Attainable payoffs in repeated games with interdependent private information

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Abstract

In this paper I prove two folk theorems for repeated games with private information and communication, in which signal spaces may be arbitrary, signals may be statistically interdependent, and payoffs for each player may depend on the signals of other players. (1) In games with transferable utility, if an outcome rule (i) is potentially individually rational and can be implemented by an (ii) interim incentive compatible and (iii) ex post budget balanced mechanism in the stage game, then the level of aggregate utility it provides can also be implemented as the average aggregate utility of a stationary perfect public equilibrium in the infinitely repeated game, given a sufficiently high discount factor. (2) In games without transferable utility, if an expected payoff profile (i) is individually rational and (ii) can be implemented by an interim incentive compatible mechanism in the stage game, if (iii) the convex hull of such payoff profiles has non-empty interior, and (iv) if the probability measure over signals satisfies a condition guaranteeing ex post weighted budget balance, then the payoff profile can be approximated as the average utility profile of a PPE in the infinitely repeated game, as the discount factor approaches unity. Finally, I show that the condition guaranteeing ex post weighted budget balance, (iv), is satisfied by both independent and globally interdependent probability measures.

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1 Introduction

Consider a class of infinitely repeated games with private information and communication. Each stage game begins with each player observing his own private signal. (Signals are identically and independently distributed over time, but not necessarily across players.) The players then have the opportunity to send each other simultaneous public messages, and then each selects an action. In such games, players face the tension of collectively wanting to share information in order to coordinate their activities, while each of them individually faces an incentive to misrepresent his own information to the others. This tension complicates the problem of coordination by requiring that the players give each other incentives to reveal their private information.

Fudenberg, Levine, and Maskin (1994, hereafter FLM) prove a folk theorem (Theorem 8.1) for such games: as players become more and more patient, they can approximate, as average payoffs in a *perfect public equilibrium (PPE)* in the repeated game, any stage game payoffs that are feasible and individually rational. The main focus of FLM is games with imperfect public monitoring, in which actions, rather than information, are private. FLM's approach to games with private information is to interpret each player's mapping from her private signal to her public message as her "action" in a public monitoring game. The realization of her message is then interpreted as a public indicator of her "action." To apply their public monitoring results to this reinterpreted game, FLM invokes three limiting assumptions: (i) the signal space is finite, (ii) the private signals are statistically independent, (iii) each player's payoff does not depend on other players' signals ("independent payoffs").

In this paper I prove two folk theorems that bypass these limitations. The first is for games with monetary transfers, with which players can exchange utility at the end of each stage. The direct transfer of utility simplifies the provision of incentives, and so the result is somewhat stronger: sufficiently patient players can support any incentive compatible and individually rational payoffs in the stage game as the average payoffs in a stationary PPE of the repeated game. The second folk theorem is for games without monetary transfers. Here, I reach a conclusion similar to that of FLM, but under a much weaker set of assumptions—specifically, I allow an arbitrary signal space, statistically interdependent signals, and interdependent payoffs. The main limiting assumption is that the probability measure over signals should satisfy a condition that guarantees budget balance, but when there are three or more players, this condition is generally satisfied.[1]

In addition to the folk theorems, this paper makes two further contributions. The first is to extend the "mechanism design approach," first developed by Athey and Bagwell (2001) for the case of collusion with hidden costs, to apply to a general class of games. In the mechanism

[1] More specifically, the condition is satisfied if the probability measure over signals is either independent or "globally interdependent." See Section 5.

design approach, the problem of designing a PPE in the repeated private information game is converted to the problem of constructing a *recursive mechanism*, which in each stage receives the players' messages and then instructs them how to act. The mechanism is "recursive" because in each period it promises them a certain level of utility in the continuation game, and when the next period begins this promised utility determines the mechanism to be used in the continuation game. The combination of monetary transfers at the end of the period and changes in promised future utility provide the incentives for players to reveal their information truthfully. The problem of constructing a recursive mechanism, in turn, can be expressed (in part) as a static mechanism design problem, to which standard techniques apply. The mechanism design approach is compared with the FLM approach in Section 1.1, below, and defined formally in Section 4.

The second additional contribution is to extend a result from the static mechanism design literature, due to d'Aspremont, Crémer, and Gérard-Varet (2003, 2004, hereafter ACGVa and ACGVb, respectively), to the case of arbitrary signal spaces. This result, which indicates that interim incentive compatible mechanisms can generally be implemented with ex post budget balance, is discussed in Section 5.

1.1 Comparison with the FLM approach

A comparison with FLM's approach shows why the mechanism design approach yields a broader set of conclusions. In both approaches, the problem of constructing a PPE in the repeated game is successively converted into simpler and simpler problems, until the result is attained. At each step of simplification, there are conditions that must be imposed. The FLM approach, as described above, begins by converting a private information game to a public monitoring game. Specifically, FLM convert a private information game to a public monitoring game with a "product structure." That is, each player's action gives rise to a separate monitoring indicator, and these indicators are statistically independent. As described above, the monitoring indicator for each player is the realization of her message, but for these messages to be statistically independent there can be no interdependence among the private signals that the players observe. Hence FLM's first step imposes statistical independence on the signals.[2]

The second step of the FLM approach is to prove that, in games with product structures, any smooth set $\mathcal{W}$ of attainable and individually rational[3] payoff profiles is "decomposable on

[2]A more minor limitation is that for this step FLM need to assume that the players take no additional actions subsequent to sending their messages—that an outcome is automatically selected based on the realized messages. Here, I allow the players to take actions after communicating, but I impose constraints to assure that they will not deviate in equilibrium. See Section 4 for details.

[3]In the context of FLM, an "individually rational" payoff is one that weakly exceeds the minimax payoff for each player. This is different from the way I use the term here; see below.

tangent hyperplanes." That is, for each payoff profile on the boundary of $\mathcal{W}$, the hyperplane tangent to $\mathcal{W}$ at that profile separates $\mathcal{W}$ from some extreme payoff profile, and the outcome function that yields this extreme payoff profile can be supported with appropriate incentives by a set of continuation rewards that lie in the hyperplane. FLM are able to prove this by assuming finite signal spaces and independent payoffs.

Finally, FLM prove that if $\mathcal{W}$ is decomposable on tangent hyperplanes, then if the players are sufficiently patient they can support any point in $\mathcal{W}$ as the average payoffs of a PPE. Since $\mathcal{W}$, as an arbitrary smooth set of attainable and individually rational payoff profiles, can include points arbitrarily close to any attainable and individually rational payoff profile, this last step yields the folk theorem.

The mechanism design approach follows a different path. As described above, the approach begins by shifting the focus from PPE to recursive mechanisms. This step is naturally similar to the dynamic programming approach developed by Abreu, Pearce, and Stacchetti (1990), FLM, and others, but the differences are twofold. First, when participating in the recursive mechanism the players are assumed to reveal their information directly to each other, whereas in the game they can send any arbitrary messages. Second, the players are assumed to obey the mechanism's instructions. In Appendix A, I show if the players are sufficiently patient then it is without loss of generality to focus on recursive mechanisms rather than PPEs, as long as the mechanism is *individually rational (IR)*—i.e., it provides average payoffs that exceed the payoffs of the stage game equilibrium.

The second step of the mechanism design approach is to convert the problem of designing a recursive mechanism into a static mechanism design problem. Restrict attention for a moment to games with monetary transfers; in such games this conversion is accomplished by lumping together the monetary transfers and changes in promised future utility into a "static transfer function." In the context of this static mechanism design problem, I look for static mechanisms that satisfy *interim incentive compatibility (IIC)*[4]—the requirement that each player be willing to reveal his information truthfully given his interim beliefs about other players—and *ex post budget balance (EPBB)*—the requirement that the static transfers sum to zero across players after every realization of the private signals. By imposing a condition on the probability measure over private signals ("Condition C", after ACGVa), we can ensure that any IIC-implementable outcome function is also implementable with EPBB. When there are two players, Condition C is equivalent to statistical independence, but when there are three or more players Condition C is generally satisfied, as I show in Appendix C.

For games with monetary transfers, it is then possible to prove the folk theorem directly, by explicitly constructing a stationary equilibrium. Given a static mechanism that satisfies

[4]Often called "Bayesian incentive compatibility."

IIC, EPBB, and IR, a recursive mechanism can be constructed that implements the same outcome function in each period. Because the static mechanism satisfies IIC and EPBB, the players can settle up the provision of IIC incentives using monetary transfers at the end of each period, so in the succeeding period they can start over from the beginning. If the players are sufficiently patient, the path of play mapped out by this recursive mechanism can be supported as an equilibrium path of a PPE; the only restrictions are that the resulting payoffs should be individually rational and the static mechanism should satisfy EPBB (if Condition C is not satisfied, EPBB can be verified directly).

For games without monetary transfers the process is somewhat more complicated. As above, we convert the problem to a static mechanism design setting, and look for static mechanisms that satisfy IIC. However, without monetary transfers the mechanism needs to satisfy more than just EPBB, since we want to find a set of payoffs that is decomposable on tangent hyperplanes. (EPBB implies decomposability only on hyperplanes orthogonal to the vector $(1, \ldots, 1)$.) By imposing "Strong Condition C," we can ensure decomposability on almost any hyperplane. Like Condition C, Strong Condition C is satisfied under statistical independence, and is generally satisfied when there are three or more players.

With Strong Condition C imposed, it is a simple task to show that any smooth set of IIC-implementable and individually rational payoffs is decomposable on almost all tangent hyperplanes. For the final step (in Appendix B), I extend FLM's conclusion that decomposability on tangent hyperplanes implies the folk theorem to the case of arbitrary type spaces and when there may be isolated hyperplanes on which decomposability fails. This extension comes with one additional restriction, which is that each of the IIC mechanisms used in the construction must employ static transfers that are uniformly bounded.[5]

In sum, the mechanism design approach allows me to impose much milder conditions than FLM: Condition C and Strong Condition C are generally satisfied when there are three or more players; individual rationality with respect to a stage game equilibrium does not impinge on supporting high payoffs, as long as a stage game equilibrium exists; and the notion that transfers should be uniformly bounded is implicit in any plausible application.

1.2 Other related literature

Since FLM, a number of papers in the literature have investigated the possibility of efficient equilibria in repeated games with private information and communication. None of them, however, have relaxed more than two of the three key limitations in FLM's folk theorem, and each has been tied to the structure of a particular game.

[5]That is, for any particular mechanism, transfers must be uniformly bounded across all realizations of the signal vector. Each mechanism may employ a different uniform bound.

Athey and Bagwell (2001), recognized above as the originators of the mechanism design approach, studied a collusion game between two firms, each with two possible private cost types that are distributed independently. Although their setting is covered by the FLM folk theorem for games with private information, Athey and Bagwell's use of the mechanism design approach enabled them to construct optimal equilibria for firms that are not fully patient. They also construct near-efficient collusive equilibria without communication in which private information is fully revealed through the actions that the firms take, but the existence of such equilibria depends on the special structure of collusion in price-setting markets, and does not generalize to alternative settings (e.g., collusion in quantity-setting markets).

Aoyagi has considered the case of collusion in repeated auctions in a pair of papers (2003a; 2003b). In 2003a he allows a continuous signal space, statistical interdependence, and interdependent payoffs, but the equilibrium he constructs is not asymptotically efficient. In 2003b he restricts the signal space to be finite, and constructs an equilibrium that is asymptotically efficient as the players become more patient. In work conducted simultaneously with the present paper, Martin and Vergote (2004) expand on Aoyagi's results by constructing an asymptotically efficient equilibrium for repeated auctions in which each player's signal is drawn from an interval of $\mathbb{R}$, but with statistical independence and independent payoffs. All of the papers mentioned in this paragraph construct equilibria using bid rotation, in which, at various times along the equilibrium path, one or more firms are excluded from participating in the auction. This method of constructing equilibria does not extend to general games.

Athey and Miller (2004), also concurrent with Martin and Vergote (2004) and the present paper, considers the setting of repeated trade between two players, using the mechanism design approach. The model assumes statistical independence and independent payoffs, but draws each player's valuation from an interval of $\mathbb{R}$, and allows the players to make monetary transfers. Efficient equilibria are constructed under several combinations of assumptions, including the assumptions adopted in the present paper.

Recently, Athey and Bagwell (2004) have looked at collusion with hidden costs when each firm's costs are correlated over time. With two possible costs for each firm, they find that efficiency can be supported if the firms are patient relative to the persistence of their cost shocks.

In Miller (2004), I consider the same setting of repeated games with private information and communication as in the present paper, but with the added restriction that the equilibrium be robust to the possibility that players may spy on each other's information, or that they may have arbitrary higher order beliefs. Under this restriction, I show that efficiency cannot be approximated by any PPE, and I characterize optimal equilibria under this restriction.

Several papers have investigated collusion in repeated auctions without communication, when the auctioneer releases only the identity of the winner. Skrzypacz and Hopenhayn (2004) show that in such games efficiency cannot be approximated by any PPE. Blume and Heidhues (2002) note that each player has private information about whether he himself won the auction, and can use this information together with the information released by the auctioneer to try to infer who may have deviated. But this means that the monitoring indicators are not common knowledge, so Blume and Heidhues construct an equilibrium in private strategies that outperforms any PPE. It is not yet known whether there exist asymptotically efficient equilibria in their setting.

2 The model

The dynamic game consists of infinite repetitions of a stage game. All players share a common discount factor $\delta \in (0,1)$. Let $\mathcal{N}$ be the set of players, $i = 1, \ldots, N$.

At the beginning of each stage, each player i observes a private signal drawn from the set Θ_i, according to the common prior probability measure ϕ on $\Theta \equiv \Theta_1 \times \cdots \times \Theta_N$.[6] The players then send simultaneous public messages; each player i's message space is $\mathcal{M}_i \supset \Theta_i$,[7] with $\mathcal{M} \equiv \mathcal{M}_1 \times \cdots \times \mathcal{M}_N$. After the communication is concluded, each player i simultaneously chooses an action from $\mathcal{X}_i$, with $\mathcal{X} \equiv \mathcal{X}_1 \times \cdots \times \mathcal{X}_N$; all actions are publicly observed. Each player i receives a payoff from the actions and signals; this is given by $\pi_i : \Theta \times \mathcal{X} \to \mathbb{R}_+$, with $\pi \equiv \{\pi_1, \ldots, \pi_N\}$. Assume that π_i is integrable and uniformly bounded for all i.

For most of this paper, I also allow the players to voluntarily transfer utility to one another. I model this by assuming that they have utility that is quasilinear in "money," which is a continuous good in zero net supply. Monetary transfers are bilateral and unidirectional: each player can transfer any non-negative amount of money to each other player, and such transfers cannot be refused by the recipient. All such transfers are publicly observed. Player i's utility in the stage game is equal to her payoff plus any monetary transfers she receives minus any monetary transfers she gives to other players.

This setup allows a virtually arbitrary measurable signal space Θ_i for each player and statistical interdependence among signals; it also allows each player's payoffs to depend on the entire vector of signals. For simplicity, I assume that randomizations over actions may be publicly observed, so that $\mathcal{X}_i$ can be thought of as a space of mixtures over pure actions.

To ensure the existence of a suitable trigger punishment, I assume the following:

[6] Assume that the conditional probability measures $\big\{\{\phi_{-i|i}(\cdot|\theta_i)\}_{\theta_i \in \Theta_i}\big\}_{i=1}^{N}$ are well defined.

[7] More generally, since messages can be relabeled, what is actually required is that $|\mathcal{M}_i| \geq |\Theta_i|$. Assuming $\Theta_i \subset \mathcal{M}_i$ simplifies the notation.

Assumption 1. *There exists a perfect Bayesian equilibrium in the stage game.*

3 Equilibrium

The equilibrium concept is *perfect public equilibrium* (PPE), which is a refinement of perfect Bayesian equilibrium in which players play behavior strategies which are conditioned only on the public history and their current private information. The public history comprises the messages, actions, and transfers that have transpired, which are common knowledge.

This section provides the notation necessary to describe strategies and equilibria. Since I will use the mechanism design approach to simplify matters in following sections, readers who are more interested in results may wish to skip ahead.

A public outcome in period τ can be represented as $h^\tau \in \mathcal{M} \times \mathcal{X} \times (\mathbb{R}_+^{N-1})^N$, while a private outcome for player i is $h_i^\tau \equiv (\theta_i, h^\tau)$. A pure stage strategy for player i is a triplet $s_i = \langle \hat{m}_i, \hat{x}_i, \hat{t}_i \rangle$ that contains a reporting rule $\hat{m}_i : \Theta_i \to \mathcal{M}_i$, an action rule $\hat{x}_i : \Theta_i \times \mathcal{M} \to \mathcal{X}$, and a transfer rule $\hat{t}_i : \Theta_i \times \mathcal{M} \times \mathcal{X} \to \mathbb{R}_+^{N-1}$, with $\hat{t}_{i,j}(\cdot) \in \mathbb{R}_+$ indicating the amount that player i transfers to player j. A pure stage strategy profile is a vector $s \equiv (s_1, \ldots, s_N)$, or, equivalently, $s \equiv \langle \hat{m}, \hat{x}, \hat{t} \rangle$. The public history at the end of period τ is $H^\tau \equiv (h^1, \ldots, h^\tau)$, and the private history for player i is $H_i^\tau \equiv (h_i^1, \ldots, h_i^\tau)$; H^0 and H_i^0 are null histories. A "stage-behavior" strategy for player i is a sequence of functions $\sigma_i \equiv \{\sigma_i^\tau\}_{\tau=1}^\infty$ (with $\sigma^\tau \equiv (\sigma_1^\tau, \ldots, \sigma_N^\tau)$), where σ_i^τ maps player i's current private history $H_i^{\tau-1}$ to a probability distribution over pure stage strategies.[8] For convenience, I allow players to choose their stage strategies using an arbitrary public randomization device, so that (indulging in some abuse of notation) $\sigma^\tau(\{H_i^{\tau-1}\}_{i\in\mathcal{N}})$ need not be statistically independent. Given a stage-behavior strategy profile σ and a set of private histories $\{H_i^{\tau-1}\}_{i\in\mathcal{N}}$, the ex post stage game payoff for player i in period τ is

$$\begin{aligned} \hat{\pi}_i(\theta_i; \sigma^\tau(\{H_i^{\tau-1}\}_{i\in\mathcal{N}})) = \pi(\theta, \hat{x}(\theta, \hat{m}(\theta))) &+ \sum_{j\neq i} \hat{t}_{j,i}\big(\theta_j, \hat{m}(\theta), \hat{x}(\theta, \hat{m}(\theta))\big) \\ &- \sum_{j\neq i} \hat{t}_{i,j}\big(\theta_j, \hat{m}(\theta), \hat{x}(\theta, \hat{m}(\theta))\big), \end{aligned} \tag{1}$$

where the understanding is that $\langle \hat{m}, \hat{x}, \hat{t} \rangle$ is the outcome of the randomization specified by $\sigma^\tau(\{H_i^{\tau-1}\}_{i\in\mathcal{N}})$. A public strategy for player i is a stage-behavior strategy such that $\sigma_i^\tau(H_i^{\tau-1}) = \sigma_i^\tau(\tilde{H}_i^{\tau-1})$ whenever $H^{\tau-1} = \tilde{H}^{\tau-1}$; i.e., a strategy in which player i ignores her private history. When σ_i is a public strategy, I sometimes write $\sigma_i^\tau(H^{\tau-1})$. In the repeated

[8] Strictly speaking, a behavior strategy would specify a probability distribution over reports as a function of private type, a probability distribution over actions as a function of private type and message vector, and so on. However, I will construct PPEs from stage-behavior strategies, and a more general formulation of behavior strategies offers no deviation that can be more profitable than any stage-behavior deviation.

game, given a profile of stage-behavior strategies $\sigma = (\sigma_1, \ldots, \sigma_N)$ the value to player i of any private history H_i^τ is

$$\hat{v}_i(H_i^\tau; \sigma) = (1-\delta)\,\mathbb{E}\left[\sum_{\tilde{\tau}=\tau+1}^{\infty} \delta^{\tilde{\tau}-1} \hat{\pi}_i(\theta_i^{\tilde{\tau}}; \sigma^{\tilde{\tau}}(\{H_i^{\tilde{\tau}-1}\}_{i\in\mathcal{N}})) \middle| H_i^\tau \right]. \tag{2}$$

Definition 1. A *perfect public equilibrium* (PPE) is a public strategy profile σ such that

$$\hat{v}_i(H_i^\tau; \sigma) \geq \hat{v}_i(H_i^\tau; (\sigma_i', \sigma_{-i})) \tag{3}$$

for all strategies σ_i', for all private histories H_i^τ, and for all i.

4 The mechanism design approach

This section develops a "mechanism design approach" to repeated games with private information. PPEs are naturally unwieldy objects to work with, because in each stage a player's strategy takes the entire public history as an argument. Fortunately, Abreu, Pearce, and Stacchetti (1986, 1990) showed that PPEs can be constructed recursively without loss of generality with respect to attainable utilities. The recursive construction yields a dynamic program in which the future path of play is summarized by a profile of average continuation utilities. I interpret this program as a recursive mechanism design problem.

In order for a solution to the dynamic program to constitute a PPE, it must satisfy certain constraints so that no player can profitably deviate. Since there are several types of deviations, there are several types of constraints. The static literature usually assumes the presence of an outside authority who designs the mechanism and enforces its prescriptions, allowing two kinds of constraints to be ignored. For one, although the authority may collect detailed information from each player, after computing the desired outcome it might simply issue instructions to the players rather than reveal all the information. Thus the players may not have complete information when they take their actions and make their monetary transfers, and so their actions might not need be immune to deviations premised on complete information. For a second, the authority is usually assumed to force the players to commit to their specified monetary transfers at the time they choose their actions, or, equivalently, to force the players to choose their actions and make their transfers simultaneously.

Although no such outside authority exists in the context of a repeated game, my approach uses the same set of constraints that would be appropriate if such an authority existed. In particular, I focus on the class of mechanisms in which players fully reveal their information to each other. This approach greatly simplifies the notation and analysis, but its validity must be proven. This is accomplished in Theorem 1, which shows that if the players are sufficiently patient then this approach yields (nearly) the same set of utilities that are attainable in PPE. Sufficient patience is required because the assumption that players reveal their information directly to each other is restrictive for low discount factors. Recursive mechanisms and their properties are defined below; Theorem 1 is stated at the end of the section.

4.1 Recursive mechanisms

A *stage mechanism* consists of an outcome function x, a transfer function t, and a continuation reward function w.

Definition 2. A *stage mechanism* is a measurable triplet $\langle x, t, w\rangle : \Theta \to \mathcal{X} \times \mathbb{R}^N \times \mathbb{R}^N$ such that t and w are integrable and $\sum_i t_i(\theta) = 0$ for all $\theta \in \Theta$.

Although this notation does not reflect it, I also allow a stage mechanism to be chosen by any public randomization, so that the set of IIC-implementable payoff profiles is convex.

In addition to a collection of stage mechanisms, a recursive mechanism also specifies a set $\mathcal{V}$ of promised (average) utility state variables over which it is defined, as well as an initial condition v^0. Formally:

Definition 3. A *recursive mechanism* is a triplet $\langle \mathcal{V}, \{\langle x(\cdot; v), t(\cdot; v), w(\cdot; v)\rangle : v \in \mathcal{V}\}, v^0\rangle$, abbreviated as $\langle \mathcal{V}, \{\langle x, t, w\rangle (\cdot; v)\}, v^0\rangle$, such that:

(i) $\mathcal{V} \subset \mathbb{R}^N$,

(ii) $\{\langle x, t, w\rangle (\cdot; v)\}$ is a collection of stage mechanisms indexed by $v \in \mathcal{V}$,

(iii) $v^0 \in \mathcal{V}$.

Such a mechanism is called "recursive" because in each period the stage mechanism is selected based on the promised utility v carried over from the previous period, and the stage mechanism generates a continuation reward $w(\cdot; v)$ that becomes the promised utility for the next period. In contexts that do not invite confusion, I drop the $(\cdot; v)$ notation for clarity.

Given a signal profile $\theta \in \Theta$ and a vector of announcements $\hat{\theta} \in \Theta$ in a stage mechanism $\langle x, t, w\rangle$, each player's total utility is her stage game utility plus the present value of her continuation reward:

$$u_i(\theta, \hat{\theta}; \delta, \langle x, t, w\rangle) \equiv \pi_i(\theta, x(\hat{\theta})) + t_i(\hat{\theta}) + \frac{\delta}{1-\delta} w_i(\hat{\theta}). \tag{4}$$

The space of promised utilities $\mathcal{V}$ must be feasible: the utilities that are promised must actually be attainable in the game, and the mechanism must actually deliver what it promises.

Definition 4. Given $\delta \in (0,1)$, a recursive mechanism $\langle \mathcal{V}, \{\langle x, t, w\rangle(\cdot; v)\}, v^0\rangle$ is *feasible* if the following are satisfied:

(i) Attainability: $\mathcal{V} \subset \text{co}\left(\left\{v \in \mathbb{R}^N : \sum_i v_i = \sum_i \mathbb{E}_\theta\left[\pi_i\left(\theta, x(\theta)\right)\right] \text{ for some } x : \Theta \to \mathcal{X}\right\}\right)$;

(ii) Promise keeping: $v = (1-\delta)\,\mathbb{E}_\theta\left[u(\theta, \theta; \delta, \langle x, t, w\rangle(\cdot; v))\right]$ for all $v \in \mathcal{V}$;

(iii) Coherence: $w(\theta; v) \in \mathcal{V}$ for all $\theta \in \Theta$ and all $v \in \mathcal{V}$.

where co$(\cdot)$ is the convex hull operator.

To support the recursive mechanism as an equilibrium of the game, deviations must be deterred. Some deviations are observable to the other players; for instance, sending a nonsense message, taking the wrong action, or refusing to make the required transfer. These are termed "off-menu" deviations,[9] and are punished by terminating the use of stage mechanisms and switching to a punishment PPE in the continuation game. The trigger punishment is not considered part of the recursive mechanism; instead, it enters the individual rationality constraint, defined below.

4.2 The equivalent static mechanism

The following definition transforms the dynamic problem into an equivalent static problem.

Definition 5. A *static mechanism* is a measurable pair $\langle x, y\rangle : \Theta \to \mathcal{X} \times \mathbb{R}^N$ such that y is integrable and uniformly bounded. Given $\delta \in (0,1)$ and a stage mechanism $\langle x, t, w\rangle$, the *equivalent static mechanism* is a static mechanism $\langle x, y\rangle$ such that

$$y(\theta) \equiv t(\theta) + \frac{\delta}{1-\delta} w(\theta). \tag{5}$$

Accordingly, y is termed the *equivalent static transfer function.* Note that by construction y is integrable. Naturally, in a static mechanism players seek to maximize $\pi_i(\theta, x(\hat{\theta})) + y_i(\hat{\theta})$.

Definition 6. A static transfer function y is *ex post budget balanced (EPBB)* if

$$\sum_i y_i(\theta) = 0 \tag{6}$$

for all θ. A static mechanism $\langle x, y\rangle$ is EPBB if y is EPBB.

Definition 7. A static mechanism $\langle x, y\rangle$ is *uniformly bounded* if there exists a bound $B < \infty$ such that $\left|y_i(\theta)\right| < B$ for all $\theta \in \Theta$ and all $i \in \mathcal{N}$.

[9] Here, "on-menu" and "off-menu" have the same meaning as the terminology "on-schedule" and "off-schedule" developed by Athey, Bagwell, and Sanchirico (2004) and Athey and Bagwell (2001).

4.3 Incentive compatibility

Incentive compatibility is the central notion that enables the mechanism design approach to repeated games of hidden information. In a standard repeated game equilibrium, deviations from the equilibrium strategy profile must be discouraged by credibly threatened punishments. In games of hidden information, deviations in which players misrepresent their private information cannot be detected directly; such deviations are termed "on-menu" deviations. To deter on-menu deviations even players who truthfully reveal their private information must be "punished" or "rewarded" so as to prevent other types of players from impersonating them. In a mechanism design environment, the notion of incentive compatibility imposes the same restrictions: each player must prefer to make a truthful announcement rather than deviate to some other on-menu announcement.

Interim incentive compatibility (IIC) requires that each player prefer to make a truthful announcement before the other players have made their announcements. IIC can be thought of as incentive compatibility that applies when all announcements are made simultaneously.

Definition 8. Given a discount factor $\delta \in (0,1)$, a stage mechanism $\langle x,t,w\rangle$ is *interim incentive compatible (IIC)* if

$$\mathbb{E}_{\theta_{-i}}\left[u_i(\theta,\theta;\delta,\langle x,t,w\rangle)\middle|\theta_i\right] \geq \mathbb{E}_{\theta_{-i}}\left[u_i(\theta,(\hat{\theta}_i,\theta_{-i});\delta,\langle x,t,w\rangle)\middle|\theta_i\right] \tag{7}$$

for all $\hat{\theta}_i \in \Theta_i$, for ϕ_i-almost all $\theta_i \in \Theta_i$, and for all $i \in \mathcal{N}$. A recursive mechanism $\langle \mathcal{V}, \{\langle x,t,w\rangle(\cdot;v)\}, v^0\rangle$ is IIC if $\langle x,t,w\rangle(\cdot;v)$ is IIC for all $v \in \mathcal{V}$.
A static mechanism $\langle x,y\rangle$ is IIC if

$$\mathbb{E}_{\theta_{-i}}\left[\pi_i(\theta,x(\theta))+y_i(\theta)\middle|\theta_i\right] \geq \mathbb{E}_{\theta_{-i}}\left[\pi_i(\theta,x(\hat{\theta}_i,\theta_{-i}))+y_i(\hat{\theta}_i,\theta_{-i})\middle|\theta_i\right] \tag{8}$$

for all $\hat{\theta}_i \in \Theta_i$, for ϕ_i-almost all $\theta_i \in \Theta_i$, and for all $i \in \mathcal{N}$.

Note that a static mechanism is IIC if and only if every static mechanism that it is equivalent to is IIC.

4.4 Individual rationality

In mechanism design problems, players are often assumed to have outside options that they can choose in lieu of participating in the mechanism. This possibility imposes "participation constraints," or individual rationality constraints, on the mechanism designer. In a game, however, there are no outside options; every possibility is included within the game. Still, there are individual rationality constraints on the equilibrium, because players can deviate from it. In this paper I use a simple perfect Bayesian threat version of individual rationality for simplicity: when any player makes an off-menu deviation from the equilibrium path, the players revert to a perfect Bayesian equilibrium in the stage game that they then play for ever after.[10]

Definition 9. Given a discount factor $\delta \in (0,1)$, a stage mechanism $\langle x, t, w \rangle$ is *individually rational (IR)* if there exists a perfect Bayesian equilibrium in the stage game with expected payoff profile p such that, for all $i \in \mathcal{N}$,

$$(1-\delta)\ \mathbb{E}_\theta \left[u_i(\theta, \theta; \delta, \langle x, t, w \rangle) \right] > p_i. \tag{9}$$

A recursive mechanism $\langle \mathcal{V}, \{\langle x, t, w \rangle (\cdot; v)\}, v^0 \rangle$ is IR if there exists $\varepsilon > 0$ such that, for all $v \in \mathcal{V}$,

$$(1-\delta)\ \mathbb{E}_\theta \left[u_i(\theta, \theta; \delta, \langle x, t, w \rangle (\cdot; v)) \right] > p_i + \varepsilon. \tag{10}$$

A static mechanism $\langle x, y \rangle$ is IR if there exists a perfect Bayesian equilibrium in the stage game with expected payoff profile p such that, for all $i \in \mathcal{N}$,

$$(1-\delta)\ \mathbb{E}_\theta \left[\pi_i(\theta, x(\theta)) + y_i(\theta) \right] > p_i. \tag{11}$$

I also say that an outcome function x is IR if $\mathbb{E}_\theta \left[\pi_i(\theta, x(\theta)) \right] > p_i$ for all i, and that an average utility profile $v \in \mathbb{R}^N$ is IR if $v_i > p_i$ for all i.

Since utility is transferable, there is the possibility that even if a mechanism violates the IR constraint for player i, the other players' IR constraints may be slack. Then a transfer of utility to player i from players with slack constraints can yield a new mechanism that satisfies IR and implements the same outcome function x. An outcome function is thus "potentially IR" if it can be implemented by a mechanism satisfying IR, even if not all mechanisms that implement it satisfy IR.

Definition 10. An outcome function x is *potentially individually rational (PIR)* if there exists a perfect Bayesian equilibrium in the stage game with expected payoff profile p such that

$$\sum_i \mathbb{E}_\theta \left[\pi_i(\theta, x(\theta)) \right] > \sum_i p_i \tag{12}$$

for all $i \in \mathcal{N}$. A static mechanism $\langle x, y \rangle$ is PIR if x is PIR.

[10] It is possible to impose punishments close to the minimax (cf. FLM), but the minimax cannot necessarily be identified using the mechanism design approach taken here. The approach here assumes that off-menu deviations are punished by Nash reversion, which is inappropriate for minimax punishments.

4.5 Justifying the mechanism design approach

The following theorem justifies the mechanism design approach to repeated games with private information. The proof, in Appendix A, is conceptually simple. When the players select their recursive mechanism (this selection is not modeled), they are collectively acting as their own mechanism designer. Their recursive mechanism specifies the equilibrium path they should follow; incentive compatibility discourages on-menu deviations while the trigger punishment discourages off-menu deviations. Hence a feasible, IR, and IIC recursive mechanism describes an equilibrium path and implies the existence of a PPE that supports that path. In the other direction, the equilibrium path of most any PPE can be described by a history-dependent dynamic mechanism. Similarly to the logic of Abreu, Pearce, and Stacchetti (1986, 1990), it is without loss of generality with respect to attainable payoffs to substitute a recursive mechanism for a dynamic mechanism.

Theorem 1. *Let $p \in \mathbb{R}^N$ be the expected utility profile of some stage game perfect Bayesian equilibrium. For any $\varepsilon > 0$, there exists $\underline{\delta} < 1$ such that, for all $\delta > \underline{\delta}$:*

(i) If a feasible, IR, and IIC recursive mechanism $\langle \mathcal{V}, \{\langle x, t, w\rangle (\cdot; v)\}, v^0\rangle$ satisfies $v_i > p_i + \varepsilon$ for all $i \in \mathcal{N}$ and all $v \in \mathcal{V}$, then there exists a PPE that yields the same announcements, actions, and net transfers along the equilibrium path;

(ii) If $\mathcal{V} \subset \mathbb{R}^N$ is the set of average utility profiles yielded at the beginning of any period along the equilibrium path of some PPE, and $v_i > p_i + \varepsilon$ for all $i \in \mathcal{N}$ and all $v \in \mathcal{V}$, then for any $v \in \mathcal{V}$ there exists a feasible, IR, and IIC recursive mechanism with initial promised utility v.

5 Budget balanced static mechanisms

Leave aside for a moment the repeated game structure, and consider the problem of designing static mechanisms that satisfy both IIC and EPBB. It is widely known that in many situations efficient IIC static mechanisms can be implemented with EPBB if IR constraints are ignored (see, e.g., d'Aspremont and Gérard-Varet 1979, Johnson, Pratt, and Zeckhauser 1990, and Fudenberg, Levine, and Maskin 1995). Most generally, ACGVb showed that, in games with finite type spaces, any IIC-implementable outcome function can be implemented with EPBB if and only if their "Condition C" is satisfied.

Note that an EPBB mechanism yields payoff profiles that lie in a hyperplane with normal vector $(1, \ldots, 1)$. For a game with transferable utility, such hyperplanes are parallel to the Pareto frontier. In a game without transferable utility, however, the convex hull of attainable payoffs may have any arbitrary convex shape. Since the folk theorem for games without transferable utility relies on continuation rewards that lie in tangent hyperplanes, it is useful to define a stronger condition so that ACGVb's result can be extended to (almost) any arbitrary hyperplane.

Definition 11. A game satisfies *Strong Condition C* if, for any $\lambda \in [-1,1]^N$ with $\lambda_i \neq 0$ for at least three players (or for both players if $N = 2$), any function $R : \Theta \to \mathbb{R}$ can be expressed as $R(\theta) = \sum_i \lambda_i r_i(\theta)$, where $r : \Theta \to \mathbb{R}^N$ satisfies

$$\mathbb{E}_{\theta_{-i}}\left[r_i(\theta)\middle|\theta_i\right] \geq \mathbb{E}_{\theta_{-i}}\left[r_i(\hat{\theta}_i, \theta_{-i})\middle|\theta_i\right] \tag{13}$$

for ϕ_i-almost all $\theta_i \in \Theta_i$, for all $\hat{\theta}_i$ in the support of ϕ_i, and for all $i \in \mathcal{N}$. A game satisfies *Condition C* if these requirements are satisfied for $\lambda = (1, \ldots, 1)$.

Lemma 1. *If Strong Condition C is satisfied and the static mechanism $\langle x, y\rangle$ is IIC and uniformly bounded, then, for any $\lambda \in [-1,1]^N$ with $\lambda_i \neq 0$ for at least three players (or for both players if $N = 2$), there exists a static mechanism $\langle x, y^\lambda\rangle$ that is IIC, uniformly bounded, and satisfies $\sum_i \lambda_i y_i^\lambda(\theta) = 0$ for all θ. For $\lambda = (1, \ldots, 1)$, this implies EPBB.*

Proof. The proof follows the logic of ACGVb's Lemma 1. Suppose that Strong Condition C is satisfied, and that the static mechanism $\langle x, y\rangle$ is IIC and uniformly bounded. Given λ as described, let $R = -\sum_i \lambda_i y_i$. Note that since y is uniformly bounded, R is uniformly bounded. Then by Strong Condition C there exists a uniformly bounded function $r : \Theta \to \mathbb{R}^N$ such that, for all $\theta \in \Theta$, $\sum_i \lambda_i r_i(\theta) = R(\theta)$, and such that, for all $i \in \mathcal{N}$, for almost all $\theta_i \in \Theta_i$, and for all $\hat{\theta}_i \in \Theta_i$, $\mathbb{E}_{\theta_{-i}}\left[r_i(\theta)\middle|\theta_i\right] \geq \mathbb{E}_{\theta_{-i}}\left[r_i(\hat{\theta}_i, \theta_{-i})\middle|\theta_i\right]$. Then the mechanism $\langle x, y + r\rangle$ is IIC because, since $\langle x, y\rangle$ is IIC, and by Eq. 13, adding r to y in Eq. 8 yields

$$\mathbb{E}_{\theta_{-i}}\left[\pi_i(\theta, x(\theta)) + y_i(\theta) + r_i(\theta)\middle|\theta_i\right] \geq \mathbb{E}_{\theta_{-i}}\left[\pi_i(\theta, x(\hat{\theta}_i, \theta_{-i})) + y_i(\hat{\theta}_i, \theta_{-i}) + r_i(\hat{\theta}_i, \theta_{-i})\middle|\theta_i\right] \tag{14}$$

for almost all $\theta_i \in \Theta_i$, for all $\tilde{\theta}_i \in \Theta_i$, and for all $i \in \mathcal{N}$. It is also uniformly bounded, since both y and r are uniformly bounded. Finally, by construction $\sum_i \lambda_i \left(y_i(\theta) + r_i(\theta)\right) = 0$. ■

Note that the only implicit requirement for this conclusion is that the conditional expectations of the transfer function are well-defined almost everywhere, which is assured by the assumption that $\langle x, y\rangle$ is IIC. Note also that the same argument would work even without a common prior, although the players' priors must be common knowledge.

Remark 1 (Statistical independence). When $\theta_1, \ldots, \theta_N$ are statistically independent according to ϕ, the conclusion of Lemma 1 is always satisfied, for $\lambda \in [-1,1]^N$ with $\lambda_i \neq 0$ for at least two players. To see this, given an IIC and uniformly bounded mechanism $\langle x, y\rangle$ and

$\lambda \in \mathbb{R}^N$, let

$$y_i^\lambda(\theta) = \begin{cases} \mathbb{E}_{\theta_{-i}}\left[y_i(\theta)\middle|\theta_i\right] - \dfrac{1}{\lambda_i(\hat{N}-1)} \sum_{j\neq i} \lambda_j \mathbb{E}_{\theta_{-j}}\left[y_j(\theta)\middle|\theta_j\right] & \text{if } \lambda_i \neq 0, \\ y_i(\theta) & \text{otherwise,} \end{cases} \tag{15}$$

where $\hat{N}$ is the number of players for which $\lambda_i \neq 0$. Then $\langle x, y^\lambda \rangle$ is uniformly bounded and satisfies $\sum_i \lambda_i y_i^\lambda(\theta) = 0$ for all $\theta \in \Theta$ and for all $i \in \mathcal{N}$. It is also IIC, since (for the case of $\lambda_i \neq 0$) the second term does not depend on θ_i while the first term satisfies

$$\mathbb{E}_{\theta_{-i}}\left[\pi_i(\theta, x(\hat{\theta}_i, \theta_{-i})) + y_i(\hat{\theta}_i, \theta_{-i})\middle|\theta_i\right] = \mathbb{E}_{\theta_{-i}}\left[\pi_i(\theta, x(\hat{\theta}_i, \theta_{-i})) + \mathbb{E}_{\theta_{-i}}\left[y_i(\hat{\theta}_i, \theta_{-i})\middle|\hat{\theta}_i\right]\middle|\theta_i\right] \tag{16}$$

for all $\theta_i \in \Theta_i$, for all $\hat{\theta}_i \in \Theta_i$, and for all $i \in \mathcal{N}$.

Remark 2 (Generality). When $N = 2$, Strong Condition C, like Condition C, can be satisfied only if θ_1 and θ_2 are statistically independent. For $N \geq 3$, ACGVb proves, in a model with finite Θ, that Condition C is satisfied for an open and dense set of probability distributions. Not only is this also the case for Strong Condition C, but also Strong Condition C is "generally" satisfied when Θ is infinite, as Appendix C shows.[11] Specifically, Strong Condition C is satisfied whenever $\theta_1, \ldots, \theta_N$ are either distributed independently or are strictly interdependent over all of Θ. Strong Condition C may be violated when $\theta_1, \ldots, \theta_N$ are locally independent in some regions of Θ and locally interdependent in other regions.

6 Folk theorems

6.1 Folk theorem with transferable utility

I now return to the repeated game context with monetary transfers to prove a folk theorem in three steps. The first step, Lemma 2, shows that if an outcome function can be implemented by an IIC, EPBB, and uniformly bounded static mechanism, then it can be implemented by an IIC and IR mechanism that achieves a fixed budget imbalance across all possible realizations of θ. The second step, Theorem 2, is then to show that the static mechanism in Lemma 2 is equivalent to a stationary recursive mechanism. Finally, Corollary 1 cites Theorem 1 to extend this conclusion to PPEs.

EPBB mechanisms generally violate IR, so it is necessary to show that it is possible to achieve the same ex post sum of transfers after every realization of θ, where the imbalance is high enough to satisfy IR.

[11] There is no clean definition of "genericity" in the space of probability distributions on Θ when Θ may be arbitrary.

Lemma 2. *If a static mechanism $\langle x, y\rangle$ is IIC, EPBB, and uniformly bounded, then for any $\delta \in (0,1)$ there exists some $\underline{C} < \infty$ such that there exists a static mechanism $\langle x, y'\rangle$ that is IIC, IR, uniformly bounded, and satisfies*

$$\sum_i y_i' (\theta) = \underline{C} \tag{17}$$

for all $\theta \in \Theta$. The same is true for all $C \geq \underline{C}$.

Proof. Suppose the mechanism $\langle x, y\rangle$ is IIC, EPBB, and uniformly bounded. To ensure IR, it is necessary to add an additional lump sum transfer

$$y_i' (\theta) = y_i (\theta) + \sup_{\theta_i \in \Theta_i} \mathbb{E}_{\theta_{-i}} \big[p_i - (1-\delta) (\pi_i(\theta, x(\theta)) + y_i(\theta))\big] + e, \tag{18}$$

where $e > 0$, so that $(1-\delta) \mathbb{E}_\theta [\pi_i(\theta, x(\theta)) + y_i'(\theta)] > p_i$ for all $i \in \mathcal{N}$. (Since $\langle x, y\rangle$ is uniformly bounded, the supremum term is finite.) Since this additional term does not vary with θ, the first part of the claim is satisfied:

$$\sum_i y_i' (\theta) = \sum_i \left(\sup_{\theta_i \in \Theta_i} \mathbb{E}_{\theta_{-i}} \big[p_i - (1-\delta) (\pi_i(\theta, x(\theta)) + y_i(\theta))\big] \right) + Ne \equiv \underline{C}. \tag{19}$$

For $C \in (\underline{C}, \infty)$, let

$$y_i'' (\theta) = y_i' (\theta) + \frac{C - \underline{C}}{N}, \tag{20}$$

so that $\langle x, y''\rangle$ satisfies the second part of the claim. ■

The fixed sum C described in Lemma 2 above can be recast in a recursive mechanism as the benefit the players receive from the fact that the trigger punishment is not imposed when they cooperate with the mechanism. This yields a "folk theorem for recursive mechanisms."

Theorem 2. *If a static mechanism $\langle x, y\rangle$ is PIR, IIC, EPBB, and uniformly bounded, then there exists $\underline{\delta} \in (0,1)$ such that, for all $\delta > \underline{\delta}$, there exists a stationary, feasible, IR, and IIC recursive mechanism $\left\langle \mathcal{V}, \{\langle \bar{x}, t, w\rangle (\cdot; v)\}, v^0 \right\rangle$ with $\sum_i v_i^0 = \sum_i \mathbb{E}_\theta [\pi_i (\theta, x(\theta))]$.*

Recall that v_i^0 is a player's average utility in the entire game, while $\mathbb{E}_\theta [\pi_i (\theta, x(\theta))]$ is her payoff in the stage game, not including transfers. The significance of the conclusion is that any level of aggregate payoffs in the stage game that can be implemented by a PIR, IIC, and uniformly bounded static mechanism can also be implemented as average aggregate payoffs in the repeated game by a feasible, IR, and IIC recursive mechanism. This statement is similar to a typical folk theorem except that it is stated in aggregate terms. The conclusion is stated in aggregate terms because if $\mathbb{E}_\theta [\pi_i (\theta, x(\theta))] < p_i$ for player i then his IR constraints will be violated unless he receives recurring payments from the other players.

Proof. The mechanism I construct is stationary on the equilibrium path; i.e., it uses the same stage mechanism in every period. This is a consequence of Lemma 2, because a fixed budget imbalance implies that all zero-sum payments required for IIC can be settled using monetary transfers in the current period, so there need be no transfers of future utility from one player to another.

Suppose the mechanism $\langle x, y\rangle$ is IIC, EPBB, PIR, and uniformly bounded. I begin by constructing the stage mechanisms $\{\langle \bar{x}, t, w\rangle (\cdot; v)\}$ and establishing that they satisfy IR and IIC for sufficiently high δ. By Lemma 2, there exists $\underline{C} \in (0, \infty)$ such that, for any $C \geq \underline{C}$, there exists a static mechanism $\langle x, y^C\rangle$ that is IIC, IR, uniformly bounded, and satisfies $\sum_i y_i^C (\theta) = C$ for all $\theta \in \Theta$. Define

$$\delta^* (C) = \frac{C}{C + \sum_i \mathbb{E}_\theta \left[\pi_i (\theta, x (\theta))\right]} < 1, \tag{21}$$

where the strict inequality is satisfied because π_i is non-negative–valued for all i and x is PIR. Given any $C \geq \underline{C}$, for each player i define

$$\bar{x}_i (\theta; v) = x_i (\theta) \tag{22}$$

$$t_i (\theta; v) = y_i^C (\theta) - \delta^*(C) \, \mathbb{E}_\theta \left[\pi_i (\theta, x (\theta)) + y_i^C (\theta)\right] \tag{23}$$

$$w_i (\theta; v) = (1 - \delta^*(C)) \, \mathbb{E}_\theta \left[\pi_i (\theta, x (\theta)) + y_i^C (\theta)\right] \tag{24}$$

for all $\theta \in \Theta$ and any v. Note that by construction $\sum_i t_i (\theta; v) = 0$ for all $\theta \in \Theta$ and all $v \in \mathbb{R}^N$. Observe that, for any v, $\langle x, y^C\rangle$ is the equivalent static mechanism for $\langle \bar{x}, t, w\rangle (\cdot; v)$:

$$\begin{aligned} t_i (\theta) + & \frac{\delta^*(C)}{1 - \delta^*(C)} w_i (\theta) \\ & = y_i^C (\theta) - \left(\delta^*(C) - \frac{\delta^*(C)}{1 - \delta^*(C)} (1 - \delta^*(C))\right) \mathbb{E}_\theta \left[\pi_i (\theta, x (\theta)) + y_i^C (\theta)\right] \\ & = y_i^C (\theta) . \end{aligned} \tag{25}$$

Hence, given $\delta^*(C)$, $\langle \bar{x}, t, w\rangle (\cdot; v)$ is IR and IIC for any v. For any $\delta \in [\delta^*(\underline{C}), 1)$, simply choose $C = C^*(\delta)$, where $C^*(\delta)$ is defined as the inverse of $\delta^*(C)$.

Second, I construct the recursive mechanism $\left\langle \mathcal{V}, \{\langle \bar{x}, t, w\rangle (\cdot; v)\}, v^0\right\rangle$ and show that it is feasible. Given $\delta \in [\delta^*(\underline{C}), 1)$, choose $v^0 = (1 - \delta) \, \mathbb{E}_\theta \left[\pi_i (\theta, x (\theta)) + y_i^{C^*(\delta)} (\theta)\right]$ and $\mathcal{V} = \left\{v^0\right\}$. Observe that the coherence condition for feasibility is satisfied, since $w \left(\theta; v^0\right) = v^0 \in \mathcal{V}$ for all θ. The promise keeping condition for feasibility is also satisfied:

$$\begin{aligned} v^0 &= (1-\delta)\ \mathbb{E}_\theta \left[\pi_i\left(\theta, x\left(\theta\right)\right) + y_i^{C^*(\delta)}\left(\theta\right)\right] \\ &= (1-\delta)\ \mathbb{E}_\theta \left[\pi\left(\theta, x\left(\theta\right)\right) + t\left(\theta; v^0\right) + \frac{\delta}{1-\delta} w\left(\theta; v^0\right)\right] \\ &= (1-\delta)\ \mathbb{E}_\theta \left[u(\theta,\theta;\delta,\langle x,t,w\rangle)\right] \end{aligned} \tag{26}$$

for all $\theta \in \Theta$ and all $v \in \left\{v^0\right\}$. The aggregate value of the recursive mechanism is as claimed:

$$\begin{aligned} \sum_i v_i^0 &= (1-\delta)\left(\sum_i \mathbb{E}_\theta\left[\pi_i\left(\theta, x\left(\theta\right)\right)\right] + C^*(\delta)\right) \\ &= \frac{\sum_i \mathbb{E}_\theta\left[\pi_i\left(\theta, x\left(\theta\right)\right)\right]}{C^*(\delta) + \sum_i \mathbb{E}_\theta\left[\pi_i\left(\theta, x\left(\theta\right)\right)\right]}\left(\sum_i \mathbb{E}_\theta\left[\pi_i\left(\theta, x\left(\theta\right)\right)\right] + C^*(\delta)\right) \\ &= \sum_i \mathbb{E}_\theta\left[\pi_i\left(\theta, x\left(\theta\right)\right)\right]. \end{aligned} \tag{27}$$

This also implies that the attainability condition of feasibility is satisfied, so the desired conclusion holds, with $\underline{\delta} = \delta^*(\underline{C})$. ■

Theorem 1 states that, for δ sufficiently high, any feasible, IR, and IIC recursive mechanism yields messages, actions, and net transfers that can be supported as the equilibrium path of a PPE. Hence Theorem 2 implies a folk theorem for PPE.

Corollary 1 (Folk theorem with transfers)**.** *Suppose that (i) the static mechanism $\langle x, y\rangle$ is IIC, PIR, and uniformly bounded, and (ii) either $\langle x, y\rangle$ is EPBB or Condition C is satisfied. Then there exists $\underline{\delta} \in (0,1)$ such that, for all $\delta > \underline{\delta}$, there exists a stationary PPE that yields aggregate average utility of $\sum_i \mathbb{E}_\theta\left[\pi_i\left(\theta, x\left(\theta\right)\right)\right]$.*

Proof. Suppose that Condition C is satisfied, and the mechanism $\langle x, y\rangle$ is IIC, PIR, and uniformly bounded. By Lemma 1, $\langle x, y\rangle$ is also EPBB without loss of generality. Then, by Theorem 2, for any $\delta \geq \delta^*(\underline{C})$ there exists a feasible, IR, and IIC recursive mechanism $\left\langle \mathcal{V}, \left\{\langle \bar{x}, t, w\rangle\left(\cdot; v\right)\right\}, v^0\right\rangle$ with $\sum_i v_i^0 = \sum_i \mathbb{E}_\theta\left[\pi_i\left(\theta, x\left(\theta\right)\right)\right]$. By Theorem 1, there exists some $\hat{\underline{\delta}} \in (0,1)$ such that, for any $\delta > \hat{\underline{\delta}}$, for any feasible, IR, and IIC recursive mechanism there exists a PPE that yields the same messages, outcomes, and net transfers on the equilibrium path. Choose $\underline{\delta} = \max\{\hat{\underline{\delta}}, \delta^*(\underline{C})\}$ to obtain the conclusion. ■

Remark 3. We can calculate a value for $\hat{\underline{\delta}}$ (as in the proof above) as follows. Since π is uniformly bounded, there exists an upper bound D on the payoff of the best possible off-menu deviation strategy for any player in a single stage of the PPE along the equilibrium path. Since x is PIR, there exists a stage game equilibrium with payoff vector p that can be used as a grim trigger punishment. Let $\delta = \hat{\underline{\delta}}$ solve

$$\delta\,\mathbb{E}_\theta\left[\pi_i\left(\theta, x\left(\theta\right)\right) + y_i\left(\theta\right)\right] = \left(1-\delta\right)\,D + \delta p_i, \tag{28}$$

where i is the player with the best possible deviation. Because $D < \infty$, the solution will satisfy $\hat{\underline{\delta}} \in (0,1)$, and no deviation from the equilibrium path is profitable when $\delta > \hat{\underline{\delta}}$.

6.2 Folk theorem without transferable utility

Allowing monetary transfers greatly simplifies the proof of the folk theorem, but in many important settings there may not be opportunities to transfer money. For instance, in an industry attempting to collude illegally, monetary transfers may invite scrutiny from antitrust authorities, imposing an expected cost on the firms in the industry. In such cases, the players would like to be able to use changes in continuation rewards to substitute for monetary transfers. In the case of collusion in a price-setting market with a finite number of cost types, this can take the form of what Athey and Bagwell (2001) call "market share favors." In general, without monetary transfers it is not possible to design stationary equilibria except for trivial or knife-edge cases, because a range of outcome functions may be necessary in subsequent periods to provide a range of continuation rewards in order to support incentive compatibility for the outcome function in the current period.

To describe recursive mechanisms and PPEs without monetary transfers, simply let $t_i(\theta) = 0$ for all $\theta \in \Theta$ and all $i \in \mathcal{N}$, and then drop t from all definitions. Most definitions then carry over unchanged, except for feasibility. Without monetary transfers, the set of attainable utilities is limited, since transfers of utility from one player to another must take place through variations in the outcome functions employed along the equilibrium path, rather than through money.

Definition 12. Given $\delta \in (0,1)$, a recursive mechanism without monetary transfers $\left\langle \mathcal{V}, \left\{\langle x, w\rangle(\cdot; v)\right\}, v^0\right\rangle$ is *feasible* if the following are satisfied:

(i) Attainability: $\mathcal{V} \subset \mathrm{co}\left(\left\{v \in \mathbb{R}^N : v = \mathbb{E}_\theta\left[\pi\left(\theta, x(\theta)\right)\right] \text{ for some } x : \Theta \to \mathcal{X}\right\}\right)$;

(ii) Promise keeping: $v = (1-\delta)\,\mathbb{E}_\theta\left[u(\theta, \theta; \delta, \langle x, w\rangle\left(\cdot; v\right))\right]$ for all $v \in \mathcal{V}$;

(iii) Coherence: $w\left(\theta; v\right) \in \mathcal{V}$ for all $\theta \in \Theta$ and all $v \in \mathcal{V}$.

Among all initial values that can be implemented by a feasible recursive mechanism without monetary transfers, the goal is to find a subset that can be supported by a PPE. As in FLM, the set I identify will be "smooth."

Definition 13. A set $\mathcal{W} \subset \mathbb{R}^N$ is *smooth* if it is closed and convex, has non-empty interior, and its boundary is a twice differentiable submanifold of $\mathbb{R}^N$.

FLM's Theorem 4.1 shows that, in games with finite type spaces, any smooth subset of the set of attainable stage game payoff profiles consists entirely of equilibrium values (for high enough $\delta < 1$) if it is "decomposable on tangent hyperplanes" and it offers at least the minimax payoff to each player. In order to apply this theorem, we must confirm that the conclusions of FLM's Theorem 4.1 apply to arbitrary type spaces. In the context of the mechanism design approach, this first requires that we restrict attention to payoff profiles that are individually rational, because off-menu deviations are punished by reversion to the stage game equilibrium. Second, since under Strong Condition C it may be that decomposability fails on isolated hyperplanes, we must allow for such failures. Finally, we also need an extra condition that all IIC mechanisms used in the construction must be uniformly bounded. The validity of FLM, Theorem 4.1 under these conditions is proven in Appendix B.

Let $\mathcal{G}$ be the set of static mechanisms that are IIC and uniformly bounded.[12] Let $\mathcal{P}^*$ be the set of payoff profiles implemented by mechanisms in $\mathcal{G}$ and that are IR with respect to some stage game perfect Bayesian equilibrium:

$$\begin{aligned} \mathcal{P}^* \equiv \text{co}\left(\left\{v \in \mathbb{R}^N : \exists \langle x, y\rangle \in \mathcal{G} \text{ s.t. } v = \mathbb{E}_\theta\left[\pi\left(\theta, x(\theta)\right)\right]\right\}\right) \\ \cap \left\{v \in \mathbb{R}^N : v_i > p_i \; \forall i \in \mathcal{N}\right\}, \end{aligned} \tag{29}$$

where p is the payoff profile of the stage game equilibrium.

Theorem 3. *Suppose Strong Condition C is satisfied. If $\mathcal{W}$ is a smooth subset of the interior of $\mathcal{P}^*$, then for any v on the interior of $\mathcal{W}$ there exists $\underline{\delta} \in (0,1)$ such that, for all $\delta > \underline{\delta}$, v is an average payoff profile of a PPE without monetary transfers.*

Proof. By Theorem 4 (in Appendix B), it suffices to show that $\mathcal{W}$ is boundedly decomposable on almost tangent hyperplanes (Definition 14 in Appendix B). That is, it suffices to show that, for any v on the boundary of $\mathcal{W}$, there exists an outcome function x such that (i) $\mathbb{E}_\theta[\pi(\theta, x(\theta))]$ is separated from $\mathcal{W}$ by the hyperplane $\mathcal{L}_v$ that is tangent to $\mathcal{W}$ at v; and (ii) for any $\delta \in (0,1)$, there exists a continuation reward function w such that $\left\langle x, \frac{\delta}{1-\delta} w\right\rangle$ is IIC and $\{w(\theta)\}_{\theta \in \Theta}$ is a bounded subset of either $\mathcal{L}_v$ or a hyperplane arbitrarily close to it.

For (i), note that $\mathcal{W}$ is a smooth subset of the interior of $\mathcal{P}^*$. Hence for any v on the boundary of $\mathcal{W}$ there exists a point in $\left\{\mathbb{E}_\theta\left[\pi(\theta, x^\ell(\theta))\right]\right\}$ that is separated from $\mathcal{W}$ by $\mathcal{L}_v$.

[12] Again, the bound applies to the transfer function in each mechanism individually; each mechanism may feature a different uniform bound.

For (ii), since x (selected as in (i)) can, by definition of $\mathcal{P}^*$, be chosen so as to be implementable by a uniformly bounded IIC mechanism, by setting $\hat{\underline{\delta}} > 0$ we assure that $\{w(\theta)\}_{\theta\in\Theta} = \left\{\frac{1-\delta}{\delta}y(\theta)\right\}_{\theta\in\Theta} \subset \mathbb{R}^N$ is bounded. Recall from Lemma 1 that under Strong Condition C, for almost any hyperplane $\mathcal{L}$, *any* outcome function that can be implemented by a uniformly bounded IIC mechanism $\langle x, y\rangle$ can also be implemented by a uniformly bounded IIC mechanism $\langle x, y^{\mathcal{L}}\rangle$ with continuation values confined to $\mathcal{L}$. Since $y^{\mathcal{L}} = \frac{\delta}{1-\delta}w^{\mathcal{L}}$, if $y^{\mathcal{L}}$ is uniformly bounded by B, then $\{w^{\mathcal{L}}(\theta)\}_{\theta\in\Theta}$ is bounded by $\frac{1-\delta}{\delta}B$ for any $\delta \in (0, 1)$. ■

The conclusions of this theorem are limited in several ways in comparison to the case with monetary transfers. First, there is no stationarity, since changes in continuation rewards are used in place of monetary transfers. Second, this theorem does not imply that it is possible to attain a level of aggregate utility yielded by an outcome function that is merely PIR without being IR. This is because monetary transfers cannot be used to shift utility to the player whose IR constraint is violated, and so that player cannot be induced to cooperate with the proposed path of play. Third, since $\mathcal{W}$ must be a smooth set, $\mathcal{P}^*$ must have non-empty interior for the the conclusion to be substantive. Fourth, since $\mathcal{W}$ must be a smooth set, the stage game payoff profile of an IIC-implementable and IR outcome function can be approximated by a PPE—and the approximation can be arbitrarily close if δ may be arbitrarily close to 1—but it cannot necessarily be achieved exactly. Fifth, Strong Condition C is invoked, because for the construction in the proof it is not sufficient that a mechanism be EPBB—it must be decomposable on the particular hyperplane tangent to $\mathcal{W}$ at v (or one arbitrarily close to it). When Strong Condition C is not satisfied, there is no guarantee that EPBB implies decomposability on a relevant hyperplane. The following corollary summarizes this discussion and provides a statement of the folk theorem that is comparable to Corollary 1. The proof is omitted.

Corollary 2 (Folk theorem without transfers)**.** *Suppose that (i) Strong Condition C is satisfied; (ii) a static mechanism $\langle x, y\rangle$ is IIC, IR, and uniformly bounded; and (iii) $\mathcal{P}^*$ has non-empty interior. Then for any $\varepsilon > 0$ there exists $\underline{\delta} \in (0, 1)$ such that, for all $\delta > \underline{\delta}$, there exists a PPE without monetary transfers that yields aggregate average utility v^0 with $\left\|\sum_i \mathbb{E}_\theta\left[\pi_i\left(\theta, x(\theta)\right)\right] - v^0\right\| < \varepsilon$.*

7 Discussion

This paper presents two folk theorems for games with private information and communication. Based on the work of Abreu, Pearce, and Stacchetti (1986, 1990), FLM, Athey and Bagwell (2001), and others, I adapt static mechanism design definitions and techniques to the analysis of repeated game equilibria. This mechanism design approach greatly simplifies the description of equilibria, by focusing on the equilibrium path and ignoring off-menu deviations. My contribution to the methodology developed in this literature is to generalize and formalize the approach for repeated games with private information and communication.

This mechanism design approach not only simplifies the description of equilibria, but also leads to a simpler and more broadly applicable extension of FLM, Theorem 8.1. Rather than adapt results from games with hidden action, where the statistical identifiability of actions is of serious concern, it is simpler to directly prove that under Strong Condition C (an extension of ACGVa and ACGVb's Condition C), IIC-implementability implies enforceability with respect to almost any hyperplane. This direct approach allows the folk theorem to be extended to games with interdependent signals, interdependent payoffs, and arbitrary signal spaces.

Along the road to the folk theorem, I proved that Strong Condition C is generally satisfied under global interdependence when the signal space may be arbitrary—a result that in its own right is a contribution to the study of static mechanism design.

Finally, note that the notions of IIC and IR that I employ are "almost sure" in the sense of Balder (1996). It is not necessary to impose stronger conditions under the assumption (heretofore maintained implicitly) that there exists some perfect Bayesian equilibrium in the continuation game after any event that some $\theta \in \Theta$ is realized on which IIC or IR fails for some player. This is consistent with the construction of a perfect Bayesian equilibrium in the repeated game because failures of IR or IIC occur with zero probability, and so affect neither interim incentives nor the value of the recursive mechanism.

Appendix A Justifying the mechanism design approach

Proof of Theorem 1 (page 14). To prove part (i), suppose that $G = \big\langle \mathcal{V}, \big\{\langle x, t, w\rangle(\cdot; v)\big\}, v^0 \big\rangle$ is a feasible, IR, and IIC recursive mechanism given δ. Construct a strategy profile σ as follows. First, consider any public history H^τ that contains no off-menu deviations with respect to G. Let $\{\sigma_i^{\tau+1}(\cdot; H^\tau)\}_{i=1}^N$ specify a pure public strategy profile such that, for all $i \in \mathcal{N}$ and all $j = 2, \ldots, N-1$,

$$\hat{m}_i(\theta_i; H^\tau) = \theta_i, \tag{30}$$

$$\hat{x}_i(\theta_i, \theta; H^\tau) = x_i(\theta; v(H^\tau; \sigma)), \tag{31}$$

$$\hat{t}_{1,2}\big(\theta_1, \theta, \hat{x}_i(\theta_i, \theta; H^\tau); H^\tau\big) = -t_1(\theta; v(H^\tau; \sigma)), \tag{32}$$

$$\hat{t}_{j,j+1}\big(\theta_j, \theta, \hat{x}_i(\theta_i, \theta; H^\tau); H^\tau\big) = -t_j(\theta; v(H^\tau; \sigma)) + \hat{t}_{j-1,j}\big(\theta_{j-1}, \theta, \hat{x}_i(\theta_i, \theta; H^\tau); H^\tau\big), \tag{33}$$

with $\hat{t}_{i,j}\big(\theta_i, \theta, \hat{x}_i(\theta_i, \theta; H^\tau); H^\tau\big) = 0$ for any $(i, j) \in \mathcal{N} \times \mathcal{N}$ not specified above.

In addition, for any consistent beliefs following the realization of any message profile $\mu \notin \Theta$, choose $\hat{t}(\theta, \mu, \hat{x}(\theta, \mu; H^\tau); H^\tau) = (0, \ldots, 0)$ and choose $\hat{x}(\theta, \mu; H_i^\tau)$ so as to constitute a perfect Bayesian equilibrium in the remainder of the stage game (assume that such an equilibrium exists); and for any consistent beliefs following the realization of any $\chi \neq \hat{x}(\theta, \mu; H^\tau)$, choose $\hat{t}(\theta, \mu, \chi; H^\tau) =$

$(0, \ldots, 0)$ so as to constitute a perfect Bayesian equilibrium in the remainder of the stage game.

Next, consider any public history H^τ off the equilibrium path. Since G is feasible, there exists $\varepsilon > 0$ and a perfect Bayesian equilibrium s^* in the stage game that yields an expected average payoff $p_i < v_i - \varepsilon$ for all $i \in \mathcal{N}$ and all $v \in \mathcal{V}$. After any off-menu deviation by any player, implement any perfect Bayesian equilibrium in what remains of the period in which the deviation occurred (as specified in the previous paragraph), and implement s^* in all subsequent periods.

This strategy profile yields an equilibrium path equivalent to that of G by construction, and it yields a PPE in every subgame after off-menu deviations since it specifies that the players play a stage game perfect Bayesian equilibrium in every stage without regard to history. Note that off-menu deviations are followed by a trigger punishment that imposes a net penalty of at least $\frac{\delta}{1-\delta}\varepsilon$. Since the most profitable possible deviation from any equilibrium path is uniformly bounded by some $D < \infty$, if $\delta > \frac{D}{D+\varepsilon}$ then no off-menu deviation is profitable. On-menu deviations are also unprofitable, because G is IIC. Every subgame after an on-menu deviation is followed by a subgame that is equivalent to equilibrium path subgames. Since the proposed strategies do not depend on private histories, they form a PPE in the repeated game.

To prove part (ii), define a *dynamic mechanism* $\Gamma = \{\langle x, t\rangle(\cdot; H^\tau)\}$ as a collection of pairs $\langle x, t\rangle$ indexed by H^τ. Let $v(H^\tau)$ be the vector of average utilities provided by Γ in the continuation game after history H^τ. Given $\delta \in (0, 1)$, I say that Γ is IIC if the stage mechanism $\big\langle x(\cdot; H^\tau), t(\cdot; H^\tau), \mathbb{E}_\theta\big[v(H^{\tau+1})\big|\cdot; H^\tau\big]\big\rangle$ is IIC for all equilibrium path public histories H^τ. I say that Γ is IR if there exists $\varepsilon > 0$ such that

$$(1-\delta)\ \mathbb{E}_\theta\left[u_i(\theta, \theta; \delta, \big\langle x(\cdot; H^\tau), t(\cdot; H^\tau), \mathbb{E}_\theta\big[v(H^{\tau+1})\big|\cdot; H^\tau\big]\big\rangle)\right] > p_i + \varepsilon. \tag{34}$$

for all $i \in \mathcal{N}$ and all equilibrium path public histories H^τ.

Fix $\varepsilon > 0$. I begin by showing that there exists $\underline{\delta} \in (0, 1)$ such that, for all $\delta > \underline{\delta}$, if a PPE σ yields, after any equilibrium path history H^τ, an average utility profile $\hat{v}(H^\tau; \sigma)$ such that $\hat{v}_i(H^\tau; \sigma) > p_i + \varepsilon$ for all $i \in \mathcal{N}$, then there exists an IIC dynamic mechanism that yields the same announcements, actions, and net transfers on the equilibrium path. Suppose $\sigma = \{\langle \hat{m}, \hat{x}, \hat{t}\rangle(\cdot; H^\tau)\}$ is a pure strategy PPE satisfying these conditions. Construct a dynamic mechanism Γ as follows. Consider any public history H^τ on the equilibrium path. For all $i \in \mathcal{N}$, let

$$x_i(\theta; H^\tau) = \hat{x}_i(\theta_i, \hat{m}(\theta; H^\tau); H^\tau), \tag{35}$$

$$t_i(\theta; H^\tau) = \sum_{j \neq i} \hat{t}_{j,i}\big(\theta_j, \hat{m}(\theta; H^\tau), x(\theta; H^\tau); H^\tau\big) - \sum_{j \neq i} \hat{t}_{i,j}\big(\theta_i, \hat{m}(\theta; H^\tau), x(\theta; H^\tau); H^\tau\big). \tag{36}$$

Although Γ is equivalent to σ in terms of equilibrium path actions and net transfers, the players may have additional information at the time they choose actions and transfers in any period, since true signals are announced publicly. Nevertheless, since π is uniformly bounded, there exists some $\underline{\delta} \in (0, 1)$ at which the most profitable off-menu deviation for any player i in any stage game is exactly balanced by the threat of the trigger punishment, by the same reasoning as in part (i). When $\delta > \underline{\delta}$ no player can can profit from such a deviation, so Γ is IR. Since the same actions, net transfers, and continuation rewards result from truthful announcements as under σ, no on-menu deviation is profitable, so Γ is IIC. This argument extends in a natural way to mixed strategy PPEs, using randomized dynamic mechanisms.

Next I show that there exists a feasible, IIC, and IR recursive mechanism $\langle \mathcal{V}, \{\langle x, t, w\rangle (\cdot; v)\}, v^0\rangle$ such that $v^0 = \hat{v}(H^0)$. Note that for each equilibrium path public history H^τ, the portion of Γ that applies following H^τ is itself an IIC and IR dynamic mechanism. Thus the promised utility $\hat{v}(H^\tau)$ is supported by a range of continuation rewards generated by dynamic mechanisms; let $w(\cdot; H^\tau)$ be the mapping from announcements to continuation rewards. To convert Γ to a recursive mechanism that yields the same value, first construct the set $\mathcal{V} = \{\hat{v}(H^\tau)\}$ over all equilibrium path public histories H^τ. For each $\tilde{v} \in \mathcal{V}$, select from among the histories $\{H^\tau : \hat{v}(H^\tau) = \tilde{v}\}$ an arbitrary history, and call it $H^*(\tilde{v})$. Then construct a recursive mechanism G by setting $\langle x, t, w\rangle(\cdot; v) = \big\langle x(\cdot; H^*(v)), t(\cdot; H^*(v)), w(\cdot; H^*(v))\big\rangle$ for all $v \in \mathcal{V}$ and setting $v^0 = \hat{v}(H^0)$. G is IR and IIC because, for each $v \in \mathcal{V}$, it specifies an action function, transfer function, and continuation reward function that offer incentives that are identical to the incentives offered by some dynamic mechanism drawn from an equilibrium path subgame of Γ. Finally, G is feasible by construction. ■

Appendix B Extending FLMa Theorem 4.1

First, some notation used in this section. Given a set $\mathcal{W} \subset \mathbb{R}^N$, let $\text{bd}(\mathcal{W})$ be the boundary of $\mathcal{W}$, and let $\text{int}(\mathcal{W})$ be the interior of $\mathcal{W}$. Suppose $\mathcal{W}$ is smooth and $\text{int}(\mathcal{W}) \neq \varnothing$; then, given a point $v \in \text{bd}(\mathcal{W})$, let ι_v be the unit vector orthogonal to $\mathcal{W}$ at v. Let $\mathcal{L}_v \equiv \{w \in \mathbb{R}^N : \iota_v \cdot w = 0\} + v$; i.e., the hyperplane tangent to $\mathcal{W}$ at v.

Definition 14. A set $\mathcal{W} \subset \mathbb{R}^N$ is *boundedly decomposable on tangent hyperplanes* if, for every point $v \in \text{bd}(\mathcal{W})$ there exists an outcome function x such that (i) $\mathbb{E}_\theta[\pi(\theta, x(\theta))]$ is separated from $\mathcal{W}$ by the hyperplane $\mathcal{L}_v$ that is tangent to $\mathcal{W}$ at v; and (ii) for any $\delta \in (0, 1)$, v is decomposable with respect to x, a bounded subset of $\mathcal{L}_v$, and δ.

$\mathcal{W} \subset \mathbb{R}^N$ is *boundedly decomposable on almost tangent hyperplanes* if, for every point $v \in \text{bd}(\mathcal{W})$, there exists an outcome function x such that (i) above is satisfied, and either (ii) above is satisfied or (ii′) for any $\bar{\varepsilon} > 0$ sufficiently small, there exists $\varepsilon \in (0, \bar{\varepsilon})$, $\delta \in (0, 1)$, and a unit vector ξ orthogonal to ι_v such that v is decomposable with respect to x, δ, and a bounded subset of the hyperplane $\{w \in \mathbb{R}^N : (\iota_v + \varepsilon\xi) \cdot w = 0\}$.

Theorem 4 (cf. FLM Theorem 4.1)**.** *If a smooth set $\mathcal{W} \subset \mathcal{P}^*$ is boundedly decomposable on almost tangent hyperplanes, then it is locally self decomposable. Hence there exists $\underline{\delta} \in (0, 1)$ such that, for all $\delta > \underline{\delta}$, every point in $\mathcal{W}$ is supported as the average payoffs of a PPE.*

Proof. FLM Theorem 4.1 relies on two lemmas, which I restate here.

> **FLM, Lemma 4.2** If $\mathcal{W} \subset \mathbb{R}^N$ is compact, convex, and locally self-decomposable,[13] then there exists $\underline{\delta} \in (0, 1)$ such that, for all $\delta > \underline{\delta}$, every point in $\mathcal{W}$ is supported by a PPE.

[13] (FLM, Definition 4.2) A subset $\mathcal{W} \subset \mathbb{R}^N$ is *locally self-decomposable* if for each $v \in \mathcal{W}$ there is a $\delta < 1$ and an open set $\mathcal{U}$ containing v such that each point in $\mathcal{U} \cap \mathcal{W}$ is decomposable with respect to $\mathcal{W}$ and δ.

FLM, Lemma 4.3 Suppose $\mathcal{L}$ is a hyperplane in $\mathbb{R}^N$ containing the origin. If x is enforceable with respect to $\mathcal{L}$ and some δ, then (i) x is enforceable with respect to any translate $\mathcal{L}' = v' + \mathcal{L}$ and any $\delta' > 0$; and (ii) there is a constant κ such that, for all δ' and $v' \in \mathbb{R}^N$, there exists a continuation reward function $w' : \Theta \to \mathcal{L}'$ such that $\{w(\theta)\}_{\theta\in\Theta} \subset \mathcal{E}'$ enforces x with respect to v' and δ', $v' = \mathbb{E}_\theta [w(\theta)]$, and $\|w(\theta) - v'\| < \kappa \left(1 - \delta'\right) / \delta'$ for all $\theta \in \Theta$.

The first task is to translate between from language of FLM, which was designed for games with imperfect public monitoring, to the language of the present paper. FLM shows that the public monitoring framework can also encompass games with hidden information, where the private action each player takes is to choose a mapping from his private signal to his public announcement; the public monitoring signal is the vector of realized announcements. Below I will show that the trigger punishment that is imposed after off-menu deviations need not be considered part of $\mathcal{W}$. Hence we can interpret "w enforces x with respect to δ" to mean that $\left\langle x, \frac{\delta}{1-\delta} w\right\rangle$ is IIC; and "v is decomposable with respect to x, $\mathcal{W}$, and δ" to mean that there exists w such that $\left\langle x, \frac{\delta}{1-\delta} w\right\rangle$ is IIC, $v = \mathbb{E}_\theta[(1-\delta)\pi(\theta, x(\theta)) + \delta w(\theta)]$, and $w(\theta) \in \mathcal{W}$ for all $\theta \in \Theta$.

For simplicity I use a trigger punishment for off-menu deviations, whereas FLM allows more general punishments so that minimax payoffs can be approximated along the equilibrium path. Hence, I must restrict the conclusion of the theorem to payoffs that are individually rational with respect to the trigger punishment. Thus the statement of the theorem restricts $\mathcal{W}$ to be a subset of $\mathcal{P}^* \subset \mathbb{R}^N$. Clearly, under this restriction the payoff profile of the trigger punishment cannot be contained in $\mathcal{W}$. But since $\mathcal{W}$ is compact, there exists a minimum "average penalty" $\min_{v\in\mathcal{W}} [v_i - p_i] > 0$ that is imposed on any player i by the trigger punishment when, in the absence of deviation, v_i would be player i's average continuation payoff. In total terms, there exists $\delta < 1$ sufficiently high that the penalty outweighs the benefit of any off-menu deviation. Thus, the conclusion of FLM, Lemma 4.2 still holds for this restricted $\mathcal{W}$ once we redefine decomposability as above.

The only remaining differences between the FLM framework and mine are that I do not restrict the signal space to be finite, and that I allow decomposability on *almost* tangent hyperplanes.[14] Hence it remains only to show that FLM Lemmas 4.2 and 4.3 and Theorem 4.1 continue to hold under these conditions.

- FLM Lemma 4.2 does not invoke finiteness of the signal space. Instead, it relies on the assumptions of compactness and local self-decomposability, which are maintained here. The proof uses the fact that $\mathcal{W}$ is self-decomposable to ensure that there is an open cover $\{\mathcal{U}\}$ on which each payoff profile in each $\mathcal{U} \cap \mathcal{W}$ is decomposable with respect to some x, $\mathcal{W}$, and some $\delta_\mathcal{U} < 1$. Then, since $\mathcal{W}$ is compact, $\{\mathcal{U}\}$ can be chosen finite, so there exists $\delta' = \max_\mathcal{U} \delta_\mathcal{U} < 1$ such that each payoff profile in $\mathcal{W}$ is decomposable with respect to some x, $\mathcal{W}$, and any $\delta \in (\delta', 1)$. With this property and given $\delta \in (\delta', 1)$, it is possible to construct a PPE to support any $v \in \mathcal{W}$ by specifying that the players should employ the outcome function and continuation reward function with respect to which v is decomposable. Since each such continuation reward function maps into $\mathcal{W}$, this approach can be applied iteratively to map

[14] FLM invokes private valuations only to show that $\mathcal{W}$ is decomposable on tangent hyperplanes, which is maintained as an assumption of FLM Theorem 4.1 and the present Theorem 4.

out an entire equilibrium path.

- Part (i) of FLM Lemma 4.3 is as obvious here as in FLM, since v' can be added to the entire schedule of continuation rewards without altering incentives.
- Part (ii) of FLM, Lemma 4.3 is the critical point at which FLM implicitly invokes finiteness of Θ. Here, since a continuation reward function w maps from a signal space Θ that may be infinite, its range $\{w(\theta)\}_{\theta\in\Theta}$ may be unbounded. If it is in fact unbounded, then there clearly cannot exist a constant $\kappa < \infty$ as described in the statement of the lemma. However, FLM Lemma 4.3 holds if it is revised to require that x be enforceable with respect to some $\delta \in (0,1)$, a bounded subset of some tangent hyperplane $\mathcal{L}$, and some δ, using some continuation reward function w. Then $\sup_{\theta\in\Theta} \|w(\theta) - \mathbb{E}_\theta[w(\theta)]\| < \infty$. In this way, for any $\delta' \in (0,1)$, we can choose
$$w'(\theta) = v' + \frac{\delta\left(1-\delta'\right)}{\delta'\left(1-\delta\right)}\left(w(\theta) - \mathbb{E}_\theta[w(\theta)]\right) \tag{37}$$
so that x is enforceable with respect to δ' and $\mathcal{L}' = \mathcal{L} + v'$. By construction, $v' = \mathbb{E}_\theta[w(\theta)]$, while
$$\|w(\theta) - v'\| \le \left(\frac{\delta}{1-\delta} \cdot \sup_{\theta\in\Theta} \|w(\theta) - \mathbb{E}_\theta[w(\theta)]\|\right)\frac{1-\delta'}{\delta'}. \tag{38}$$
- The conclusion of FLM Theorem 4.1 then holds if we can show that $\mathcal{W}$ is locally self-decomposable if it is boundedly decomposable on tangent hyperplanes. Since FLM's proof of Theorem 4.1 is argued entirely in terms of geometry in $\mathbb{R}^N$,[15] it holds whenever the conclusions of FLM Lemmas 4.2 and 4.3 hold. As argued above, these conclusions are assured by the added requirements that $\mathcal{W}$ be boundedly decomposable on tangent hyperplanes and be a subset of $\mathcal{P}^*$.
- It now remains only to show that the conclusion of FLM Theorem 4.1 also holds under bounded decomposability on *almost* tangent hyperplanes.
  - First, by the same argument as in FLM Theorem 4.1, for any open set $\mathcal{T}$ with $\mathrm{bd}(\mathcal{T}) \subset \mathrm{int}(\mathcal{W})$ there exists $\delta^* < 1$ and such that any point in $\mathcal{T}$ is decomposable with respect to the stage game equilibrium payoff p, any $\delta \in (\delta^*, 1)$, and $\mathrm{int}(\mathcal{W})$.
  - Next, assume for a moment that every point in $\mathrm{bd}(\mathcal{W})$ has a different tangent hyperplane. Under bounded decomposability on almost tangent hyperplanes, almost every point $v \in \mathrm{bd}(\mathcal{W})$ is decomposable with respect to x, any $\delta \in (0,1)$, and a bounded subset of its tangent hyperplane. As FLM argue in the proof of their Theorem 4.1, this implies that for any such point there exists an open neighborhood $\mathcal{U} \supset v$ sufficiently small and $\delta < 1$ sufficiently high such that any $v' \in \mathcal{U}$ is decomposable with respect to x, δ, and $\mathcal{W}$. Since the collection of all such neighborhoods $\mathcal{U}$ combined with the collection of sets $\mathcal{T}$ described above covers $\mathcal{W}$, and $\mathcal{W}$ is compact, there exists $\underline{\delta} < 1$ such that every point in $\mathcal{W}$ is decomposable with respect to some outcome function, any $\delta \in (\underline{\delta}, 1)$, and $\mathcal{W}$; i.e., $\mathcal{W}$ is locally self-decomposable.

[15]Except when it makes use of the fact that a stage game equilibrium is sure to exist due to finiteness. Existence of a stage game equilibrium (which is not assured when Θ may be arbitrary) is assumed throughout the present paper.

– Finally, suppose that there exists some hyperplane $\mathcal{L}^*$ such that $\mathcal{K}^* \equiv \mathcal{L}^* \cap \text{bd}(\mathcal{W})$ is non-empty and not a singleton, which must be the case if not every point in $\text{bd}(\mathcal{W})$ has a different tangent hyperplane. If $\text{int}(\mathcal{K}^*) = \varnothing$, then the argument above applies naturally, so assume that $\text{int}(\mathcal{K}^*)$ is non-empty. Still, by the argument above, there exists $\underline{\delta} < 1$ such that each $v \in \text{bd}(\mathcal{K}^*)$ is decomposable with respect to some outcome function x, any $\delta \in (\underline{\delta}, 1)$, and $\text{int}(\mathcal{W})$. Any other point $v' \in \text{int}(\mathcal{K}^*)$ can be obtained by public randomization among the points in $\text{bd}(\mathcal{K}^*)$. The result then follows. ■

Appendix C Generality of Strong Condition C

From Remark 1 we know that Strong Condition C is satisfied when ϕ is independent. In this section I demonstrate that Strong Condition C is also satisfied under "global interdependence." ACGVa demonstrate that Condition C is generically satisfied for finite Θ, but their method of proof relies crucially on finiteness. Furthermore, there is no clean notion of "genericity" when Θ may be arbitrary, so I prove a more specific result that makes direct use of the interdependence that is "generally" found in probability measures.

Definition 15. A probability measure ϕ is *globally interdependent* if, for any $\hat{\Omega} \subset \Theta$, if $\phi(\hat{\Omega}) > 0$ then there exists a rectangular set $\Omega \equiv \prod_{i=1}^N \Omega_i \subset \hat{\Omega}$ such that, for each pair of players (i, j), $\phi_{ij}(\Omega_i \times \Omega_j) \neq \phi_i(\Omega_i)\phi_j(\Omega_j)$.

Theorem 5 (Conjecture)**.** *If $N \geq 3$ and ϕ is globally interdependent, then Strong Condition C is satisfied.*

Proof (in progress). Strong Condition C is satisfied if, for any $\lambda \in [-1, 1]^N$ with $\lambda_i \neq 0$ for at least two players i, for any uniformly bounded function $R : \Theta \to \mathbb{R}$ there exists a uniformly bounded function $r : \Theta \to \mathbb{R}^N$ such that the following two constraints are satisfied:

(i) "Adding up": $\sum_i \lambda_i r_i(\theta) = R(\theta)$ for all θ;

(ii) "EqIIC": $\mathbb{E}_{\theta_{-i}}\left[r_i(\theta)\middle|\theta_i\right] = \mathbb{E}_{\theta_{-i}}\left[r_i(\hat{\theta}_i, \theta_{-i})\middle|\theta_i\right]$ for all $\hat{\theta}_i$, for almost all θ_i, and for all i.

Note that the two constraints are linear equalities in $\{r_i(\theta)\}_{\theta\in\Theta}$. Although Θ may be infinite, we can heuristically express these constraints in a block matrix equation as follows:

$$\begin{bmatrix} \lambda_1 \mathbf{I} & \dots\dots\dots & & \lambda_N \mathbf{I} \\ \mathbf{M}_1 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_2 & \dots & \mathbf{0} \\ \dots & \dots & \dots & \dots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{M}_N \end{bmatrix} \cdot \begin{bmatrix} r_1(\Theta) \\ \vdots \\ r_N(\Theta) \end{bmatrix} = \begin{bmatrix} R(\Theta) \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}, \tag{39}$$

where $\mathbf{I}$ is an identity matrix, $\mathbf{0}$ is a matrix (or vector) of zeroes, and $r_i(\Theta)$ and $R(\Theta)$ are the expressions of r_i and R in vector form. The first row represents the $|\Theta|$ adding up constraints, and each succeeding row represents the $|\Theta_i \times \Theta_i|$ EqIIC constraints for player i. For the theorem to fail, there must exist some globally interdependent probability measure and some vector $\lambda \in \mathbb{R}^N$ (other than those specifically excluded from Definition 11) for which some linear combinations of these constraints are mutually contradictory.

Along these lines, two facts are immediately evident in Eq. 39: (i) there can be no contradictions among the adding up constraints, and (ii) there can be no contradictions across EqIIC constraints for different players. I will further demonstrate that there can be no contradictions among the EqIIC constraints for any one player. Then I will show that under global interdependence there can be contradictions among combinations of the adding up and EqIIC constraints only if $N = 2$.

To see that there can be no contradictions among the EqIIC constraints for player i, note that these constraints require, for any pair $(\theta_i, \hat{\theta}_i) \in \Theta_i^2$, only that

$$\mathbb{E}_{\theta_{-i}}\left[r_i(\theta)\middle|\theta_i\right] = \mathbb{E}_{\theta_{-i}}\left[r_i({\theta_i}', \theta_{-i})\middle|\theta_i\right] \tag{40}$$

$$\mathbb{E}_{\theta_{-i}}\left[r_i(\theta)\middle|\theta_i'\right] = \mathbb{E}_{\theta_{-i}}\left[r_i({\theta_i}', \theta_{-i})\middle|\theta_i'\right]. \tag{41}$$

If $\phi_{-i|i}(\cdot|\theta_i)$ and $\phi_{-i|i}(\cdot|\theta_i')$ differ (as they do generically when $\theta_i \neq \theta_i'$), then the constraints are not collinear and there is no contradiction. If $\phi_{-i|i}(\cdot|\theta_i)$ and $\phi_{-i|i}(\cdot|\theta_i')$ are identical, then the constraints are identical. The same is true for linear combinations of incentive constraints.

Hence if a contradiction arises, it must be that a combination of player i's incentive constraints with the adding up constraints is a linear combination of the incentive constraints of players $-i$. In fact, this is the case when $N = 2$. In this case, player 1's incentive constraints on $\{r_1(\theta)\}_{\theta\in\Theta}$ can be converted to constraints on $\{r_2(\theta)\}_{\theta\in\Theta}$ by combining them with the adding up constraints. In heuristic block matrix notation, the effect is to reduce Eq. 39 to

$$\begin{bmatrix} -\lambda_2 \mathbf{M}_1 \\ \mathbf{M}_2 \end{bmatrix} \cdot \begin{bmatrix} r_2(\Theta) \end{bmatrix} = \begin{bmatrix} -\mathbf{R}' \\ \mathbf{0} \end{bmatrix}, \tag{42}$$

where $\mathbf{R}'$ is a $|\Theta_1 \times \Theta_1|$ vector derived from linear operations on R. From this expression the potential for contradiction is clear. To begin with, the cardinality of the constraints, $|\Theta_1 \times \Theta_1| + |\Theta_2 \times \Theta_2|$, is greater than the cardinality of the unknowns, $|\Theta_1 \times \Theta_2|$ (although constraints may be collinear without giving rise to a conflict). Furthermore, since the constraints must be satisfied for any choice of R, the potential extends even to the case in which the EqIIC constraints are relaxed to inequality. More formally, expressing player 1's (inequality) IIC constraints in terms of r_2 via the adding up constraints yields

$$\begin{aligned} &\int\limits_{\Theta_2} r_1(\theta)\, d\phi_{2|1}(\theta_2|\theta_1) \geq \int\limits_{\Theta_2} r_1({\theta_1}', \theta_2)\, d\phi_{2|1}(\theta_2|\theta_1) \\ &\implies \quad \int\limits_{\Theta_2} \big(R(\theta) - \lambda_2 r_2(\theta)\big)\, d\phi_{2|1}(\theta_2|\theta_1) \geq \int\limits_{\Theta_2} \big(R({\theta_1}', \theta_2) - \lambda_2 r_2({\theta_1}', \theta_2)\big)\, d\phi_{2|1}(\theta_2|\theta_1). \end{aligned} \tag{43}$$

Then observe that appropriate choice of R will cause the following convex combinations of constraints to conflict:

$$\begin{aligned}
&\int\limits_{\Theta_1}\int\limits_{\Theta_1}\int\limits_{\Theta_2} \big(-r_2(\theta_1,\theta_2)+r_2(\theta_1',\theta_2)\big)\ d\phi_{2|1}(\theta_2|\theta_1)d\phi_1(\theta_1)d\phi_1(\theta_1') \\
&\qquad = -\int\limits_{\Theta}\int\limits_{\Theta_1} d\phi_1(\theta_1')\, r_2(\theta_1,\theta_2)\, d\phi(\theta_1,\theta_2) + \int\limits_{\Theta}\int\limits_{\Theta_1} d\phi_{1|2}(\theta_1)\, r_2(\theta_1',\theta_2)\, d\phi_1(\theta_1')d\phi_2(\theta_2) \\
&\qquad = -\int\limits_{\Theta} r_2(\theta_1,\theta_2)\, d\phi(\theta_1,\theta_2) + \int\limits_{\Theta} r_2(\theta_1',\theta_2)\, d\phi_1(\theta_1')d\phi_2(\theta_2) \\
&\qquad \geq \frac{1}{\lambda_2}\Big(-\int\limits_{\Theta} R(\theta_1,\theta_2)\, d\phi(\theta_1,\theta_2) + \int\limits_{\Theta} R(\theta_1',\theta_2)\, d\phi_1(\theta_1')d\phi_2(\theta_2)\Big),
\end{aligned} \tag{44}$$

$$\begin{aligned}
&\int\limits_{\Theta_2}\int\limits_{\Theta_2}\int\limits_{\Theta_1} \big(r_2(\theta_1,\theta_2)-r_2(\theta_1,\theta_2')\big)\ d\phi_{1|2}(\theta_1|\theta_2)d\phi_2(\theta_2)d\phi_2(\theta_2') \\
&\qquad = \int\limits_{\Theta}\int\limits_{\Theta_2} d\phi_2(\theta_2')\, r_2(\theta_1,\theta_2)\, d\phi(\theta_1,\theta_2) - \int\limits_{\Theta}\int\limits_{\Theta_2} d\phi_{2|1}(\theta_2|\theta_1)\, r_2(\theta_1,\theta_2')\, d\phi_1(\theta_1)d\phi_2(\theta_2') \\
&\qquad = \int\limits_{\Theta} r_2(\theta_1,\theta_2)\, d\phi(\theta_1,\theta_2) - \int\limits_{\Theta} r_2(\theta_1,\theta_2')\, d\phi_1(\theta_1)d\phi_2(\theta_2') \\
&\qquad \geq 0.
\end{aligned} \tag{45}$$

(The first inequality combines player 1's IIC constraints, expressed in terms of r_2 by means of the adding up constraints, while the second inequality combines player 2's IIC constraints.) Since $\frac{1}{\lambda_2}\big(-\int_\Theta R(\theta_1,\theta_2)\, d\phi(\theta_1,\theta_2) + \int_\Theta R(\theta_1',\theta_2)\, d\phi_1(\theta_1')d\phi_2(\theta_2)\big)$ is arbitrary when θ_1 and θ_2 are distributed interdependently (it is exactly zero when they are independent), Condition C must be violated in such cases.

When $N \geq 3$, however, the case is much different, due to the interaction of each pair of players' signals with the remaining players' signals. Suppose, without loss of generality, that $\lambda_1 = 1$. Since any contradiction must arise from a combination of the incentive constraints with the adding up constraints, if a contradiction arises it must be that some linear combination of player 1's incentive constraints (stated in terms of $r_2,\ldots,r_N$),

$$\int\limits_{\Theta_{-1}} \sum_{i=2}^{N} \lambda_i\big(r_i({\theta_1}',\theta_{-1}) - r_i(\theta)\big)\, d\phi_{-1|1}(\theta_{-1}|\theta_1) \ \geq \int\limits_{\Theta_{-1}} \big(R({\theta_1}',\theta_{-1}) - R(\theta)\big)\, d\phi_{-1|1}(\theta_{-1}|\theta_1), \tag{46}$$

is collinear with some linear combination of the incentive constraints of players $2,\ldots,N$. In block matrix terms, this is to say that Eq. 39 can be expressed equivalently as:

$$\begin{bmatrix} -\lambda_2\mathbf{M}_1 & \ldots\ldots & & -\lambda_N\mathbf{M}_1 \\ \mathbf{M}_2 & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_3 & \ldots & \mathbf{0} \\ \ldots & \ldots & \ldots & \ldots \\ \mathbf{0} & \ldots\ldots & & \mathbf{M}_N \end{bmatrix} \cdot \begin{bmatrix} r_2(\Theta) \\ \vdots \\ r_N(\Theta) \end{bmatrix} = \begin{bmatrix} \mathbf{R}' \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}, \tag{47}$$

where $\mathbf{R}'$ is a $|\Theta_1 \times \Theta_1|$ vector that is a linear function of $R(\Theta)$. Note that when $N \geq 3$ the cardinality of the constraints can easily be less than the cardinality of the unknowns, although this fact will not play a role in the proof since many of the constraints may be collinear without contradiction.

Formally, a contradiction arises only if there exist probability measures $\mu_1, \ldots, \mu_N$ on $\Theta_1 \times \Theta_1, \ldots, \Theta_N \times \Theta_N$, and constants $k_2, \ldots, k_N$ such that, for any r,

$$\begin{aligned} &\int_{\Theta_1^2} \int_{\Theta_{-1}} \sum_{i=2}^{N} \lambda_i \big(r_i(\theta_1', \theta_{-1}) - r_i(\theta)\big)\, d\phi_{-1|1}(\theta_{-1}|\theta_1) d\mu_1(\theta_1, \theta_1') \\ &\quad = \int_{\Theta_i^2} \int_{\Theta_{-i}} \sum_{i=2}^{N} k_i \big(r_i(\theta) - r_i(\theta_i', \theta_{-i})\big)\, d\phi_{-i|i}(\theta_{-i}|\theta_i) d\mu_i(\theta_i, \theta_i'), \end{aligned} \tag{48}$$

with both integrals non-zero for at least some r. For the case of statistical independence Eq. 48 is satisfied for all r by setting $\mu_i(\cdot, \cdot\cdot) = \phi_i(\cdot)\phi_i(\cdot\cdot)$ and $k_i = \lambda_i$ for all i, but this sets both integrals equal to exactly zero for all r. This is consistent with the fact that we already know Strong Condition C holds under independence, by Remark 1.

If there is statistical interdependence, then without loss of generality assume that $\lambda_2 \neq 0$. Then, given any particular set of measures $\mu^1, \ldots, \mu^N$ and constants $k_2, \ldots, k_N$, we can show that Eq. 48 is violated for

$$\begin{aligned} r_2(\theta) = r^{\Omega}(\theta) &\equiv \begin{cases} 1 & \text{if } \theta \in \Omega \\ 0 & \text{otherwise,} \end{cases} \\ r_3(\theta) = \cdots &= r_N(\theta) = 0 \end{aligned} \tag{49}$$

if we can find $\Omega \subset \Theta$ that satisfies

$$\begin{aligned} 0 &\neq \int_{\Theta_1^2} \int_{\Theta_{-1}} \lambda_2 \big(r^{\Omega}(\theta_1', \theta_{-1}) - r^{\Omega}(\theta)\big)\, d\phi_{-1|1}(\theta_{-1}|\theta_1) d\mu^1(\theta_1, \theta_1') \\ &\neq \int_{\Theta_2^2} \int_{\Theta_{-2}} k_2 \big(r^{\Omega}(\theta) - r^{\Omega}(\theta_2', \theta_{-2})\big)\, d\phi_{-2|2}(\theta_{-2}|\theta_2) d\mu^2(\theta_2, \theta_2'). \end{aligned} \tag{50}$$

In particular, restrict attention to rectangular sets of the form $\Omega = \prod_i \Omega_i$, with $\mu^1(\Omega_1 \times \Omega_1) > 0$. Then the second inequality in Eq. 50 can be rewritten as

$$\begin{aligned} &\lambda_2 \int_{\alpha_1 \in \Theta_1} \phi_{-1|1}(\Omega_{-1}|\alpha_1) \mu^1_{1'|1}(\Omega_1|\alpha_1)\, d\mu^1_1(\alpha_1) + k_2 \int_{\alpha_2 \in \Theta_2} \phi_{-2|2}(\Omega_{-2}|\alpha_2) \mu^2_{2'|2}(\Omega_2|\alpha_2)\, d\mu^2_2(\alpha_2) \\ &\quad \neq \lambda_2 \int_{\alpha_1 \in \Theta_1} \phi_{-1|1} \mu^1_{1|1'}(\Omega|\alpha_1) d\mu^1_{1'}(\alpha_1) + k_2 \int_{\alpha_2 \in \Theta_2} \phi_{-2|2} \mu^2_{2|2'}(\Omega|\alpha_2) d\mu^2_{2'}(\alpha_2), \end{aligned} \tag{51}$$

where $\phi_{-i|i}\mu^i_{i|i'}(\Omega|\alpha_i) \equiv \int_{\Omega_i} \phi_{-i|i}(\Omega_{-i}|\theta_i)\, d\mu^i_{i|i'}(\theta_i|\alpha_i)$. Note that if ϕ is globally interdependent, then $\phi_{-i|i}\mu^i_{i|i'}(\cdot|\alpha_i)$ is globally interdependent as well, for any α_i. Observe that the left hand side of Eq. 51 is a linear combination of measures for which θ_1 and θ_2 must be independent, whereas the right hand side is a linear combination of measures for which θ_1 and θ_2 are interdependent.

Hence under global interdependence, for any $\hat{\Omega}$ for which $\phi(\hat{\Omega} \cap \text{support}(\mu_1)) > 0$, we can find some $\Omega \subset \hat{\Omega} \cap \text{support}(\mu_1)$ for which the inequality is satisfied.[16] Unlike in the case with $N = 2$, for appropriate choice of Ω the two terms with θ_1 and θ_2 independent (the terms on the left hand side) cannot be equal under global interdependence, due to the interaction between θ_2 and θ_3 in the first term and between θ_1 and θ_3 in the second term. ■

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[16] Note that if ϕ satisfies neither global interdependence nor independence, then there may be some values of θ and θ' for which θ_1 and θ_2 are distributed independently of $(\theta_3, \ldots, \theta_N)$ according to $\phi_{-1|1}$ and $\phi_{-2|2}$, and yet there is (non-global) interdependence in ϕ. It is evident from Eq. 51 that, under these conditions, Strong Condition C may be violated.

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