproportionately large as $\Delta \to 0$.

The unifying feature of the two models is that when $\Delta$ gets smaller, a deviation changes the mean of the signal by an amount that is $O(\sqrt{\Delta})$ relative to the standard deviation of this signal. The signal is the innovation in price in the nonstationary model presented in this section, and the price itself in the stationary model.

# 8. Discussion and Further Extensions

So far we have established the impossibility result and shown that the same reasoning can be applied when we allow for monetary transfers or in a model with serially correlated demand. There are many ways one may modify the benchmark model and many other applications in which the presented reasoning can be employed. We conclude the paper by informally discussing some of them.

A stationary model with nonnegative prices

In our basic model of Section 2 the average price over any interval of time $\Delta$ is normally distributed. This allows us to divide the timeline into arbitrarily fine grid at which the firms adjust quantities, and show that collusion is impossible in the limit. The basic model has an unattractive feature that prices can become unboundedly negative over short intervals of time. We can construct an alternative model with bounded prices as follows. Suppose that spot-market prices, observed every second, are given by

$$p_t = g(s_t),\ \ t = 0, 1, \ldots$$

where $g$ is an increasing nonnegative function, $s_t = f(Q) + \varepsilon_t$ is a signal with $\varepsilon_t \sim N(0,\sigma^2)$ and $f$ is a decreasing function. For example, if $g(s) = \exp(s)$, then prices would be lognormal. In this model, the expected inverse demand is
$P(Q) = E[g(f(Q) + \varepsilon_t)]$. Assume that $g$ and $f$ are such that $P(Q)$ satisfies A1 and A2. Such a model separates the monitoring variable ($s$) from the prices.

Assume $\sigma$ is sufficiently large. If firms have a production technology that allows them change output flows every $\Delta$ seconds, then the expected per-period profits are $\Delta q_t\, P(Q)$. After observing one-second prices over the time period, an optimal way to test for deviations is to invert $g$ and calculate average $s_t$ over that period. The average $s_t$ has mean $f(Q)$ and variance $\sigma^2/\Delta$. Applying the same reasoning as before, we can show that as $\Delta$ gets small, the highest average payoffs that can be sustained in equilibrium become very close to static Nash payoffs.

Frequency of market data

In the game we have studied there are in fact two frequencies that determine the scope of collusion: the frequency of moves ($1/\Delta$) and the frequency at which the players observe prices. In Section 2 we have assumed that these two frequencies are the same but remarked that nothing would change if firms observed prices at higher frequencies (in particular, the continuous price process). One could also imagine a market in which firms can move frequently, but the market data are collected by a third party and available only infrequently.

In general, what is important for the scope of collusion is the minimum of the two frequencies. The intuition is that if more prices are observed while quantities are fixed, then the sufficient statistic is the mean. On the other hand, if frequencies can be adjusted in between price observations, then given that there is no new information available to act on (and marginal revenue of residual demand is downward sloping making output smoothing optimal) such deviations would not be profitable.

Endogenous selection of $\Delta$

We have assumed that $\Delta$ is fixed exogenously. What would happen if it were chosen by the firms before any output game? Assume that the different choices of technology have the same costs (in the sense that lowering $\Delta$ is costless). It is easy to consider the following two scenarios.

First, suppose that choice of $\Delta$ by each firm is easily monitored. Then as a corollary to Propositions I and II in the payoff-maximizing equilibrium firms would choose an interior $\Delta$ - neither too small, nor too large.

Second, suppose that the firms can secretly choose their $\Delta$ and expect to play a collusive equilibrium afterwards. Then each will choose $\Delta$ as small as possible. The reason is that having a lower $\Delta$ gives the firm the option to react to the path of prices. As a result, endogenously firms will destroy any hope for collusion. The same reasoning applies if $\Delta$=0 but the firms can choose the frequency (or delay) at which they observe prices.

Additional Signals

Assume Δ is close to zero and prices are observed continuously. Suppose there exists another source of information $y_t = G(Q_t) + \eta_t$ observed at intervals $\Delta_y > 0$, where $\eta_t$ is normal noise independent of $\varepsilon_t$ with variance inversely proportional to $\Delta_y$.

If time interval between observations $\Delta_y$ is fixed independently of Δ, then profitable collusion would be achievable in equilibrium: the players should disregard prices and focus on *Y* to provide incentives to restrict supply. This situation could arise if $y_t$ is a signal from an industry survey released by an independent agency at regular time intervals.

However if $\Delta_y = O(\Delta)$ then as $\Delta \to 0$ still no collusion would be achievable. This could be the case if $y_t$ is the price of an input commonly used in production by the two competing firms.

Delay of Information

Finally, suppose $\Delta \cong 0$ and that the information arrives continuously but with a delay of length *d*. Like in AMP such delay allows for some information aggregation and hence can improve the scope of collusion. To see why collusion above Nash is possible, let us construct an equilibrium of this game based on the best equilibrium of a game without delay with $\Delta = d$ and $\delta = e^{-r2d}$. Divide time into time segments of length d. In the odd segments the players are recommended to play like in the best equilibrium with $\Delta = d$ and with $\delta = e^{-r2d}$. In the even segments they are recommended to play static Nash. At the end of even segments players calculate average price observed between time *t-d* and *t*, which contains information about supply rates at time interval (*t*-2*d*, *t*-*d*), and decide on continuation payoffs based on it.

Partnerships and other applications

Our analysis is applicable beyond the realm of collusion. For example, consider the following stylized partnership game. Two partners choose effort rates $(q_{1t}, q_{2t})$ that affect profits of the partnership. The total profits are given by

$$R_t = P(q_{1t} + q_{2t}) + \varepsilon_t$$

where $P'(\cdot) > 0$ and $\varepsilon_t$ is normal noise with variance inversely proportional to Δ. The payoffs of the players are

$$(1 - e^{-r\Delta}) \sum_{t=0,\Delta,2\Delta...} e^{-rt} \left( R_{it} - c(q_{it}) \right)$$

where $R_{it}$ is the share of total profit that player $i$ receives at time $t$ and $c(q)$ is a cost of effort (increasing and convex). Effort of player $i$ benefits both players, but only player $i$ pays its cost. This externality implies that the static Nash equilibrium does not yield joint profit maximization. The results presented in this paper imply that as the $\Delta \to 0$ the best the partnership can achieve is the static-Nash equilibrium payoffs.[15]

Mixed strategy perfect public equilibria

Throughout the paper we have restricted our attention to pure strategy PPE. As we know from Fudenberg, Levine and Maskin (1994), in some games even if all pure strategy profiles are not identifiable, mixed strategy profiles may be. If some mixed strategy profiles have pairwise full rank, i.e. they allow telling apart (statistically) deviations of different players and different deviations of the same player, then those profiles can be sustained with balanced monetary transfers. In our game, the existence of such profiles depends on whether the action space (the range of quantities the firms can choose) is bounded or not. If in the mixed strategy profile both firms mix over quantities that can be locally increased, then a deviation by firm 1 to increase quantity by a unit (for all realizations of the mixing) is observationally equivalent to the analogous deviation by firm 2. Therefore if there are no bounds on quantity flows then deviations from any mixed strategy profile are not identifiable and hence all our results still hold. However, if as we assumed in Section 2 the firms have a capacity constraint $\overline{q}$, then a mixed strategy profile in which the two firms produce at capacity with some small probabilities could be both profitable and identifiable. Hence it may be possible to sustain collusion with balanced monetary transfers.

[15] Of course, one needs to make assumptions on the $P$ and $c$ functions that are analogous to A1-4.

# 9. Conclusion

Let us review what is known about the relationship between the structure of the game and the equilibria as $\Delta \to 0$. First, when signals do not carry information about the identity of the player who deviated, collusion becomes more difficult or impossible when $\Delta \to 0$. AMP study repeated 2x2 partnership games where information arrives discontinuously according to a Poisson distribution. They show that if informative signals are "bad news" (i.e. they arrive more frequently when someone deviates), then some collusion may be possible in the limit as $\Delta \to 0$. However, if signals are "good news" then collusion is impossible in the limit. We show that if information arrives continuously (i.e. there are no informative Poisson jumps), then collusion is always impossible in the limit. This is true in the case of a simple repeated duopoly game, and also in more complicated variations that have not been studied in literature, e.g. one where price is a mean-reverting process.

The assumption that deviations of different players cannot be statistically distinguished is important. If information arrives continuously and deviations by different players are statistically distinguishable from public signals, then payoffs above static Nash can be achievable (see Sannikov (2004)). The intuition is that differentiated signals allow firms to provide individual incentives by simply transferring payoffs without the need to destroy any surplus. This could be the case when products of two firms are imperfect substitutes and public signals are two prices.[16] Nevertheless, we conjecture that the effects captured in our paper are important even in the case of differentiated products when the firm's strategies are flexible in other ways (e.g. if each firm can secretly sell its competitor's product on the market). This paper can be also viewed as providing a rationale to why firms may seek signals that go beyond the average market price in order to obtain collusion. Additionally, it provides a strong intuition why asymmetric collusive schemes (like the ones observed in many recent cases, see for example Arbault (2002) or European Commission, (2002)), rather than symmetric ones (like Green and Porter (1984)) may be the key to profitable collusion.

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[16] On the other hand, if firms are selling perfect compliments then deviations (to higher prices) may be difficult to prevent, as they change the signal (total sales) in the same direction.

We see that the nature of noise and signal structure affect the possibility of collusion in games with imperfect monitoring and frequent moves. There are other factors that could affect the scope of collusion: private monitoring, non-linear payoff structure, new possibilities for strategic flexibility (e.g. ability to secretly undercut price to a fraction of customers), etc. We hope to provide insights about more general games in future research.

# 10. Appendix

**PROOF OF LEMMA 1:** First, let us prove that the static best response is unique and less than q. The static best response problem is

$$\max_{q} P\left(q+q_j\right)q$$

Let us show that this problem is concave. Differentiating twice with respect to $q$, we obtain:

$$\begin{aligned}&\text{if } P''\left(q+q_j\right)<0: \quad P''\left(q+q_j\right)q+2P'\left(q+q_j\right)<0\\&\text{else: } P''\left(q+q_j\right)q+2P'\left(q+q_j\right)\leq P''\left(q+q_j\right)\left(q+q_j\right)+2P'\left(q+q_j\right)<0\end{aligned}$$

where the last inequality follows from **A1**. So the problem is strictly concave. That establishes uniqueness. Directly from **A2** we have $q^*(q_j) < \overline{q}$ .

The best response is defined by the first order condition

$$P'\left(q+q_j\right)q+P\left(q+q_j\right)=0$$

whenever $P(q_j) > 0$ and it is equal to 0 otherwise. Now, if both firms choose positive quantities in equilibrium then the equilibrium satisfies the two FOCs:

$$\begin{aligned}P'(Q)q_i+P(Q)=0\\P'(Q)q_j+P(Q)=0\end{aligned}$$

Indeed, there does not exist an equilibrium with either of the quantities equal to 0: Suppose that $q_j = 0$. That would require $P\left(q_i\right)\leq 0$. But then the marginal profit of firm $i$

is negative and hence it is not playing a best response. So in equilibrium both FOCs have to hold. Now, subtracting them we get:

$$P'(Q)(q_i - q_j) = 0$$

which can be satisfied only if $q_i = q_j$.

Finally, using this symmetry we can write $Q = 2q$ and then the equilibrium has to satisfy:

$$P'(Q)Q + 2P(Q) = 0 \qquad (17)$$

The LHS of this equation is decreasing, as by **A1**:

$$P''(Q)Q + 3P'(Q) < P''(Q)Q + 2P'(Q) < 0$$

Furthermore, at $Q = 0$ the LHS of (17) is positive and at $Q = 2\overline{q}$ it is negative by **A2**. Therefore, equation (17) has a unique interior solution $Q = 2q_N$. ◆

**PROOF OF LEMMA 2:** There are two ways of representing the likelihood difference of a tail test on a graph, as shown on Figure 7.

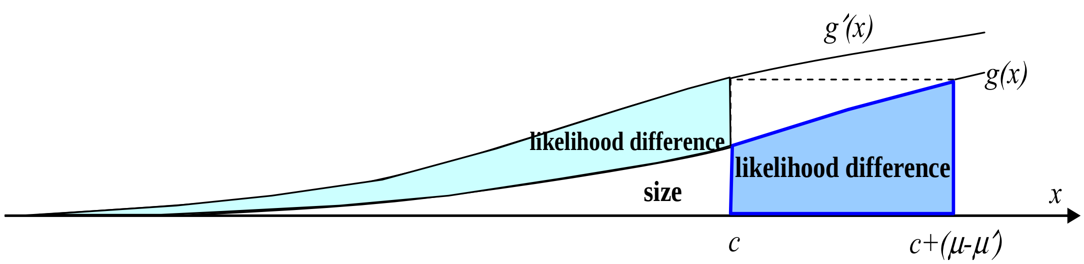


**Figure 7**

Using the area to the right, we see that there exists $x^* \in (c, c + (\mu - \mu'))$ such that

$$\text{likelihood difference } = (\mu - \mu')g(x^*) = (\mu - \mu')\frac{\sqrt{\Delta}}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(x^* - \mu)^2 \Delta}{2\sigma^2}\right)$$

Then if the likelihood difference is equal to $C_1\Delta$:[17]

[17] From now on we take $x^* < \mu'$. Otherwise, the size is strictly positive as $\Delta \to 0$ and the Lemma is trivially satisfied.

$$(\mu-\mu')\frac{\sqrt{\Delta}}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(x^*-\mu)^2\Delta}{2\sigma^2}\right)=C_1\Delta \quad\Rightarrow\quad \mu-x^*=\frac{\sigma}{\sqrt{\Delta}}\sqrt{-\log\left(2\pi\Delta\frac{C_1^{\,2}\sigma^2}{(\mu-\mu')^2}\right)}$$

Let $\alpha>1$ be a number to be specified later. Let $y^*$ satisfy $(\mu - y^*)=\alpha(\mu - x^*)$. Because $x^* - (\mu - \mu') < c$, the probability of making type I error (i.e. the size of the test) is greater than the shaded area in the Figure 8, i.e.

$$(x^*-(\mu'-\mu)-y^*)g(y^*)=\left((\alpha-1)(\mu-x^*)-(\mu-\mu')\right)\frac{\sqrt{\Delta}}{\sqrt{2\pi}\sigma}\exp\left(-\frac{\alpha^2(x^*-\mu)^2\Delta}{2\sigma^2}\right)=$$

$$\left((\alpha-1)\frac{\sigma}{\sqrt{\Delta}}\sqrt{-\log\left(2\pi\Delta\frac{C_1^{\,2}\sigma^2}{(\mu-\mu')^2}\right)}-(\mu-\mu')\right)\frac{\sqrt{\Delta}}{\sqrt{2\pi}\sigma}\left(\frac{C_1\Delta\sqrt{2\pi}\sigma}{(\mu-\mu')\sqrt{\Delta}}\right)^{\alpha^2}>O\left(\Delta^{\alpha^2/2}\right)$$

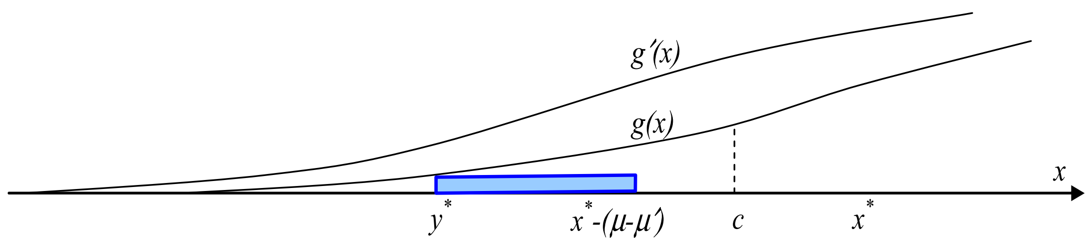


**Figure 8**

Taking $\alpha$ sufficiently close to 1 proves the Lemma.◆

**PROOF OF LEMMA 3:** Write the Lagrangian for the maximization problem

$$L=\int_{-\infty}^{\infty}v(x)g(x)dx+\lambda\left(\delta\int_{-\infty}^{\infty}v(x)\left(g(x)-g'(x)\right)dx-D\Delta\right)$$
$$+\int_{-\infty}^{\infty}\rho_1(x)\left(v(x)-\underline{v}\right)dx+\int_{-\infty}^{\infty}\rho_2(x)\left(\overline{v}-v(x)\right)dx,$$

where $\rho_1(x)>0$ only if $v(x)=\underline{v}$ and $\rho_2(x)>0$ only if $v(x)=\overline{v}$. First-order condition with respect to $v(x)$ gives

$$g(x)+\lambda\delta\left(g(x)-g'(x)\right)+\left(\rho_1(x)-\rho_2(x)\right)=0$$

If $g(x)+\lambda\delta(g(x)-g'(x))<0$ then $\rho_1(x)>0$ and $v(x)=\underline{v}$. If $g(x)+\lambda\delta(g(x)-g'(x))>0$ then $\rho_2(x)>0$ and $v(x)=\overline{v}$. We have

$$g(x)+\lambda\delta(g(x)-g'(x))<0 \Leftrightarrow \frac{g'(x)}{g(x)}>\frac{\lambda\delta+1}{\lambda\delta}$$

Because $g'(x)/g(x)$ is decreasing in *x*, the last inequality holds when $x < c$ for some constant *c*. Moreover, since $g'(x)/g(x)=1<(\lambda\delta+1)/\lambda\delta$ when $x=(\mu+\mu')/2$, $c<(\mu+\mu')/2$. We conclude that the solution to the maximization problem satisfies (1).◆

Lemma **4** is technical. It investigates profitable deviations.

**LEMMA 4.** Consider symmetric supply rates (*q*,*q*) for $q\in[0,q_N-\varepsilon]$. For any $\varepsilon > 0$, the gain from the most profitable deviation is bounded away from 0 i.e. there exists $\varepsilon_\pi > 0$ such that

$$\forall q\in[0,q_N-\varepsilon],\quad P\left(q^*(q)+q\right)q^*(q)-P(2q)q>\varepsilon_\pi,$$

where $q^*(q)$ is a static best response to *q*.

Also, the price reduction in the most profitable deviation is bounded i.e. there exists *M* such that

$$\forall q\in[0,q_N-\varepsilon],\quad P(2q)-P\left(q^*(q)+q\right)>M.$$

**PROOF OF LEMMA 4:** First, the gains from deviations are bounded away from 0. Indeed, for all $q\in[0,q_N-\varepsilon]$, $q^*(q)\neq q$, so

$$P\left(q^*(q)+q\right)q^*(q)-P(2q)q>0.$$

Also, $P\left(q^*(q)+q\right)q^*(q)-P(2q)q$ is continuous. (In fact, differentiable by the Envelope theorem, with derivative $P'\left(q^*(q)+q\right)q^*(q)$.) A positive continuous function on a compact set $[0,q_N-\varepsilon]$ must be bounded away from 0.

Second, the price reduction in the most profitable deviation is bounded because $q\in[0,q_N-\varepsilon]$, $q^*(q)\in[0,\overline{q}]$, and prices are continuous over this range. ◆

**PROOF OF REMARK IN SECTION 4:** i) Fix $\Delta > 0$ and $\varepsilon > 0$. Denote by $q_M$ the monopoly output. Suppose that the transfers are unbounded, in particular that the players can commit to burning any amount of money $T$ each (in the sense that they each pay $T$ to the third party). Then we can provide incentives statically, period-by-period. Suppose that the players sign a contract

$$t_i(x) = \begin{cases} 0 & \text{if } x \geq c \\ T & \text{if } x < c \end{cases}$$

which acts as a tail test that prescribes the players to burn $T$ if the average price in the given period drops below $c$. The continuation payoff is independent of current prices.

Let us fix $c$ and show that firms will choose quantity pair $(q_M/2, q_M/2)$ for an appropriate choice of $T$. We have to make sure that $q' = q_M/2$ maximizes

$$\Delta q' P(q' + q_M/2) - T \int_{-\infty}^{c-P(q'+q_M/2)} \phi(x)dx\,, \tag{18}$$

where $\phi$ is the density of a normal distribution with mean 0 and variance $\sigma^2/\Delta$. The choice of $q'$ affects the current-period profit and the probability of punishment. Taking the first-order condition, we get

$$T = -\Delta \frac{P(q_M) + P'(q_M)q_M/2}{\phi(c - P(q_M))P'(q_M)} > 0$$

Since there is no bound on $T$, for any small $c$ we can find $T$ that provides the players with incentives to choose $(\frac{q_M}{2}, \frac{q_M}{2})$. Then the per-period expected payoffs are

$$\frac{1}{2}\Delta v_M - T\Phi(c - P(q_M)) = \Delta\left(\frac{1}{2}v_M - K\frac{\Phi(c - P(q_M))}{\phi(c - P(q_M))}\right),$$

where $K = -\frac{P(q_M) + P'(q_M)q_M/2}{P'(q_M)}$ does not depend on $c$ ($\Phi$ is the *cdf* of the standard normal distribution). As $\lim_{z\to-\infty} \frac{\Phi(z)}{\phi(z)} = 0,$ the firms can achieve arbitrarily close to monopoly profit by choosing sufficiently small $c$.

ii) We sketch the reasoning behind this proof: Suppose that we want to obtain collusion with $\bar{v}_a(\Delta) > 2v_N + \varepsilon$. From the previous part we know that to obtain payoffs

$\bar{v}_a(\Delta) > 2v_N + \frac{1}{n}\varepsilon$ we need to be able to burn at least $T(\Delta)$ in a given period with $\lim_{\Delta \to 0} T(\Delta) = \infty$. Now, we pick $\Delta$ small enough that the initial wealth is smaller than $T(\Delta)$ and that with probability $1-\varepsilon'$ the wealth of the players at any time $t \in [0, \tau]$ is smaller than $T(\Delta)$ as well. Then the expected payoffs at time 0 are bounded by

$$(1-e^{-r\tau})\left(2v_N + \frac{1}{n}\varepsilon\right) + \varepsilon' v_M + (1-\varepsilon')e^{-r\tau} v_M,$$

which as $n, \tau \to \infty$ and $\varepsilon' \to 0$ converges to $2v_N$. ◆

SKETCH OF **PROOF OF PROPOSITION V.** Let $p^*$ be the price that maximizes $\bar{F}(p) - F(p)$, which implies that $\bar{F}'(p^*) = F'(p^*)$ and $\bar{F}''(p^*) \le F''(p^*)$. Let $\tilde{q} = \tilde{q}(p^*)$. Let us show that players can not achieve payoffs higher than $F(p^*)$ by choosing quantity pair $(q, q) \ge (\tilde{q}, \tilde{q})$ in the current period. Intuitively, if players wanted to collude, they would need to cut quantities and not increase them. Indeed, if $(q, q)$ is used currently, then the Bellman equation implies

$$\bar{F}(p_t) = (1-\delta)\tilde{q}p_t + \delta\left(\bar{F}(p_t) + \Delta f(2q, p_t)\bar{F}'(p_t) + \frac{\Delta\sigma^2(p_t)}{2}\bar{F}''(p_t) - \text{exp. punishment}\right) + O(\Delta^2) \Rightarrow$$

$$(1-\delta)\bar{F}(p_t) \overset{\boxed{1}}{\le} (1-\delta)qp_t + \delta\left(\Delta f(2q, p_t)\bar{F}'(p_t) + \frac{\Delta\sigma^2(p_t)}{2}\bar{F}''(p_t)\right) + O(\Delta^2) \overset{\boxed{2}}{\le}$$

$$(1-\delta)qp_t + \delta\left(\Delta f(2q, p_t)F'(p_t) + \frac{\Delta\sigma^2(p_t)}{2}F''(p_t)\right) + O(\Delta^2) \overset{\boxed{3}}{\le}$$

$$(1-\delta)\tilde{q}p_t + \delta\left(\Delta f(2\tilde{q}, p_t)F'(p_t) + \frac{\Delta\sigma^2(p_t)}{2}F''(p_t)\right) + O(\Delta^2) = (1-\delta)F(p_t),$$

where the inequality 1 follows from ignoring punishment and inequality 2 follows because $\bar{F}'(p^*) = F'(p^*)$ and $\bar{F}''(p^*) \le F''(p^*)$. For inequality 3, the first-order condition in a MPE at price $p^*$ implies that

$$(1-\delta)p_t + \delta\Delta f_1(2\tilde{q}, p_t)F'(p_t) + O(\Delta^2) = 0 \quad \Rightarrow$$
$$\forall q \ge \tilde{q}, \quad (1-\delta)p_t + 2\delta\Delta f_1(2q, p_t)F'(p_t) + O(\Delta^2) < 0,$$

so $(1-\delta)qp_t + \delta\Delta f(2q, p_t)F'(p_t)$ must be decreasing in $q$ for all $q \ge \tilde{q}$.

We claim that it is impossible to achieve payoffs larger than $F(p_t)$ even by using quantity pair $(q,\ q) < (\tilde{q}, \tilde{q})$. Intuitively, this happens because a statistical test to detect potential deviations is too costly. Let $M$ be an upper bound on the size of punishment possible in an equilibrium. If firms use quantity pair $(\hat{q}, \hat{q}) < (\tilde{q}, \tilde{q})$ to achieve $\bar{F}(p_t)$, they must perform a statistical test on $p_{t+\Delta}$ to decide when to go to punishment. Let $q^*(\hat{q})$ be a supply rate that maximizes

$$(1-\delta)qp_t + \delta\Delta f(q+\hat{q}, p_t)\bar{F}'(p_t)$$

i.e. that maximizes expected payoffs (modulo constants and terms of order $O(\Delta^2)$) without taking into account the possibility of switching to the punishment phase. Let $T$ be the best tail test on $\frac{p_{t+\Delta} - p_t}{\Delta} \sim N\left( f(q+\hat{q}, p_t), \frac{\sigma^2(p_t)}{\Delta} \right)$ that prevents deviation towards $q^*(\hat{q})$ through a punishment of magnitude $M$. Then

$$(1-\delta)\bar{F}(p_t) \leq (1-\delta)\hat{q}p_t + \delta\left( \Delta f(2\hat{q}, p_t)F'(p_t) + \frac{\Delta\sigma^2(p_t)}{2}F''(p_t) - M \cdot size(T) \right) + O(\Delta^2)$$

and the incentive compatibility:

$$(1-\delta)q^*(\hat{q})\,p_t + \delta\Delta f(q^*(\hat{q}) + \hat{q}, p_t)\bar{F}'(p_t) - (1-\delta)\hat{q}p_t - \delta\Delta f(2\hat{q}, p_t)\bar{F}'(p_t) \leq M \cdot LD(T).$$

Unless $\hat{q} \approx \tilde{q}$, the likelihood difference $LD(T)$ must be on the order of Δ. But then $size(T)$ is on the order of $\Delta^{1/2+\varepsilon}$ and it becomes impossible to sustain $\bar{F}(p_t) > F(p_t)$. ◆

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