

# A Dual Self Model of Impulse Control<sup>1</sup>

Drew Fudenberg and David K. Levine<sup>2</sup>

First Version: August 5, 2004

This Version: July 28, 2005

**Abstract:** We propose that a simple “dual-self” model gives a unified explanation for several empirical regularities, including the apparent time-inconsistency that has motivated models of hyperbolic discounting and Rabin’s paradox of risk aversion in the large and small. The model also implies that self-control costs imply excess delay, as in the O’Donoghue and Rabin models of hyperbolic utility, and it explains experimental evidence that increased cognitive load makes temptations harder to resist. Finally, the reduced form of the base version of our model is consistent with the Gul-Pesendorfer axioms.

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<sup>1</sup> We thank David Laibson and Matt Rabin for many years of instructive conversations on related topics, Stefano DellaVigna and Ulrike Malmendier for helpful discussion of their research, and Drazen Prelec and Antonio Rangel for detailed comments on an earlier draft. We are grateful to NSF grants SES-01-12018, SES-03-14713, and SES-04-26199 for financial support.

<sup>2</sup> Department of Economics Harvard University, and UCLA/Federal Reserve Bank of Minneapolis.

“The idea of self-control is paradoxical unless it is assumed that the psyche contains more than one energy system, and that these energy systems have some degree of independence from each other.”

(McIntosh [1969])

## **1. Introduction**

This paper argues that a simple “dual-self” model gives a unified explanation for a number of empirical regularities related to self-control problems and a value for commitment in decision problems. This includes the apparent time inconsistency that has motivated economists’ models of hyperbolic discounting: Faced with a choice between consuming some quantity today and a greater quantity tomorrow, some people will choose to consume the lesser quantity today. However, when these same individuals are faced with the choice between the same relative quantities a year from now and a year and a day from now, they choose to consume the greater quantity a year and a day from now.<sup>3</sup> The second regularity is Rabin’s [2000] paradox of risk aversion in the large and small. The paradox is that the risk aversion experimental subjects show to very small gambles implies hugely unrealistic willingness to reject large but favorable gambles. In addition, the model explains the effect of cognitive load on self-control that is noted by Shiv and Fedorkin, and it predicts that increased costs of self-control lead to increased delay in stationary stopping-time problems, as in O’Donoghue and Rabin [2001].

Our theory proposes that many sorts of decision problems should be viewed as a game between a sequence of short-run impulsive selves and a long-run patient self. This is consistent with recent evidence from MRI studies such as McClure, Laibson, Loewenstein, and Cohen [2004] that suggest that short-term impulsive behavior is associated with different areas of the brain than long-term planned behavior.<sup>4</sup> The

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<sup>3</sup> The economics literature on hyperbolic discounting, following Strotz [1955] and Laibson [1997], uses the now-familiar  $(\beta, \delta)$  form. This is called “quasi-hyperbolic discounting” in the psychology literature to distinguish it from the discounting function  $f(t) = (1 + \alpha t)^{\beta/\alpha}$ , which actually is hyperbolic. See Prelec [2004] for a characterization of these functions in terms of “decreasing impatience.”

<sup>4</sup> They say “Parts of the limbic system associated with the midbrain dopamine system... are preferentially activated by decisions involving immediately available rewards. In contrast, regions of the lateral prefrontal

findings of this research are reinforced also by introspection – we are all aware of the internal conflict when our “rational self” is faced with short-term indulgences that lead to bad long-term consequences. We argue that our theory explains a broad range of behavioral anomalies, and that it is a better fit for the modular structure of the brain than the hyperbolic model, which posits a game between multiple “selves,” one in each period.<sup>5</sup> Moreover, the dual-selves model is analytically simpler than the hyperbolic discounting model, as the equilibria of the model can be calculated as the solution to a decision problem. In addition, in settings where the hyperbolic model has multiple equilibria, the dual-self equilibrium is unique. While multiple equilibria seem a good way of describing some situations, the unique equilibria of the dual-self model are an advantage to the extent that it can explain the empirical facts just as well as models with less precise predictions. The only model of hyperbolic preferences we know of that has similar properties is the Harris and Laibson [2004] model of instantaneous gratification in a consumption-savings problem. That model seems to us to be more complicated and specialized than our own.

In our model, the patient long-run self and a sequence of myopic short-run selves share the same preferences over stage-game outcomes; they differ only in how they regard the future. Specifically, we imagine that the short-run myopic self has “base-preferences” in the stage game that depend only on the outcome in the current stage. That is, the short-run players are completely myopic.<sup>6</sup>

The stage game is played in two phases. In the first phase, the long-run self chooses the utility function of the myopic self. At some reduction in utility (for both selves – who share the same stage game utility function) the long-run self can choose preferences other than the “base preferences. In the second phase of the stage game, after the short-run player preferences have been chosen, the short-run player takes the final

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cortex and posterior parietal cortex are engaged uniformly by intertemporal choices irrespective of delay. Furthermore, the relative engagement of the two systems is directly associated with subjects’ choices, with greater relative fronto-parietal activity when subjects choose longer term options.”

<sup>5</sup> At the same time, we recognize that our model is only a very loose approximation of the brain’s structure. Recent studies suggest both that reward-related information may be processed in many different brain regions, and that the links between these regions are more complex than the top-down control assumed in our model. See for example O’Doherty [2004] and Platt and Glimcher [1999].

<sup>6</sup> This is a very stark assumption, but it leads to a much simpler model, and may be a reasonable approximation in some cases of interest. The conclusion discusses the complications introduced by forward-looking “short run selves.”

decision. It is important that we do not allow the long-run self to precommit for the entire dynamic game. Instead, she begins each stage game facing the choice of which preferences to give the myopic self – or equivalently, how much self-control to exert. Note also that while the hyperbolic discounting model emphasizes the conflict between present and future selves, we emphasize that the long-run self has the same stage game preferences as the short-run self, and so wishes to serve the interests of future short-term selves.

Games with long run versus short run players are relatively simple to analyze. This particular class is especially simple. Imposing a minimal perfection requirement that the short-run self must always play a best response, the long-run self implicitly controls the short-run self, albeit at some cost. Equilibria of this game are equivalent to the solution to an optimization problem. In this respect, the long-run versus short-run player model is more conservative than hyperbolic discounting, preserving many of the methods and insights of existing theory, as well as delivering strong predictions about behavior.

Our model is similar in spirit to that of Thaler and Shefrin [1981] (from whom we have taken the McIntosh quotation at the start of the paper.) Like them, we view our model as “providing a simple extension of orthodox models that permits [self-control behavior] to be viewed as rational.” One difference is that their model is defined only for the consumption-savings problem we study in section 3, while we develop a more general model that can be applied to other situations. Also, we work with more precise specifications of the costs of self-control, and show how to reduce the game between the selves to a single decision problem. This makes the model analytically tractable, and enables us to make more precise predictions. Independent work by O'Donoghue and Lowenstein [2004] describes a similar but more general model, with less focus on tractability and applications. Benabou and Pycia [2002] analyze a two-period model where the long-run self and the short-run self compete for control by expending resources, with the probability that a given self takes control equal to its share of the total expenditure. Brocas and Carrillo [2005] analyze a two-period dual self model of a consumption-leisure choice; they focus on the case where the short-run self has private information, so that the long-run self offers a “menu” of consumption-effort pairs to the short-run self, as in agency models. Bernheim and Rangel [2004], and Benahib and Bisin [2004] consider multi-period models where a long-run self is only sometimes in control,

either because it is unable to take control (Bernheim and Rangel) or chooses not to do so (Benahib and Bisin.) We discuss these papers further in the conclusion. Section 5 discusses Miao [2005], who applies the dual-self model to a variant of the waiting-time problem we analyze in that section.

Although our point of departure is different, the reduced form of the dual self model is closely connected to the representations derived in Gul and Pesendorfer [2001], [2004] and Dekel, Lipman, and Rustichini [2005]. These papers consider a single player who has preferences over choice sets that includes the desire to limit the available alternatives. Under various axioms over choices over menus of lotteries, they show that the decision process can be represented by a utility function with a cost of self-control.<sup>7</sup> Although the reduced form of our model leads to a similar decision problem, we have a concrete interpretation of preferences in terms of those of a myopic self, and as a result are able to bring both introspective and physiological evidence to bear on what those preferences might be. This leads us to a model that is more restrictive in some ways, and less restrictive in others; we discuss the relationship in detail in Section 7. Krusell, Kuruscu and Smith [2005] examine a variation on the infinite-horizon Gul-Pesendorfer model in the setting of a consumption-savings problem. Within this setting, they consider a wider range of preferences; we say more about this in Section 3.

The dual self model predicts a preference for commitment, just as the hyperbolic model does, but through a different mechanism. When dealing with decisions that effect only future options, the short-run self is indifferent, hence can be manipulated by the long run self at minimal cost. The long-run self, then, has two different sorts of mechanisms through which to change the behavior of future short-run selves. She can intervene directly in a future stage game by choosing an appropriate utility function, but to do so requires a substantial utility cost. Alternatively, in some settings it may be possible for the current short-run self to limit the alternatives available to the future short-run selves; manipulating these decisions has negligible cost. Finally, in some cases the long-run self may be willing to incur short run costs to reduce the future cost of self control.

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<sup>7</sup> More precisely, Gul and Pesendorfer [2001] and Dekel, Lipman, and Rustichini [2005] give axioms on preferences over static choice sets; Gul and Pesendorfer [2001] complement this with a conditions for a representation of joint preferences over both choice sets and the choice from a given set. Gul and Pesendorfer extends their earlier representation to an infinite horizon, but only for preferences over choice sets.

As an application, we examine a simple one-person savings problem. We show that if the short-run self has access to all available wealth, the savings rate is reduced to keep the cost of self-control low. On the other hand, when wealth is kept in a bank account, and the short-run self that withdraws the money is different from the short-run self who (at a later time) spends the money, savings are exactly those predicted in the absence of self-control costs. However, the dual self model predicts that the propensity to spend out of unanticipated cash receipts is greater than out of unanticipated bank-account receipts. In particular, a sufficiently small unanticipated cash receipt will be spent in its entirety, and so winnings from sufficiently small cash gambles are evaluated by the short-run self's preferences, which are over consumption. These preferences are more risk-averse than the preferences over long-term consumption that are used to evaluate large gambles, so this "cash effect" provides an explanation of Rabin's [2000] paradox of risk aversion in the small and in the large.<sup>8</sup>

We also apply the dual-self model to the study of procrastination and delay in a stationary stopping-time environment that is very similar to that of O'Donoghue and Rabin [2001]. Like them, we find that self-control costs lead to longer delays, but our model yields a unique prediction, in contrast to their finding of multiple equilibria. Our model also suggests some qualifications to the interpretations that DellaVigna and Malmendier [2003] give to their data on health-club memberships. We examine the effect of cognitive load on self-control, and use this to motivate the assumption that the cost of self-control is convex, as opposed to linear. Finally, we relate our model to the axiomatizations of Gul and Pesendorfer [2001, 2004] and of Dekel, Lipman, and Rustichini [2005], and conclude with a discussion of some modeling issues and open questions.

## **2. The Model**

Time is discrete and potentially unbounded,  $t = 1, 2, \dots$ . There is a fixed, time-and history invariant set of actions  $A$  for the short-run selves; this is assumed to be

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<sup>8</sup>As we explain below, the cash effect has the same impact in the hyperbolic discounting mode. Note also that this result is in the opposite direction from those of Gul and Pesendorfer [2001,2004], who do not consider environments with mechanisms such as banks that substitute for self-control.

a compact subset of Euclidean space.<sup>9</sup> There is a measure space  $Y$  of states that will be used to encode all effects of past history on current and future payoff possibilities, for example the state will correspond to wealth in the consumption-savings application. Finally there is a set  $R$  of self-control actions for the long-run self;  $R$  is a compact convex subset of Euclidean space. The point  $0 \in R$  is taken to mean that no self-control is used. A finite history of play  $h \in H$  consists of the past states and actions,  $h = (y_1, a_1, r_1, \dots, y_t, a_t, r_t)$  or the null history  $0$ . The length of the history is denoted by  $t(h)$ , the final state in  $h$  by  $y(h)$ . There is an initial state  $y_1$ . The probability distribution over states at time  $t + 1$  depends on the time- $t$  state and action  $y_t, a_t$  according to the exogenous probability measure  $\mu(y, a)$ .

Thus the short-run self is both the short-run utility assessor and the “doer”; all interactions with the outside world are handled by the short-run self. The long-run self’s action  $r$  has no direct effect on the future state and serves only to influence the actions of the short-run player through its effect on the short-run player’s payoff function  $u(y, r, a)$ . Physiologically, we imagine that self-control corresponds to the release of chemicals that determine “mood” and other variables relevant to the preferences of the impulsive myopic self.

The game is between the long-run self, whose pure strategies are maps from histories and the current game to self-control actions  $\sigma_{LR} : H \times Y \rightarrow R$ , and the sequence of short-run selves. Each short-run self plays in only one period, and observes the self-control action chosen by the long-run self prior to moving. Denote by  $H_t$  the set of  $t$ -length histories  $H_t$ . A strategy from the time- $t$  short-run self is a map  $\sigma_t : H_t \times Y \times R \rightarrow A$ ; we denote the collection of all of these strategies by  $\sigma_{SR}$ . The strategies together with the measure  $\mu$  give rise to a measure  $\pi_t$  over histories of length  $t$ .

The utility of the long-run self is given by

$$U = \sum_{t=1}^{\infty} \delta^{t-1} \int u(y, r, a) d\pi_t(y(h)).$$

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<sup>9</sup> This assumption is a modeling convenience, but it can be relaxed to allow for history-dependent action sets.

In this formulation, the self-control cost (that is, the difference between  $u(y, r, a)$  and  $u(y, 0, a)$ ) is borne by both selves. However, since the short-run self cannot influence that cost, all that matters is the influence of self-control on the marginal incentives of the short-run self and thus on its decisions.

Since each move begins a proper subgame, the strategies are a *subgame perfect equilibrium* if each self's strategy is optimal following every history, and the short-run self's strategy is also optimal following the move of the long-run self.

**Assumption 1 (Costly Self-Control):** *If  $r \neq 0$  then  $u(y, r, a) < u(y, 0, a)$ .*

**Assumption 2 (Unlimited Self-Control):** *For all  $y, a$  there exists  $r$  such that for all  $a'$ ,  $u(y, r, a) \geq u(y, r, a')$ .*

Under the assumptions that self-control is costly and unlimited, we may define the cost of self-control

$$C(y, a) \equiv u(y, 0, a) - \sup_{\{r \mid u(y, r, a) \geq u(y, r, \cdot)\}} u(y, r, a)$$

**Assumption 3 (Continuity):**  *$u(y, r, a)$  is continuous in  $r, a$ .*

This assures that the supremum in the definition of  $C$  can be replaced with a maximum.

**Assumption 4 (Limited Indifference):** *for all  $a' \neq a$ , if  $u(y, r, a) \geq u(y, r, a')$  then there exists a sequence  $r^n \rightarrow r$  such that  $u(y, r^n, a) > u(y, r^n, a')$ .*

This means that when the short-run self is indifferent, the long-run self can break the tie for negligible cost.

Notice that from Assumptions 1 and 2, if  $a = \arg \max_{a'} (u(y, 0, a'))$  then  $C(y, a) = 0$ , and  $C(y, a') > 0$  for  $a' \neq a$ . In addition by Assumption 3,  $C(y, a)$  is continuous in  $a$ . Conversely, if we have given functions  $u(y, 0, a)$  and  $C(y, a)$  satisfying these properties, then we can take  $R = A$  and construct

$$u(y, r, a) = \begin{cases} u(y, 0, r) - C(y, r) - \|r - a\| & a \mid u(y, 0, a) \geq u(y, 0, r) \\ u(y, 0, a) - C(y, r) - \|r - a\| & a \mid u(y, 0, a) < u(y, 0, r) \end{cases}$$

which gives rise to the target cost of self-control function, while satisfying Assumptions 1-4.

Now consider the following *reduced form* optimization problem, of choosing a strategy from histories and states to actions  $\sigma_{SC} : H \times Y \rightarrow A$  to maximize the objective function

$$\sum_{t=1}^{\infty} \delta^{t-1} \int [u(y, 0, a) - C(y, a)] d\pi_t(y(h)).$$

**Theorem 1 (Equivalence of Subgame Perfection to the Reduced Form):** *A subgame perfect equilibrium exists if and only if a solution to the reduced form problem exists; if a solution exists then for every optimal  $\sigma_{SC}$  there are equilibrium strategies  $\sigma_{LR}, \sigma_{SR}$  such that  $\sigma_{SC} = \sigma_{LR} \circ \sigma_{SR}$  and vice versa. Moreover, every Nash equilibrium in undominated strategies is a subgame-perfect equilibrium.*

*Proof:* This is a game of perfect information, in which each short-run player  $t$  plays once, and cares only about utility in period  $t$ . Applying the one-stage deviation principle, we see that subgame-perfect equilibrium is equivalent to the condition that the long-run player maximizes and that at each history  $h_t$  and for every  $y_t, r_t$ , the short run players must play an action that maximizes  $u(y_t, r_t, a)$ . Thus if a subgame-perfect equilibrium exists, the long run player's strategy maximizes

$$\begin{aligned} & \sum_{t=1}^{\infty} \delta^{t-1} \int \sup_{\{r | u(y, r, a) \geq u(y, r, r)\}} u(y, r, a) d\pi_t(y(h)) \\ & = \sum_{t=1}^{\infty} \delta^{t-1} \int [u(y, 0, a) - C(y, a)] d\pi_t(y(h)) \end{aligned} ,$$

The last claim in the theorem follows because the constraints that subgame perfection imposes on the play of the short-run players can be obtained by a single round of deletion of weakly dominated strategies.

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*Remark 1:* We have not imposed sufficient assumptions on  $Y$  and  $\mu$  to guarantee the existence of a solution to the optimization problem. If  $Y$  is finite, it is well known that this problem has a solution; however we wish to examine cases where  $Y$  is infinite, and

although in our examples existence of a solution is unproblematic, it is complicated to give general conditions guaranteeing the existence of an optimum in the infinite case.

*Remark 2:* In the economic applications we consider, the solution to the reduced problem is unique for generic parameter values. We believe that this is a consequence of the structure of the general reduced-form problem, but we have not established a “generic uniqueness” result at this level of generality.

While Assumptions 1-4 are sufficient for the equivalence result in Theorem 1, they are too general to be of much use in applications, so our next order of business is to specialize the model in a way that still lets it cover the intended applications.. One step in that direction is to assume that the state influences the cost of self-control only through the current utility possibilities. This is a restrictive assumption, as it rules out the possibility that some states might make self-control more difficult without having any effect on the utility possibilities when  $r=0$ . We explain how to adapt the model to handle this in section 6, where we discuss how cognitive load influences self-control. Until then, however, we will assume this possibility away, and will further specialize to the case where the cost of self-control depends only on the foregone utility, so that two states with the same maximum possible short-run utility have the same self-control costs.

**Assumption 5 (Opportunity Based Cost of Self Control)** *If*

$$\max_a u(y, 0, a') \geq \max_a u(y', 0, a') \text{ and } u(y, 0, a) \leq u(y', 0, a) \text{ then } c(y, a) \geq c(y', a).$$

This assumption says that the cost of self control depends only on the utility of the best foregone utility and the utility of the option chosen; this rules out cases where the long-run self is uncertain about which option the short-run self will find more tempting. The assumption implies that  $c(y, a) = c(u(y, 0, a), \max_a u(y, 0, a'))$  and that this function is decreasing in its first argument and increasing in the second. Because we have assumed that the long-run self cares only about the utility of the short-run self, Assumptions 1-5 are in some ways stronger than those of Gul and Pesendorfer, as they allow the cost of self control to depend on  $\max_a v(y, a)$ , where the utility function  $v$  need not be the utility function used to evaluate choices. At the same time, Assumptions 1-5 are in some ways

less restrictive than the axioms of Gul and Pesendorfer and Dekel, Lipman, and Rustichini as our assumptions do not imply their independence axioms. We discuss the connection between our model and these papers in some detail in Section 7.

In many of our applications, it is convenient to strengthen Assumption 5 by requiring that  $c$  is linear:

**Assumption 5' (Linear Self-Control Cost):**  $C(y, a) = \gamma[\max_{a'} u(y, 0, a') - u(y, 0, a)]$ ,

so that

$$\begin{aligned} U &= \sum_{t=1}^{\infty} \delta^{t-1} \int [u(y, 0, a) - C(y, a)] d\pi_t(y(h)) \\ &= \sum_{t=1}^{\infty} \delta^{t-1} \int [(1 + \gamma)u(y, 0, a) - \gamma \max_{a'} u(y, 0, a')] d\pi_t(y(h)) \end{aligned}$$

This assumption provides a tractable and tightly parameterized functional form for the cost of self-control, namely that it is proportional to the difference in utility between the best available action and that actually taken. Under Assumption 5', improving the best available alternative does not change the marginal cost of self-control. In this sense, these assumptions are conservative, maintaining as much of the standard model as is consistent with an interesting theory of self-control. Moreover, as we show in section 7, under Assumption 5' our model is consistent with the Gul and Pesendorfer axioms, essentially because it implies their independence axiom. However we argue in Sections 6 and 7 that both Assumption 5' and the independence axiom may be too strong, as they rule out the idea that self-control is a limited resource. They are also not well motivated in settings where uncertainty will be resolved after the action of the short-run self. To accommodate these aspects of behavior, we do not need to revert to the full generality of Assumption 5. Instead, we will use a functional form with one more parameter, namely

$$C(y, a) = \gamma[\max_{a'} u(y, 0, a') - u(y, 0, a)]^{\psi}.$$

### 3. A Simple Savings Model

To start, consider the simple case of an infinite-lived consumer making a savings decision. The state  $y \in \mathfrak{R}_+$  represents wealth, which may be divided between consumption and savings according to the action  $a \in [0,1]$  representing the savings rate. Borrowing is not allowed. Savings are invested in an asset that returns wealth  $y_{t+1} = Ra_t y_t$  next period, there is no other source of income.<sup>10</sup>

In each period of time, the base preference of the short-run self has logarithmic utility,<sup>11</sup>

$$u(y, 0, a) = \log((1 - a)y)$$

where we define  $\log(0) = -\infty$ .

The short-run self wishes to spend all available wealth on consumption. We assume a separable cost of self-control, so

$$\begin{aligned} C(y, a) &= \gamma [\max_{a'} u(y, 0, a') - u(y, 0, a)] = \gamma (\log(y) - \log(1 - a)y) \\ &= -\gamma \log(1 - a). \end{aligned}$$

The reduced form for the long-run self has preferences

$$\begin{aligned} U &= \sum_{t=1}^{\infty} \delta^{t-1} \int [u(y_t, 0, a_t) - C(y_t, a_t)] d\pi_t(y(h)) \\ &= \sum_{t=1}^{\infty} \delta^{t-1} [(1 + \gamma) \log((1 - a_t)y_t) - \gamma \log(y_t)] \end{aligned}$$

The long-run self's problem is thus to maximize this function subject to the wealth equation  $y_t = Ra_{t-1}y_{t-1}$ .<sup>12</sup> It is shown in the Appendix that there is a solution, and that

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<sup>10</sup> Because we take the short-run self's action to be the savings rate and not total savings, the feasible actions  $A$  are independent of the long-run self's actions. Note also that since the short-run self makes all consumption/savings decisions, the model satisfies our requirement that the evolution of the state depends only on the actions of the short-run self.

<sup>11</sup> The Appendix presents the extension to CRRA form of which logarithmic utility is a special case. We show that the problem has a solution, and that the solution is characterized by a first-order condition. We also solve the banking model of the next section with CRRA preferences. Because these preferences have an extra parameter of flexibility, they seem likely to better fit field data; we present the logarithmic case in the text to get across the main ideas.

<sup>12</sup> Krusell, Kurusac, and Smith [2005] consider a more general family of preferences; the model of this section is the special case of theirs in which  $\beta = 0$ . (Larger values of  $\beta$  correspond to a temptation cost that reflects hyperbolic discounting by the "doer.") Instead of showing that the solution is stationary, we do, they take as primitive a stationary value function that satisfies their equation (4); they then restrict

the solution has a constant savings rate strictly between zero and one.<sup>13</sup> Thus we compute present value utility for constant savings rates, and maximize

$$\begin{aligned}
 U &= \sum_{t=1}^{\infty} \delta^{t-1} \left[ (1 + \gamma) \left( \log(1 - a) + (t - 1) \log Ra + \ln y_0 \right) - \gamma(t - 1) \log(Ra) - \gamma \log(y_0) \right] \\
 &= \frac{\left[ (1 + \gamma) \left( \log(1 - a) \right) + \log(y_0) \right]}{(1 - \delta)} + \frac{\delta \log(Ra)}{(1 - \delta)^2}. \tag{1}
 \end{aligned}$$

From the first-order conditions we can then compute that<sup>14</sup>

$$a = \frac{\delta}{1 + \gamma - \delta\gamma}. \tag{2}$$

The comparative statics are immediate and intuitive: As  $\gamma$  increases, so self-control becomes more costly, the savings rate is reduced, to avoid the cost of self-control. As the long-run player becomes more patient, (as  $\delta$  increases) this cost of future self control becomes more important, so the effect of  $\gamma$  increases, which tends to increase the difference between the savings rate at a fixed  $\gamma$  and that at  $\gamma = 0$ . (In particular,  $\gamma$  is irrelevant when  $\delta = 0$ , as the savings rate is 0 with or without costs of self-control.) However, increasing  $\delta$  also increases the savings rate for any fixed  $\gamma$ , as is the case when  $\gamma = 0$  and there is no self-control problem. This latter effect dominates, as total saving increases.

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attention to cases where this stationary solution is the unique limit of their finite-horizon models. As they note, equation (4) can sometime admit multiple solutions, but in our case of  $\beta = 0$  equation (4) is a contraction and has a unique solution. However, since infinite-horizon games can have equilibria that are not limits of exact finite horizon equilibria, their argument seems to leave open the possibility that there are other equilibria where the stationarity of (4) is not satisfied.

<sup>13</sup> As we have written the problem, with the savings rate as the control, the state evolution equation is not concave. If we change variables so that the control is the absolute level of consumption  $c_t$ , the state evolution equation is linear but the per-period payoff becomes  $(1 + \gamma) \log(c_t) - \gamma \log(y_t)$ , which is not concave in the state if  $\gamma > 0$ . For this reason our proof technique does not rely on concavity. We can extend the conclusion that savings are a constant fraction of wealth to the case where asset returns  $\tilde{R}$  are stochastic and i.i.d. provided that there is probability 0 of 0 gross return. In the more general CRRA case studied in the Appendix, the solution given there remains unchanged provided we define  $R^{1-\rho} = E(\tilde{R}^{1-\rho})$ .

<sup>14</sup> Since the solution must be interior, it must satisfy the first-order condition, and since there is a unique solution to the first order condition, this is the optimum. Krusell, Kurusucu, and Smith obtain the same formula for the savings rate when they specialize their model to our case.

Note that when  $\gamma = 0$ , so there are no self-control costs, the optimum savings rate is  $a^* = \delta$ . In this case the agent's lifetime utility as a function of initial wealth  $y_0$  is

$$\left[ \frac{\log(1 - \delta) + \log(y_0)}{(1 - \delta)} \right] + \frac{\delta \log(R\delta)}{(1 - \delta)^2} \quad (3)$$

we use this fact in the following section.

To summarize, the dual-self model has a constant savings rate for both logarithmic and CRRA utility, and the savings rate is the solution to a first-order condition; the solution is particularly simple in the case of logarithmic utility. In contrast, as Harris and Laibson [2004] emphasize, consumption need not be monotone in wealth in the usual discrete-time hyperbolic model, even in a stationary infinite-horizon environment. Moreover, the hyperbolic model typically has multiple equilibria (Krusell and Smith [2003]), which complicates both its analysis and its empirical application.

In response, Harris and Laibson [2004] propose a continuous-time model of the consumption-savings problem, where the return on savings is a diffusion process. They show that the equilibrium is unique in the limit where individuals prefer gratification in the present discretely more than consumption in the only slightly delayed future.<sup>15</sup> Moreover, in our case of constant return on assets, their results show that consumption is a constant fraction of wealth if the discount factor is sufficiently close to 1. Thus the limit form of their model makes qualitatively similar predictions to ours; we feel that the dual-self approach is more general and more direct.

#### **4. Banking, Commitment, and Risk Aversion**

In practice there are many ways of restraining the short-run self besides the use of self-control: the obvious thing to do is make sure that the short-run self does not have access to resources that would represent a temptation. This leads to a value for commitment, and can also explain some of the “preference reversals” discussed in the literature. We will shortly develop a more complex model where commitment occurs via

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<sup>15</sup> To do this, they show that equilibrium is characterized by the solution of a single-agent problem, where the agent's utility function is derived from the shadow values in the original problem. When base preferences are CRRA, the only difference between the derived utility function and that of a “fully rational” agent (an exponential discounter) is that the agent gets a utility boost at zero wealth,

a cash-in-advance constraint, but as a first illustration consider a three-period model where savings is impossible, and baseline consumption is constant. Suppose that at period 1 the consumer is offered a choice between two possible increases over base-line consumption, either an increase of 1 unit in period 2 or 2 units in period 3. Since neither option increases short-run consumption, the short-run self is indifferent, and the consumer will choose as if she were fully rational, that is, as if her self-control cost were equal to 0. On the other hand, offered the choice between a unit now and 2 units in period 2, the consumer does face a self-control cost in choosing the delayed payoff. Hence for a range of parameter values the consumer would choose the larger, later increment in the first decision and the smaller, sooner one in the second.

Before developing the banking model, we need to address a key unresolved issue in the behavioral literature, namely the correct way to model how agents view money rewards. This is important, both for our paper, and for the interpretation of empirical studies of preference reversal in humans that examine not consumption choices but monetary payoffs.<sup>16</sup> Unless current consumption is liquidity-constrained, this evidence raises a puzzle for both the dual-self and quasi-hyperbolic models: since people cannot literally consume currency, why do they act as if current monetary rewards are tempting? In brief it seems that the short-run self treats money as a “cue” for an immediate reward even though the only real consequence of earning money is in the future,<sup>17</sup> but that observation leaves open the questions of exactly which financial rewards we should expect agents to view as tempting, and what other sorts of deferred rewards will be treated in the same way. The conclusion speculates about some possible extensions of our model to learned cues, but explaining when and why money is “viewed as” consumption is beyond the scope of this paper. Instead, we will develop a model where agents get utility only from consumption, and so are subject to self-control costs only when consumption is possible.

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<sup>16</sup> This is true for example of the many studies cited in Frederick, Loewenstein, and O’Donoghue [2004], p. 173.

<sup>17</sup> This is consistent with evidence (such as Pavlov’s bell) that the impulsive short-run self responds to learned behavioral cues in addition to direct stimulus. Modern physiological research is making progress in identifying some of the brain chemistry that reflects the response to these stimuli, see, for example, Haruno et al [2004]. Camerer, Loewenstein and Prelec [2000] say that “roughly speaking, it appears that similar brain circuitry (dopaminergic neurons in the midbrain) is active for a wide variety of rewarding experiences (including) money rewards.”

In particular, we develop a simple model in which basic savings decisions are made in a bank, where consumption temptations are not present. In the bank, the decision is made how much “pocket cash” to make available for spending when a consumption opportunity arises in the following period.<sup>18</sup> Since savings decisions are made in the bank, with perfect foresight, the optimum without self-control can be implemented simply by rationing the short-run self. Thus the baseline, deterministic version of the model has an equilibrium equivalent to a model without a self-control problem. However, the consumer’s response to unanticipated cash receipts is quite different than that to anticipated receipts, or to unanticipated bank account receipts: the propensity to consume out of a small unanticipated cash receipt is 100%, while the propensity to consume out of a similar amount of money received in the bank account (for example, a small capital gain on a stock) is small.

This wedge between the propensity to consume out of pocket cash and to consume out of bank cash has significant implications for “risk aversion in the large and small.” Winnings from sufficiently small cash gambles are spent in their entirety, and so are evaluated by the short-run self’s preferences, which are over consumption. When the stakes are large, self-restraint kicks in, part of the winnings will be saved and spread over the lifetime. This leads to less risk-averse preferences, so the model explains the paradox proposed by Rabin [2000].

The implication of the pocket versus bank cash model are very important in the interpretation of experimental results: in experiments the stakes are low, but individuals demonstrate substantial curvature in the utility function. Besides exhibiting risk aversion, when given the opportunity, for example, to engage in altruistic behavior, they generally do not make the minimum or maximum donation, but some amount in between. (Similar behavior is observed on the street: many people will make a positive donation to a homeless person, but few will empty their pockets of all cash.) If utility is viewed in terms of wealth, this type of behavior makes little sense, since the effect of a small donation on the marginal utility of wealth to either the donor or recipient is miniscule. Viewed in terms of pocket cash, which is the relevant point of comparison when there is

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<sup>18</sup> It is interesting to note that Hellwig’s [1973] thesis discusses a multiple-selves model of changing preferences, and applies it to a model of banking where “in the bank, the agent has a higher preference for savings than when he sits down for dinner.” Hellwig allows the interval between visits to the bank to be endogenous, and studies how it compares to the solution to the single-agent control problem.

a wedge due to the rationing of cash to the short-run self, this behavior makes perfect sense.

Formally, we augment the simple saving model by supposing that each period consists of two subperiods, the “bank” subperiod and the “nightclub” subperiod. During the “bank” subperiod, consumption is not possible, and wealth  $y_t$  is divided between savings  $s_t$ , which remains in the bank, and cash  $x_t$  which is carried to the nightclub. In the nightclub consumption  $0 \leq c_t \leq x_t$  is determined, with  $x_t - c_t$  returned to the bank at the end of the period. Wealth next period is just  $y_{t+1} = R(s_t + x_t - c_t)$ . The discount factor between the two consecutive nightclub periods (which is where consumption occurs) is  $\delta$ ; preferences continue to have the logarithmic form.<sup>19</sup> First, consider the perfect foresight problem in which savings are the only source of income. Since no consumption is possible at the bank, the long-run self gets to call the shots; and the long-run self can implement  $a^* = \delta$ , the optimum of the problem without self-control, simply by choosing pocket cash  $x_t = (1 - a^*)y_t$  to be the desired consumption. The short-run self will then spend all the pocket cash; because the optimum can be obtained without incurring self-control costs, the long-run self does not in fact wish to exert self-control at the nightclub.

Now we turn to the problem of stochastic cash receipts (or losses). That is, we suppose that at the nightclub in the first period there is a small probability the agent will be offered a choice between several lotteries. If the lotteries are themselves drawn in an i.i.d. fashion, this will also result in a stationary savings rate that is slightly different from the  $a^*$  computed above, but if the probability that a non-trivial choice is drawn is small, the savings rate will be very close to  $a^*$ . We find it easier to consider the limit where the probability of drawing the gamble is zero.

For the agent to evaluate a lottery choice  $\tilde{z}_1$ , he needs to consider how he would behave conditional on each of its possible realizations  $z_1$ . The short-run self is constrained to consume  $c_1 \leq x_1 + z_1$ . Next period wealth is given by

$$y_2 = R(s_1 + x_1 + z_1 - c_1) = R(y_1 + z_1 - c_1) .$$

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<sup>19</sup> The appendix provides the parallel computations for the CRRA case.

The utility of the long-run self starting in period 2 is given by the solution of the problem without self control, as in equation (3):

$$U_2(y_2) = \frac{1}{(1-\delta)} \left( \log(1-\delta) + \log(y_2) + \frac{\delta}{1-\delta} \log(R\delta) \right)$$

The utility of both selves in the first period is  $(1+\gamma)\ln(c_1) - \gamma\ln(x_1 + z_1)$ , and so the overall objective of the long-run self is to maximize

$$(1+\gamma)\log(c_1) - \gamma\log(x_1 + z_1) + \frac{\delta}{(1-\delta)} \left( \log(1-\delta) + \log(R(y_1 + z_1 - c_1)) + \frac{\delta}{1-\delta} \log(R\delta) \right) \quad (4)$$

The first order condition for optimal consumption is

$$\frac{1+\gamma}{c_1} = \frac{\delta}{(1-\delta)(y_1 + z_1 - c_1)}$$

so

$$c_1\delta = (1-\delta)(y_1 + z_1 - c_1) + \gamma(1-\delta)(y_1 + z_1 - c_1)$$

and

$$c_1 = \frac{(1-\delta)(1+\gamma)(y_1 + z_1)}{\delta + (1+\gamma)(1-\delta)} = \left( 1 - \frac{\delta}{\delta + (1+\gamma)(1-\delta)} \right) (y_1 + z_1) \quad (5)$$

$$\equiv (1-B)(y_1 + z_1)$$

Note that when  $\gamma = 0$ , (5) simplifies to  $c_1 = (1-\delta)(y_1 + z_1)$ , as it should. If the solution  $c_1$  satisfies the constraint  $c_1 \leq x_1 + z_1$  it represents the optimum; otherwise the optimum is to consume all pocket cash,  $c_1 = x_1 + z_1$ . Because  $x_1$  is the solution for  $\gamma = 0$ , we know that  $x_1 = (1-\delta)y_1$ . Thus  $c_1 \leq x_1 + z_1$  if  $z_1 < z_1^*$ , where the critical value of  $z_1^*$  is derived from the solution to the problem with self-control cost  $\gamma$ :

$$\left( 1 - \frac{\delta}{\delta + (1+\gamma)(1-\delta)} \right) (y_1 + z_1^*) = (1-\delta)y_1 + z_1.$$

This yields

$$\begin{aligned}(1 - B)(y_1 + z_1^*) &= (1 - \delta)y_1 + z_1^* \\ z_1^* &= (\delta / B - 1)y_1 \\ z_1^* &= \gamma(1 - \delta)y_1\end{aligned}$$

Since  $B < \delta$  when  $\gamma > 0$  we see that for a range of positive  $z_1$  it is in fact optimal to spend the entire amount of pocket cash  $x_1$ . Note also that when  $\gamma = 0$ , so there is no self-control problem, so  $z_1^* = 0$ : it is never optimal to spend all of the increment to wealth.

The above establishes

**Theorem 2:** If  $z_1 < z_1^*$ , overall utility is

$$\log(x_1 + z_1) + \frac{\delta}{(1 - \delta)} \left( \log(1 - \delta) + \log(R(y_1 - x_1)) + \frac{\delta}{1 - \delta} \log(R\delta) \right) \quad (6)$$

If  $z_1 > z_1^*$  utility is

$$\begin{aligned}(1 + \gamma) \log\left(\frac{(1 - \delta)(1 - \gamma)}{1 + \gamma(1 - \delta)}(y_1 + z_1)\right) - \gamma \log(x_1 + z_1) \\ + \frac{\delta}{(1 - \delta)} \left( \log(1 - \delta) + \log\left(\frac{R\delta}{1 + \gamma(1 - \delta)}(y_1 + z_1)\right) + \frac{\delta}{1 - \delta} \log(R\delta) \right)\end{aligned} \quad (7)$$

To analyze risk aversion, imagine that  $\tilde{z}_1 = \bar{z} + \sigma\varepsilon_1$ , where  $\varepsilon_1$  has zero mean and unit variance, and suppose that  $\sigma$  is very small. Now consider the usual conceptual experiment of comparing a lottery with its certainty equivalent. For  $\bar{z} < z_1^*$  overall payoff is given by (6). Thus relative risk aversion is constant and equal to  $\rho$ , where wealth is  $w = x_1 + \bar{z}_1$  so risk is measured relative to pocket cash. On the other hand, for  $\bar{z} > z_1^*$ , the utility function (7) is the difference between two others, one of which exhibits constant relative risk aversion relative to wealth  $y_1 + \bar{z}$ , the other of which exhibits constant risk aversion relative to pocket cash  $x_1 + \bar{z}$ . When  $\gamma$  is small, the former

dominates, and to a good approximation for large gambles risk aversion is relative to wealth, while for small gambles it is relative to pocket cash.<sup>20</sup>

We can see this effect graphically in the case of Rabin's [2000] paradox of risk aversion in the small and in the large:

“Suppose we knew a risk-averse person turns down 50-50 lose \$100/gain \$105 bets for any lifetime wealth level less than \$350,000, but knew nothing about the degree of her risk aversion for wealth levels above \$350,000. Then we know that from an initial wealth level of \$340,000 the person will turn down a 50-50 bet of losing \$4,000 and gaining \$635,670.”

The point being of course that many people will turn down the small bet, but no one would turn down the second.

We can easily explain these facts in our model using logarithmic utility. The first bet is most sensibly interpreted as a pocket cash gamble; the experiments with real monetary choices in which subjects exhibit similar degrees of risk aversion certainly are. Moreover, if the agent is not carrying \$100 in cash, then there may be a transaction cost in the loss state reflecting the necessity of finding a cash machine or bank.

The easiest calculations are for the case where the gain \$105 is smaller than the threshold  $z^*$ . In this case, logarithmic utility requires the rejection of the gamble if pocket cash  $x_1$  is \$2100 or less. That is,  $.5 \log(2100 - 100) + .5 \log(2100 + 105) \approx \log(2100)$ . In order for \$105 to be smaller than the threshold  $z^*$ , we require  $\gamma \geq 105/x_1$ , so if  $x_1 = 2100$  is pocket cash, we need a  $\gamma > .05$ , while for<sup>21</sup>  $x_1 = 300$ ,  $\gamma$  must be of at least 0.35. However, the conclusion that the gamble should be rejected also applies in some cases where the favorable state is well over the threshold. For example, if pocket cash is \$300,  $\gamma = 0.05$ , and wealth is \$300,000, then the favorable state of \$105 will be well over the threshold is \$15, but a computation shows that the gamble should be rejected, and in fact it is not close to the

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<sup>20</sup> The former also dominates as  $\delta$  approaches  $1/R^{1-\rho}$ , which is relevant when  $\rho \leq 1$ .

<sup>21</sup> The usual daily limit in the U.S. for ATM withdrawals is \$300.

margin.<sup>22</sup> Indeed, the disutility of the \$100 loss relative to pocket cash of \$300 is so large that even a very flat utility for gains is not enough to offset it. Even if we bound the utility of gains by replacing the logarithmic utility with its tangent above \$300, not only should this gamble be rejected, but even a gamble of lose \$100, win \$110 should be rejected.

Turning to the large stakes gamble, unless pocket cash is at least \$4,000, the second gamble must be for bank cash; for bank cash, the relevant parameter is wealth, not pocket cash. It is easy to check that if wealth is at least \$4,026, then the second gamble will always be accepted. So, for example, an individual with pocket cash of \$2100,  $\gamma = 0.05$  and wealth of more than \$4,026 will reject the small gamble and take the large one, as will an individual with pocket cash of \$300,  $\gamma = 0.05$  and wealth equal to the rather more plausible \$300,000.

Sheffrin and Thaler [1988] argue that people use mental accounting to mitigate self-control problems, and that the propensity to spend out of the “current income account” is near 1. Our model provides an explicit micro-foundation for their argument and links it to the issue of risk aversion. The mechanism in our model is very simple, and covers only the case where the current account is for one day’s spending. While we do think that cash on hand has some impact on propensity to spend, and hence on risk aversion, even myopic agents may be able to smooth consumption over a few days. To accommodate this, we would need to interpret the period length in our model as the length of time over which the short-run self is willing to smooth consumption. This is easy to do if one takes the mental frames as exogenous. However, since mental frames are heuristics for responding to cognitive limits and self-control costs, it would be more satisfying to derive the nature of the frames from assumptions on those fundamentals.

We should point out that other models can yield these results. In particular, in this specific case, the hyperbolic discounting model yields the identical prediction about bank savings in the first period and second periods, and thus about the response to

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<sup>22</sup> The relationship between pocket cash and wealth depends on  $\delta$  and hence on the period length. The relationship is  $\delta = 1 - (x_1 / y_1)$ , so for example, if pocket cash is \$300 and wealth \$300,000, then the interest rate corresponding to  $\delta$  is 1/1000, or if the annual rate is 10%, the period between bank visits is 3-4 days.

unanticipated cash shock. To see this, note that our model of response to an unanticipated shock is to maximize the utility function

$$(1 + \gamma) \log c_1 - \gamma \log(x_1 + z_1) + \delta U_2(R(y_1 + z_1 - c_1)).$$

Denoting the hyperbolic discount factor by  $\beta$ , the hyperbolic discounting model says that the response to an unanticipated shock is to maximize the utility function

$$\log c_1 + \delta \beta U_2(R(y_1 + z_1 - c_1)).$$

In both cases, the utility function  $U_2$  is the utility function derived by solving the unconstrained problem, which is the same in the two cases and equal to the utility function of an agent without self-control problems ( $\beta = 1$  or  $\gamma = 0$ .) Since  $x_1 + z_1$  is not a decision variable at this “nightclub” stage of the problem, we see that if  $\beta = 1/(1 + \gamma)$  the two objective functions differ only by a linear transformation, and so necessarily yield the same preferences over lotteries at the nightclub stage.

The analysis so far has supposed that cash is only available at the banking stage. If the agent, when banking, anticipates the availability of \$300 from an ATM during the nightclub stage, it is optimal to reduce pocket cash by this amount. Of course if the goal is to have pocket cash less than \$300, then self-restraint will be necessary in the presence of cash machines. Note that this explains why we find cash machines where impulse purchases are possible: where lottery tickets are sold, for example. In equilibrium, few if any additional overall sales are induced by the presence of these machines, since their presence is anticipated, but of course the competitor who fails to have one will have few sales. So one consequence of the dual self-model is that we may see an inefficiently great number of cash machines.<sup>23</sup>

The implications of the theory for experiments are ambiguous and complicated. The theory explains why we see substantial risk aversion in experiments with immediate money payoffs. The theory also predicts that people will be less risk averse about gambles that payoff in the future. We are fairly confident that this is true about gambles a

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<sup>23</sup> Of course businesses engage in a variety of methods to induce impulse purchasing, for example a car salesman’s offer to let the purchaser drive away in the car right now.

year or more away, but rather less confident about gambles that pay off in a few days or a few weeks. This brings us back to the previously mentioned fact that the psychology of money is not well understood. Unfortunately, the theory predicts a high degree of idiosyncrasy in that risk aversion. It will depend, for example, on such factors as how much cash the subjects are carrying with them, the convenience of nearby cash machines and the like.

Credit cards and checks also pose complications in applying the theory, as for some people the future consequences of using credit cards and checks can be significantly different than the expenditure of cash. That is, it is one thing to withdraw the usual amount of money from the bank, spend it all on the nightclub and skip lunch the next day. It is something else to use a credit card at the nightclub, which, in addition to the reduction of utility from lower future consumption, may result also in angry future recriminations with one's spouse, or in the case of college students, with the parents who pay the credit card bills. So for many people it is optimal to exercise a greater degree of self-control with respect to non-anonymous expenditures such as checks and credit cards, than it is with anonymous expenditures such as cash. This conclusion is consistent with the finding of Wertenbroch, Soman, and Nunes [2002] that individuals who are purchasing a good for immediate enjoyment have a greater propensity to pay by cash, check or debit card than by credit card.

Finally, we should point out that even without a self-control problem, fear of theft can also lead agents to impose binding constraints on their ability to draw against wealth in nightclub periods, and so predicts that unanticipated losses must be absorbed from consumption. This fear-of-theft model predicts that unanticipated gains will be treated the same regardless of whether they are received in cash or in the bank, while the self-control model does not. It is true that the treatment of losses that is relatively more important for resolving the paradox of high risk aversion for small stakes gambles, but for choices between gambles that have only gains, (the usual laboratory case) the "fear of theft" model predicts little risk aversion, where the dual self and hyperbolic discounting model predict that risk aversion will continue to be substantial.

## 5. Procrastination and Delay

O'Donoghue and Rabin [1999], [2001] use quasi-hyperbolic preferences to explain how self-control problems can lead to procrastination and delay. We can use the dual-self model to make similar predictions. Specifically, consider the following model: Every period  $t = 1, 2, \dots$  the short-run self must either take an action or wait. Waiting allows the self to enjoy a leisure activity that yields a stochastic amount of utility  $x_t$ , whose value is known at the start of that period; think for example that the leisure activity is playing outside and its utility depends on the weather. We suppose that the  $x_t$  are i.i.d. with fixed and known cumulative distribution function  $P$  and associated density  $p$  on the interval  $[\underline{x}, \bar{x}]$ , and mean  $\mu = Ex$ . Taking the action ends the game, and results in a flow of utility  $v$  beginning next period, and so gives a present value of

$$\frac{\delta}{1-\delta}v \equiv \delta V.$$

If the agent waits, the problem repeats in the next period.<sup>24</sup>

Except for the use of the dual-self model, this model is very similar to that of O'Donoghue and Rabin [2001], who consider hyperbolic preferences in a stationary, a and deterministic environment.<sup>25</sup> We compare the models after deriving our conclusions.

Because the current value of  $x$  has a monotone effect on the payoff to waiting, and no effect on the payoff to doing it now, the optimal solution is a cutoff rule: If  $x \geq x^*$  then wait, and if  $x < x^*$  take the action. The maximum utility that the agent can attain in any period is  $x_t$ , which is the payoff to waiting, while doing it now requires foregoing  $x_t$ . Hence waiting incurs no self-control cost, and acting has self-control cost of  $\gamma x_t$ .

If we let  $W(x, \gamma)$  denote the value starting tomorrow when using cut-off  $x$ , and let  $W^* = W(x^*(\gamma), \gamma)$  denote the value when using the optimal cutoff  $x^*(\gamma)$ , the value of waiting today when the leisure activity is worth  $x$  is

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<sup>24</sup> Our later assumptions will imply that  $v > \mu$ , so that we can think of the agent continuing to enjoy leisure activities in every period after he acts.

<sup>25</sup> They suppose that there is a short-run cost  $c > 0$  of acting, and no short run benefit of not acting, while we assume that the ‘‘cost’’ of acting is the opportunity cost of foregone leisure, but this is only a normalization

$$x + \delta W^*,$$

and the value of acting is

$$-\gamma x + \delta V.$$

The expected present value tomorrow if the action is not taken today and cutoff rule  $x$  is used in the future is then given by

$$W(x^*, \gamma) = P(x^*)(-\gamma E(x | x < x^*) + \delta V) + (1 - P(x^*))(E(x | x > x^*) + \delta W^*),$$

so that

$$W^* = \frac{P(x^*)(-\gamma E(x | x < x^*) + \delta V) + (1 - P(x^*))E(x | x > x^*)}{1 - \delta + \delta P(x^*)} \quad (8)$$

We assume that when the opportunity cost is at its lowest possible value  $\underline{x}$ , the present value of acting is greater than the present value of waiting forever:

$$\delta V > \underline{x} + \frac{\delta \mu}{1 - \delta}, \text{ or equivalently, } v > \left( \frac{1 - \delta}{\delta} \right) \underline{x} + \mu.$$

When the reverse inequality holds, an agent without self-control costs will never act. For the time being we also assume that  $\delta v < \bar{x}$ , so that an agent without no cost of self-control would choose to delay when  $x$  is close to  $\bar{x}$ . This combination of assumptions rules out the deterministic case studied by O'Donoghue and Rabin.

**Theorem 3** (i)  $\underline{x} \leq x^* < \bar{x}$ ; if  $\delta v > \delta \mu + (1 - \delta)(1 + \gamma)\underline{x}$  then  $x^* > \underline{x}$ .  
(ii) When  $\underline{x} < x^* < \bar{x}$ ,

$$-\gamma x^* + \delta V = x^* + \delta W(x^*, \gamma) \quad (9).$$

Moreover,  $dx^*/d\gamma < 0$ , so expected waiting time is increasing in the cost of self control.

*Proof:* see appendix B.

Let  $x_n$  (for “naïve”) be the cutoff the agent would use if forced to make a choice between stopping now and never stopping. Then

$$(1 + \gamma)x_n = \delta V - \frac{\delta\mu}{1 - \delta}.$$

When the optimal rule is interior, the agent eventually does choose to stop, and so the continuation value  $W^*$  exceeds  $\delta\mu/1 - \delta$ , and equation (9) shows that  $x^* < x_n$ . This shows that the standard “option value of waiting” consideration carries over to the dual-self model.

In independent work, Miao [2005] applies a dual-self model to a very similar problem. He considers the cases of immediate costs and future benefits, as here; immediate benefits and future rewards, where the temptation is to act to soon, and immediate costs and immediate rewards. His model of future benefits differs from ours in that the long-run benefit is stochastic and the short-run cost is fixed, as in the literature on wage search.<sup>26</sup>

O’Donoghue and Rabin [2001] analyze the implications of hyperbolic discounting in a deterministic and stationary infinite-horizon stopping-time problem. They consider a range of assumptions about the agent’s beliefs about his own future preferences; we will focus on the case of “sophisticates” who are correctly perceive their own self-control problems. For these sophisticates, they show that in any pure-strategy equilibrium there is an “intended delay”  $d$  such that in every period where the agent plans to do the task, he must predict that waiting would lead to a delay of exactly  $d$  periods. We view these cyclic equilibria as artificial and unappealing; they also complicate the analysis of the

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<sup>26</sup> Unlike the difference between costs and opportunity costs of acting, this is a substantive difference and not simply a normalization.

equilibrium set.<sup>27</sup> Despite the presence of multiple equilibria, O’Donoghue and Rabin can show that sophisticates never indefinitely postpone an action that would be worth doing, taking into account the agent’s hyperbolic discounting of future returns, although they may postpone acting for a few periods on actions that an agent without self-control costs would do immediately.

At this point we would like to compare the O’Donoghue and Rabin [2001] analysis to our own. One obvious difference is that in a deterministic setting, our model predicts that the agent either acts at once or never acts; partial delay is never observed. Some readers have suggested that our model makes the wrong prediction here, but we are not convinced of this, as we are unaware of experimental or field data on deterministic stationary problems. Most real-world stopping time problems have both a stochastic component (either in costs and benefits, or simply in feasibility of acting at certain dates) and some non-stationarity, and we are unaware of relevant experimental data. Moreover, the assumptions of Theorem 3 concern only the support of the distribution of opportunity costs, and not the distribution itself; the assumptions are consistent with there being a very high probability of costs in a small interval. Thus, the appropriate question is not whether observed behavior resembles the prediction of the deterministic stationary model, but the calibration question of whether the amount of uncertainty and non-stationarity required for observed delays is plausible. In a related vein, note that if O’Donoghue and Rabin sophisticates were constrained to place a small minimum probability on each action, as in a trembling-hand perfect equilibrium, then the model predicts there would be delay cycles;” we would be surprised if such cycles were observed.

To further explore the relationship between the dual-self and quasi-hyperbolic models, we extend the O’Donoghue and Rabin analysis of sophisticates to the stochastic environment presented above. Thus, consider a game between selves, one at each date  $t$ , where the date- $t$  self has payoff  $\delta\beta V$  if it acts, payoff

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<sup>27</sup> O’Donoghue and Rabin conjecture but do not prove that their model has only one additional mixed-strategy equilibrium. The pure-strategy equilibria correspond to the set of limits of exact equilibria of the finite horizon games as the horizon goes to infinity, but this restriction relies on long chains of backwards induction and is not robust to even small payoff uncertainty, as shown by Fudenberg, Kreps, and Levine [1988].

$$x_t + \beta\delta(x_{t+1} + \delta x_{t+2} + \dots + \delta^\tau V)$$

if a future self acts at date  $t + \tau$ , and payoff 0 if no self ever acts.

First we look for a stationary equilibrium, in which each agent acts if its opportunity cost is less than a cut-off  $x^{**}$ . The agent's expected continuation value if it does not act is the solution to the functional equation

$$W^{**} = P(x^{**})(\delta V) + (1 - P(x^{**}))(E(x | x > x^{**}) + \delta W^{**}),$$

so that

$$W^{**} = \frac{P(x^{**})(\delta V) + (1 - P(x^{**}))E(x | x > x^{**})}{1 - \delta + \delta P(x^{**})}. \quad (10)$$

The agent is indifferent about acting when

$$\delta\beta V = x^{**} + \delta\beta W^{**}, \text{ or}$$

$$x^{**} = \delta\beta(V - W^{**}). \quad (11)$$

Theorem 4 below shows that this equation has an interior solution when the support of the cost distribution is sufficiently broad. When such a solution exists, it is an equilibrium of the quasi-hyperbolic model; this equilibrium is qualitatively similar to the unique outcome of the dual-self model. Unlike the dual self model, though, and like the deterministic O'Donoghue-Rabin model, this model typically has multiple equilibria. For example, there can be equilibria in which the agents in odd periods use cut-off  $x^1$  and the agents in even periods use cut-off  $x^2$ .<sup>28</sup> Of course one can get a unique stationary equilibrium from the hyperbolic model here simply by fiat, that is by deciding to rule out all other equilibria; we think it is more satisfactory to have this be a result than an assumption.

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<sup>28</sup> We suspect that there can be many other sorts of equilibria as well, but given our focus on the dual-self model a full characterization is well beyond the scope of this paper.

**Theorem 4:**

a) If  $v > \mu$  and  $\bar{x} > \delta\beta v$ , the “sophisticated quasi-hyperbolic model” has a stationary equilibrium with  $\underline{x} < x^{**} < \bar{x}$ .

b) There is an open set of parameters that satisfies the restrictions of part a) and for which there are other equilibria.

*Proof:* see appendix.

DellaVigna and Malmendier [2003] report some calibrations of the O’Donoghue-Rabin model to data on delay in canceling health club memberships, which they attribute to a combination of hyperbolic preferences and “lack of sophistication,” meaning that consumers misperceive their own hyperbolic parameter and thus incorrectly forecast their health club usage.<sup>29</sup> Our model suggests several qualifications to their analysis. First of all, as is standard in models of timing, it is not in general optimal for the agent to act whenever he is indifferent between acting now or not at all, as there is an “option value” in waiting.<sup>30</sup> Second, while there is some evidence that agents do not have perfect knowledge about themselves, we expect them to have more information about things that they have had more chances to observe.<sup>31</sup> Thus it seems natural to assume that the

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<sup>29</sup> Sophisticated, “low  $\beta$ ” agents who have correct perceptions about the costs and benefits of the club would correctly forecast that they would rarely attend but take a long time to cancel, while agents who misperceive their  $\beta$  would expect to exercise a lot. DellaVigna and Malmendier also show that agents choose monthly or annual plans with no per-visit charge when it would be cheaper to pay per visit. The use of prepayment as a commitment device is a consequence of both the hyperbolic and dual-self models.

<sup>30</sup> This factor is also present in the O’Donoghue-Rabin model, but the discussion of cancellation in DellaVigna and Malmendier [2003] seems to use a deterministic specification for the costs of cancellation. Also, the calibration measures cancellation lag by the number of full months between the last attendance and contract termination for users who hold a monthly contract at the time of termination. This is a conservative estimate *if* the customer knows that she will not attend in the future by the end of the month that included the customer’s last visit, but otherwise may exaggerate the amount of delay.

<sup>31</sup> Bodner and Prelec [2003] and Benabou and Tirole [2004] build on the idea of imperfect self-knowledge to develop models of “self-signalling” that they use to explain the use of “personal rules.” These models assume that the agent is uncertain of only one thing. In Benabou and Tirole [2004], for example, the agent knows the distribution of costs but does not know his hyperbolic parameter  $\beta$ . Both Bodner and Prelec [2003] and Benabou and Tirole [2004] analyze Bayesian equilibria of their models, which raises the question of whether a plausible non-equilibrium learning process would lead agents to learn the strategy of their “other selves” without learning the underlying value of  $\beta$ . Dekel et al [2004] analyze this issue in

misperceptions about the short-run disutility and long-run benefits of going to the health club are larger than misperceptions about their own impulsiveness, and of course these former misperceptions can also explain the excessive delay.

We can also compare the dual-self model to the deterministic, finite horizon model of O’Donoghue and Rabin [1999], which allows non-stationary costs. In their Example 1, there are 4 periods to do a report, with costs 3, 5, 8, 13, and value  $v$ . Applying our model to this problem, we see that in the final period there is no self-control problem, so the payoff to acting is  $v - 13$ . In the next to last period the short-run self gets 0 from waiting,  $-8$  from acting, so the utility if doing now is  $v - 8(1 + \gamma)$ ; in the previous periods the payoffs are  $v - 5(1 + \gamma), v - 3(1 + \gamma)$  respectively. The solution in our model depends on  $\gamma$  :. If  $\gamma < 10/3$ , the agent acts at the start, and if prevented from doing so, then act immediately when given the chance. If  $\gamma > 10/3$ , the agent never acts, Thus (except for the knife-edge case of  $\gamma = 10/3$ ) the agent never plans to act in intermediate periods. The equilibrium of the quasi-hyperbolic model with sophisticates depends on  $\beta$ . For a range of values around  $\beta = 1/2$ , as the sophisticated agent acts in the second period, but intends to wait until the fourth period if prevented from acting when planned. Basically this equilibrium corresponds to one of the unappealing cyclic equilibria in their infinite horizon model; with an odd number of periods, the equilibrium is for the sophisticated agent to act in the first, third and fifth periods.<sup>32</sup>

## **6. Cognitive Load and Self Control**

Shiv and Fedorikhin [1999] report on the following experiment. Subjects were asked to memorize either a two- or a seven-digit number, and then walk to a table with a choice of two deserts, namely chocolate cake and fruit salad. Subjects would then pick a ticket for one of the deserts, and go to report both the number and their choice in a second room. In one treatment, actual samples of the deserts were on the table, and in a second treatment, the deserts were represented by photographs. The authors’ hypothesize that

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games between multiple agents; their results show that when  $\beta$  is fixed over time, assumptions that allow the agent not to learn  $\beta$  are typically too weak to justify restricting attention to equilibrium.

<sup>32</sup> The multiple-selves version of the delay game is continuous at infinity, so from Fudenberg and Levine [1983] every limit of finite-horizon subgame-perfect equilibria is a subgame-perfect equilibrium in the infinite horizon.

subjects will face a self-control problem with respect to the cake, in the sense that it will have higher emotional appeal but be less desirable from the “cognitive” viewpoint; that the subject’s reaction is more likely to be determined by the emotional (“affective”) reaction when cognitive resources are constrained by the need to remember the longer number, and that this effect will be greater when faced with the actual deserts than with their pictures.<sup>33</sup> The experimental results confirm these predictions. Specifically, when faced with the real deserts, subjects who were asked to remember the seven-digit number chose cake 63% of the time, while subjects given the easier two-digit number chose cake 41% of the time, and this difference was statistically significant. In contrast, when faced with the pictures of the deserts, the choices were 45% and 42% respectively, and the difference was not significant.<sup>34</sup>

The finding that the increasing the cognitive load increases responsiveness to temptation can be easily captured in our model by the assumption that the marginal cost of self-control is higher when the long-run self, which we identify with cognitive processing, has other demands on its resources. Here are two ways of formalizing this. First, we can assume that the cognitive center has a fixed amount  $D$  of cognitive resources, and that the resources required for self-control are proportional to the short-run foregone utility. Then when other cognitive tasks consume  $d$  resources, the utility of the short-run self can be reduced by at most  $D - d$ ; greater self-control is simply infeasible. Alternatively, we suppose that the cognitive center does not face a fixed resource constraint, but instead has increasing marginal cost. In the former case, Assumption 2 is violated, since some actions have infinite costs. Moreover, neither version makes sense with the linearity assumption, 5’. Specifically, we want to explain how ranking are changed by self-control costs, so that we want the cognitive load  $d$  to change the ranking of alternatives.

To make a formal connection with our model, let the possible actions be  $h$  (chocolate) and  $f$  (fruit). The short-term utilities are  $u^h$  and  $u^f$ , with  $u^h > u^f$ , so there is no self-control cost in choosing  $h$ . To make this into a self-control problem we also assume that the long-term utility of fruit is higher than that of chocolate. Finally suppose

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<sup>33</sup> They base this last hypothesis on the work of Lowenstein [1996].

<sup>34</sup> Ward and Mann [2000] report a similar effect of cognitive load (a combination of a memorization task and a response-time task) on individuals who a pre-experiment survey identified as “restrained eaters.”

that the state  $y$  is simply the level of cognitive load  $d$ . Then, from assumption 5, the self-control cost of action  $f$  with load  $d$  is  $C(d, f) = c(d, u^h, u^f)$ . To explain the data we need this cost to be increasing in  $d$ . One way to do this would be to set

$$C(d, f) = d \cdot (u^h - u^f).$$

This is consistent with the linearity assumption A5', but seems unsatisfactory: If using cognitive resources for memorization increases the marginal cost of self-control, then we would expect that using these resources for self-control should also change the marginal cost of self control, and hence the ranking of alternatives. That is, we would expect that having a more attractive foregone alternative should have the same effect as a higher value of  $d$ . This motivates us to prefer the specification

$$C(d, f) = g(d + u^h - u^f),$$

where  $g$  is an increasing convex function with  $g' > 0$  and  $g'' > 0$ . Moreover, the non-linear formulation, but not the linear one, fits the psychological evidence that self-control is a limited resource, as discussed by Muraven, Tice and Baumeister [1998] and Muraven and Baumesiter [2000]. While the linear and non-linear formulations are equivalent on the data of the “deserts” experiment, they make different predictions about how behavior changes when a third alternative is added, as we show by example in the next section.

In the quasi-hyperbolic model, there is a sequence of far-sighted individuals with time-inconsistent preferences parameterized by  $\beta$ , so the most obvious way to explain the Shiv and Fedorikhin result is to assume that  $\beta$  decreases when the cognitive center has other tasks. This is roughly analogous to our proposal, but to us it seems more natural and direct to assume that cognitive load uses up self-control resources.

The effect of substituting photographs for the actual deserts shows the importance of cues and framing: The evidence supports Lowenstein’s theory that vividness influences the effect of temptation. This raises the conjecture that the use of “rules of thumb” like “only have sweets at dinner” aids self-control by reducing the vividness of the temptation; one way to investigate this would be to compare brain scans of agents who use these rules to other agents who do not.

## 7. Axiomatizations of Self Control

At this point, we would like to relate our model to the axiomatizations of Gul and Pesendorfer and Dekel, Lipman, and Rustichini.<sup>35</sup> As we indicated earlier, their axioms need not be satisfied if we make Assumption 5 but not the stronger Assumption 5'. To make this claim precise, we use a dynamic programming approach to reduce our dynamic problem to the two-period problems considered in Gul and Pesendorfer [2001] and Dekel, Lipman and Rustichini [2005].<sup>36</sup> So we now suppose that the initial period action  $a$  has no utility consequences for the short-run self, and serves solely to establish the state  $y$  determining utility possibilities starting in period 2, as the banking action does in the model of section 4. For any initial period choice of action, we can then consider a plan of action starting in period 2. This plan results in a second-period utility to the short-run self of  $u$ , and a present value of  $V$  to the long-run self starting in period 3. From the viewpoint of the long-run agent at the start of the second period, what matters are the feasible utility consequences  $(u, V)$ , so we can think of the long-run self in the initial period as choosing a set of feasible  $(u, V)$  pairs. In our model, the resulting utility to the initial long-run self choosing the set  $W$  is given by

$$\Phi(W) = \max_{(u, V) \in W} [u + \delta V - C(\max(u' \mid (u', V') \in W), u)].$$

This gives rise to a preference ordering over sets  $W$  and we can ask when and whether this preference ordering satisfies various axioms.

Gul and Pesendorfer [2001] show that their axioms are equivalent to a representation

$$\max_{(u, V) \in W} [h(u, V) + H(u, V)] - \max_{(u, V) \in W} H(u, V).$$

When

$$C((\max(u' \mid (u', V') \in W), u) = \gamma(\max(u' \mid (u', V') \in W) - u),$$

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<sup>35</sup> See also Noor [2005] for an alternative approach to axiomatization using time as a key variable.

<sup>36</sup> Gul and Pesendorfer [2004] consider an infinite-horizon dynamic programming problem. We do not compare our model with those axioms because they are more complicated, and because this paper does not provide axioms for the choices of actions to complement its axioms on choices over choice sets.

we take  $h(u, V) = (1 + \gamma)u + \delta V$  and  $H(u, V) = u$ , so under Assumption 5' our model is consistent with the axioms.<sup>37</sup>

On the other hand, when  $C$  is not linear, as in our preferred explanation of the cognitive load experiment, our model is consistent with neither their axioms nor the weaker ones of Dekel, Lipman, and Rustichini. To see this, consider a lottery that takes place after  $a$  is chosen by the short-run self, so that  $u, V$  represent random variables rather than constants. In our model, what matters to either the short- or long-run self is the expected present value of this lottery, so our ranking over choice sets is given by

$$\Phi(W) = \max_{(u, V) \in W} [u + \delta V - C(\max(Eu' \mid (u', V') \in W), Eu)].$$

But this ranking violates the independence axiom of Gul and Pesendorfer and Dekel, Lipman, and Rustichini, which is a combination of the usual independence axiom for single lotteries and an assumption about the agent's indifference to the timing of the resolution over uncertainty.<sup>38</sup> That is, *ex ante*, at the time that the long-run self chooses the set of alternatives, our ranking is not given by an expected value, but by a non-linear function of an expected value,<sup>39</sup> and so violates the independence axiom. This is closely connected to the insight of Machina [1984], who shows that the ranking over lotteries that is induced *ex ante* before a decision (such as a saving decision or self-control decision) is taken will generally violate the independence axiom, even when the independence axiom is satisfied *ex post* as it is here. Our conclusion is that in this setting the independence axiom *ex ante* is not compelling.<sup>40</sup> In terms of observed choices, the independence axiom implies that we would see the same distribution of choices if the two desert options were replaced by lotteries that gave a probability  $p$  of the chosen desert and probability  $1 - p$  of no desert at all. By way of contrast, our model with convex

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<sup>37</sup> Benabou and Pycia [2002] show that the equilibrium of their two-period "lobbying" model is consistent with these same axioms. As we remarked earlier, the Gul and Pesendorfer axioms are in some ways less restrictive than Assumption 5, and hence than Assumption 5'. Specifically, we constraint the "temptation" to be the foregone short-run utility, while their axioms are consistent with other "temptation functions."

<sup>38</sup> The axiom in question is Axiom 3 in both papers; it was first proposed in Dekel, Lipman and Rustichini [1999].

<sup>39</sup> More precisely, no monotone transformation of this function is linear in the expected value of the utilities.

<sup>40</sup> The conclusion of Dekel, Lipman and Rustichini mentions another possible reason for the failure of the independence axiom based on "guilt" as opposed to an intervening action.

costs implies that more agents should chose fruit when the probability  $p$  of a desert is lower.

In addition, there can be differences between our model the Gul and Pesendorfer axioms even in the choices over deterministic choice sets. This is shown by example 2 of Dekel, Lipman, and Rustichini: There are three possible actions, broccoli ( $b$ ), frozen yogurt ( $y$ ), and ice cream ( $i$ ), with  $\{b, y\} \succ \{y\}$  and  $\{b, i, y\} \succ \{b, i\}$ . Here the frozen yogurt is a “compromise” option that is appealing in the face of strong temptations but not when faced weaker ones. Dekel, Lipman, and Rustichini show that this preference over choice sets is not consistent with the Gul and Pesendorfer axioms, but it is consistent with their more general axioms. It is also consistent with our assumptions 1-5: take  $(u, V)(b) = (0, 100)$ ;  $(u, V)(y) = (8, 30)$ , and  $(u, V)(i) = (14, 0)$ , and let

$$\Phi(W) = \max_{(u, V) \in W} [u + .9V - .5(\max(u' \mid (u', V') \in W) - u)^2].$$

Then confronted with the choice between  $b$  and  $y$  the long-run self computes the value of broccoli to be  $90 - .5(64) = 58$ , which exceeds the value of yogurt, which is 35 in both the set  $\{b, y\}$  and the set  $\{y\}$ . However, in the choice set  $\{b, y, i\}$  we compute that the value of yogurt is  $35 - .5(36) = 17$ , while the value of ice cream is 14, and the value of broccoli is  $90 - .5(196) = -8$ .<sup>41</sup>

We should also point out that our Assumption 5 implies set-betweenness. The intuition for this is given in Gul and Pesendorfer [2001] who argue that dependence on only the best and chosen alternative lead to set in-betweenness. We state and prove a formal result for completeness. The theorem shows that Assumption 5 rules out the preferences in Dekel, Lipman, and Rustichini’s example 1, which are  $\{b\} \succ \{b, h\}$  and  $\{b, p\} \succ \{b, h, p\}$ . These preferences could describe a situation where the long-run self is uncertain which of  $h$  and  $p$  will be more tempting, so the example illustrates the fact that our model requires that no uncertainty is realized between the time that the long-run self chooses  $r$  and the time that the short-run self picks  $a$ . This is more plausible when this time interval is very short.

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<sup>41</sup> Dekel, Lipman, and Rustichini point out that these preferences are consistent with either the Gul and Pesendorfer set-betweenness axiom or their independence axiom, but not both. Note that these preferences are not consistent with our Assumption 5’.

**Theorem 6:** *Under Assumptions 1-5, the induced preferences  $\Phi$  over choice sets satisfy set-betweenness. That is, for all choice sets  $W, Z$  either  $W \succeq W \cup Z \succeq Z$  or  $Z \succeq W \cup Z \succeq W$ .*

*Proof:* Let  $(u_W, V_W)$  be the act that is chosen from  $W$ ,  $(u_Z, V_Z)$  be the act that is chosen from  $Z$ , and  $(u_Q, V_Q)$  be the act that is chosen from  $Q = W \cup Z$ , and let  $w', z'$ , and  $q'$  be the corresponding temptations, that is,  $w' = \max_{w \in W} u$ ,  $z' = \max_{z \in Z} u$  and  $q' = \max(w', z')$ .

Then

$$\begin{aligned}\Phi(Q) &= u_Q + \delta V_Q - C(q', u_Q) \\ &= \max_{(u_Q, V_Q) \in W \cup Z} \{u_Q + \delta V_Q - C(q', u_Q)\}.\end{aligned}$$

Suppose without loss of generality  $(u_Q, V_Q) \in W$ . Then

$$\begin{aligned}\Phi(Q) &= \max_{(u_Q, V_Q) \in W \cup Z} \{u_Q + \delta V_Q - C(q', u_Q)\} \\ &= \max_{(u_Q, V_Q) \in W} \{u_Q + \delta V_Q - C(q', u_Q)\} \\ &\leq \Phi(W)\end{aligned}$$

so  $W \succeq W \cup Z$ .

Moreover, if  $q' = z'$  then

$$\begin{aligned}\Phi(Q) &\geq \max_{(u_Q, V_Q) \in Z} \{u_Q + \delta V_Q - C(q', u_Q)\} \\ &= \max_{(u_Q, V_Q) \in Z} \{u_Q + \delta V_Q - C(z', u_Q)\} \\ &= \Phi(W)\end{aligned}$$

so  $W \cup Z \succeq Z$ , and set-betweenness is satisfied, while if  $q' = w'$  then

$$\begin{aligned}\Phi(Q) &= \max_{(u_Q, V_Q) \in W} \{u_Q + \delta V_Q - C(w', u_Q)\} \\ &= \Phi(W)\end{aligned}$$

so  $W \cup Z \sim W$ .



In summary, our Assumptions 1-5 imply the set-betweenness property assumed by Gul and Pesendorfer, but not their independence axiom, while the Gul and Pesendorfer axioms are more general about the nature of temptations. The extra generality obtained by dropping the independence axiom is needed to explain the effect of self-control when the short-run self is choosing between lotteries, and to model the idea that self-control is a limited resource. The usual arguments and evidence for the independence axiom no longer make sense when an intermediate decision such as self-control is involved.

## **8. Conclusion and Discussion**

Our resolution of the Rabin paradox shows how the dual self model can capture some sorts of “framing” effects, as the model makes different predictions about the response to unanticipated payments depending on whether they are received on the floor of a casino or the lobby of a bank. Cues are obviously the key to understanding framing. The dual self theory implies that it is the attention span of the short-run self that is relevant for determining what constitutes a “situation, ” which is the most difficult modeling issue in confronting these types of issues. This suggests that one might be able to use experimental and physiological data to determine what the relevant frames are. The dual self theory would then enable us to paste information about the motivation of the myopic self into the broader context in which real decision making takes place.

Most existing work on cues, such as Laibson [2001] and Bernheim and Rangel [2004], abstracts away from self-control costs. Laibson analyzes a “rational addiction” model. This is a model with a single, fully rational, agent, in a setting where there are two cues, namely “green lights ” and “red lights.” The utility from engaging in the addictive utility when a given light is on depends on one’s past behavior under that particular light. Because it is based on rational choice, the model has a unique equilibrium, but that equilibrium has four steady states: never indulge, always indulge, indulge iff green, and indulge iff red. This shows how the agent’s experience can determine the importance of the cues, but does not allow the agent to have a preference for self-control. Bernheim and

Rangel [2004] consider an addiction model where the agent can sometimes enters a “hot mode” in which he consumes the addictive good whether or not his “cool self” wants him to. The probability that this occurs depends on the observed cues, while which cues trigger the hot mode in turn depends on the agent’s past frequency of use and also on whether the cool self chooses to expend resources on avoiding the cues. This model, like ours, captures a value for self-control while avoiding the multiple equilibria of a multiple-selves model, but it differs in a number of respects. Most notably, in their model self control is costless in the cool mode and infinitely costly in the hot one. In our model the agent is in a “hot mode” whenever his actions have short-term consequences, but even in hot mode the long-run self can exert self control. The fact the self-control is always possible, albeit costly, is what underlies our finding that the agent responds differently to small versus large gambles in our banking model: a sufficiently large windfall will trigger self control and the long-term perspective.

We focus on the case of “sophisticated” agents who are aware of their own self-control costs. Many papers on self-control problems consider the case of “naïve” agents, who have a current self-control problem but incorrectly forecast that they will not have such problems in the future. For example, O’Donoghue and Lowenstein [2004] consider an extension similar to allowing agents to misperceive the future value of  $\gamma$ . O’Donoghue and Lowenstein also point out that that if the current long-run self has correct expectations but does not care about future costs of self control, the decision making process becomes a game between the long-run selves, and the result is equivalent to the usual quasi-hyperbolic discounting.<sup>42</sup>

A more complicated extension is allowing for the preferences of the short-run selves to respond in some way to future consequences. The related work of Benhabib and Bisin [2004] does allow for one form of this responsiveness. They consider a consumption-savings model in which exercising self-control is a 0-1 and costly decision, made by a “cognitive control center” that corresponds to the long-run self in our model. Temptation is stochastic, and its strength is determined by an exogenous i.i.d. sequence of cues, so that costly self-control is only used when giving in to temptation is sufficiently costly. The equilibrium again corresponds to the solution of an optimization

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<sup>42</sup> They also use their model to explain non-linear probability weighting in the assessment of risks

problem by the long-run self, who takes the behavior rule of the affective self as given. In contrast to our own paper and other related work, the behavior of the “affective self” – the mental unit that is susceptible to temptation- is not required to be myopic. Instead, the affective self’s strategy can depend on expectations of future play, but it must be independent of the distribution of temptations and the cost of self control. As an example, they suggest the case where the behavior of the affective self is a Markov-perfect equilibrium of the game where self-control is impossible. If the actual outcome is that the cognitive center does sometimes exercise self-control, this raises the question of how the adaptive selves would come to learn the equilibrium play of the wrong game.<sup>43</sup>

Instead of following the expectations-based approach, we would like to model the long-run self having “taught” the short-run self to attach positive affective weight to certain variables that have long-run consequences, as in the learning of cues. Here the use of stimulus-response models of learning may play an important role. The standard forms of these models seem to be a poor fit for many aspects of human cognition.<sup>44</sup> For example, faced with no observations people will respond differently depending on their prior knowledge of a situation.<sup>45</sup> More strikingly, we know that people can learn by “figuring things out” without any external stimulus at all. However, these cognitive activities can be sensibly regarded as aspects of the long-run self, while it makes sense to model the expectations of the short-run self as arising from process of stimulus-response learning that depends solely on the past history, and does not involve forward-looking expectations. We would then theorize, based on introspection and casual empiricism, that the long-run self can train the short-run self by manipulating this stimulus-response learning.

Finally, we should point out that the presence of self-control leads to some potentially perverse implications for public policy. For example, as Wertenbroch [1998] discusses, a common self-control strategy is to reduce temptation by avoiding stockpiling

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<sup>43</sup> It is of course possible that either the short-run or long-run self or both may misperceive the future. However, learning makes it less likely that these misperceptions will involve frequently experienced variables such as  $\gamma$ , or more broadly, one’s own ability to exercise self-control. Moreover, when little learning occurs, it is not clear why one should expect to see equilibrium play.

<sup>44</sup> By the “standard model” here we mean that reinforcements are applied directly to actions. Stimulus-response dynamics can be defined on much larger spaces of sequences of actions, hypothetical reinforcements, etc.; at that level of generality they encompass a much larger set of phenomena.

<sup>45</sup> For a particularly striking experiment where learning takes place in the absence of feedback, see Roberto Weber [2003].

of durable “temptation goods.” As an example, he points to the fact that some people buy cigarettes only by the pack even though cartons are much cheaper. The rationale is that this reduces the temptation of having lots of cigarettes around the house, and so mitigates the cost of self-control. Suppose a law was passed that cigarette store were open only on the first day of each month. Then some people who would otherwise have purchased cigarettes by the pack will instead choose to stockpile cartons for the month. In the face of the greater temptation, their smoking will be increased. Of course, other people may reduce or quit smoking as a result of the law, so the overall effect is ambiguous, but there is at least the potential for a public policy achieving the opposite of the desired effect. A more serious application is to that of illegalization of recreational drugs. Making sales illegal, while making possession legal or at most a mild offense raises the fixed cost of conducting a transaction. The more transactions, the greater the chance the dealer will get caught. Consequently the likelihood of consumers stockpiling drugs is increased, creating a subsequent increase in consumption due to the problem of self-control. For these types of goods a policy such as legalization together with a high excise tax (as is the case for cigarettes) may prove more effective in reducing consumption than abolition. Notice also that there is a difference for “temptation” goods that are durable and those, such as services, that are not. Illegalizing prostitution should have different effect than illegalizing marijuana – the latter can be stockpiled, but not so easily the former.

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## Appendix A: The Consumption-Savings Model

I. We first give a result for general per-period utility functions in the simple savings model. Consider the problem of maximizing

$$U(\vec{a}) = \sum_{t=1}^{\infty} \delta^{t-1} [(1 + \gamma)u((1 - a_t)y_t) - \gamma u(y_t)]$$

over all feasible plans  $\vec{a}$ , i.e. plans that satisfy  $a_t \in [0, 1]$  and the wealth equation  $y_t = Ra_{t-1}y_{t-1}$ . We suppose that  $u$  is non-decreasing and continuous on  $(0, \infty)$ ; we do not require continuity on  $[0, \infty)$  because we want to allow for the logarithmic case where  $u(0) = \lim_{c \rightarrow 0} u(c) = -\infty$ . Let  $\bar{U}$  be the supremum in this problem.

**Proposition A.1:** *Suppose  $R > 1$  and*

$$(A.1) \quad \sum_{t=1}^{\infty} \delta^{t-1} u(y_0 R^t) < \infty.$$

- Then: (i) For any feasible plan the sum defining  $U$  has a well defined value in the sense that either the sum converges absolutely or converges to  $-\infty$ .
- (ii) The supremum  $\bar{U}$  of the feasible values satisfies  $-\infty < \bar{U} < \infty$
- (iii) If feasible  $\vec{a}^n \rightarrow \vec{a}^*$  in the product topology then  $\vec{a}^*$  is feasible. If in addition  $U(\vec{a}^n) \rightarrow \bar{U}$  then  $U(\vec{a}^*) = \bar{U}$ .
- (iv) An optimal plan exists. That is, there is a feasible plan that attains  $\bar{U}$ .

*Proof:* (i) For any sequence  $(\vec{a}, \vec{y})$  with  $y_t = Ra_{t-1}y_{t-1}$ , let

$$\chi_+(x) = \begin{cases} x & x > 0 \\ 0 & x \leq 0 \end{cases}$$

and  $\chi_-(x) = -\chi_+(-x)$ . We can write any finite sum as the sum of negative and positive parts.

$$\begin{aligned} & \sum_{t=1}^T \delta^{t-1} [(1 + \gamma)u((1 - a_t)y_t) - \gamma u(y_t)] = \\ & \sum_{t=1}^T \delta^{t-1} \chi_+ [(1 + \gamma)u((1 - a_t)y_t) - \gamma u(y_t)] + \\ & \sum_{t=1}^T \delta^{t-1} \chi_- [(1 + \gamma)u((1 - a_t)y_t) - \gamma u(y_t)] \end{aligned}$$

The positive part of the sum is summable from (A.1), since

$$(1 + \gamma)u((1 - a_t)y_t) - \gamma u(y_t) \leq u((1 - a_t)y_t) \leq u(y_0 R^t).$$

The negative part is monotone decreasing in  $T$ , so it either converges absolutely or converges to  $-\infty$ . In the former case the entire sum converges absolutely; in the latter case the sum converges to  $-\infty$ .

(ii) Part (i) already shows that  $\bar{U} < \infty$ . To see that  $\bar{U} > -\infty$ , note that it is feasible to set  $a_t = 1/R$  for all  $t$ , and that for  $R > 1$  this plan yields a finite value.

(iii) Consider a sequence of feasible plans  $\vec{a}^n \rightarrow \vec{a}^*$ . Because the constraints are period by period and closed, it is clear that  $\vec{a}^*$  satisfies the constraints, so it is feasible. Now suppose in addition that  $U(\vec{a}^n) \rightarrow \bar{U}$ . Choose any  $\varepsilon > 0$  and pick  $n$  large enough that  $\bar{U} - U(\vec{a}^n) < \varepsilon/2$ . If we now pick  $\tau$  such that

$$\sum_{t=\tau+1}^{\infty} \delta^{t-1} |u(R^t y_0)| < \varepsilon/2,$$

we know that

$$\bar{U} - \sum_{t=1}^{\tau} \delta^{t-1} [(1 + \gamma)u((1 - a_t^n)y_t^n) - \gamma u(y_t^n)] \leq \varepsilon.$$

Since  $\bar{U}$  is finite, and payoffs in the first  $\tau$  periods are bounded above by  $u(y_0 R^t)$ , each term in this summation is bounded below (by  $\bar{U} - \varepsilon - T u(y_0 R^t)$ ). Since per-period payoffs are continuous at any  $(a, y)$  with  $a > 0$ ,  $\vec{a}^n \rightarrow \vec{a}^*$ , and  $y_t^n \rightarrow y_t^*$ , it follows that

$$\bar{U} - \sum_{t=1}^{\tau} \delta^{t-1} [(1 + \gamma)u((1 - a_t^*)y_t^*) - \gamma u(y_t^*)] \leq \varepsilon.$$

Since this is true for any  $\varepsilon > 0$  and we know that  $U(\vec{a}^*) \leq \bar{U}$ , we conclude that  $U(\vec{a}^*) = \bar{U}$ .

(iv) Now consider a feasible sequence  $(\vec{a}^n, \vec{y}^n)$  with  $U(\vec{a}^n) \rightarrow \bar{U}$ . Each savings rate  $a_t$  must lie in the compact interval  $[0, 1]$  and each  $y_t$  must lie in the compact interval  $[0, R^t y_0]$ , so the sequence  $(\vec{a}^n, \vec{y}^n)$  has an accumulation point  $(\vec{a}^*, \vec{y}^*)$  in the product topology. This accumulation point is a maximum by part (iii).

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II. Now we specialize to the CRRA utility functions

$$u(c) = \frac{(c)^{1-\rho} - 1}{1-\rho}$$

and  $u(c) = \ln(c)$ , which corresponds to the case  $\rho = 1$ . Assuming  $\delta < R^{\rho-1}$  implies

$$\sum_{t=1}^{\infty} \delta^{t-1} u(y_0 R^t) < \infty.$$

It follows from Proposition A.1 that an optimum  $\vec{a}^*$  exists.

**Proposition A.2:** *With CRRA utility a stationary optimum with  $a_t = a$  exists.*

Proof: Suppose that  $\vec{a}^*$  is an optimal plan. By homogeneity of the objective function, and the fact that plans are defined in terms of savings rates,  $\vec{a}^*$  is also an optimal plan starting in period 2 (for any initial condition). Note that the plan  $\vec{a}^2 = (a_1^*, a_1^*, a_2^*, a_3^*, \dots)$  yields wealth in period 2 of  $a_1^* R y_1$ , and let  $\bar{U}(y_1)$  denote the maximized utility when starting in the second period with wealth  $y_1$ . Then

$$U(\vec{a}^2) = (1 + \gamma)u((1 - a_1^*)y_0) - \gamma u(y_0) + \delta \bar{U}(a_1^* R y_0) = \bar{U}$$

where the first equality follows because  $\vec{a}^*$  is optimal from period 2 on, and the second equality because  $\vec{a}^*$  is optimal from the first period. Proceeding in this way we can construct sequence of feasible plans  $\vec{a}^n = (a_1^*, a_1^*, \dots, a_1^*, a_2^*, a_3^*, \dots)$  that play  $a_1^*$  for the first  $n$  periods such that  $U(\vec{a}^n) = U(\vec{a}^*) = \bar{U}$ . Clearly  $\vec{a}^n$  converges in the product topology to the plan of choosing the fixed savings rate  $a_1^*$ . Hence it follow from Proposition A.1 (iii) that this limiting plan is feasible and gives utility  $\bar{U}$ ; that is, it is optimal.



III. We have shown that it is sufficient to compute the present value utility from a fixed savings rate  $a$ , and maximize over savings rates. We have present value utility

$$\begin{aligned} U &= \frac{y_0^{1-\rho}}{1-\rho} \sum_{t=1}^{\infty} (\delta(Ra)^{1-\rho})^{t-1} [(1+\gamma)(1-a)^{1-\rho} - \gamma] - \frac{1}{(1-\delta)(1-\rho)} \\ &= \frac{y_0^{1-\rho}}{1-\rho} \frac{[(1+\gamma)(1-a)^{1-\rho} - \gamma]}{1 - \delta(Ra)^{1-\rho}} - \frac{1}{(1-\delta)(1-\rho)} \end{aligned}$$

Since the optimal savings rate cannot be 0 or 1, we may differentiate with respect to the saving rate to find

$$\begin{aligned} dU / da &= \\ y_0^{1-\rho} (1-a)^{-\rho} &\frac{[(1+\gamma)(1-a) - \gamma(1-a)^\rho] \delta R (Ra)^{-\rho} - (1+\gamma)(1 - \delta(Ra)^{1-\rho})}{(1 - \delta(Ra)^{1-\rho})^2} \end{aligned}$$

which gives necessary condition for an optimum<sup>46</sup>

$$(1+\gamma)a^\rho = R^{1-\rho} \delta ((1+\gamma) - \gamma(1-a)^\rho).$$

When  $\gamma = 0$  we get the usual solution  $a^* = R^{(1-\rho)/\rho} \delta^{1/\rho}$ . Thus we can rewrite the first order condition as

$$(a/a^*)^\rho = ((1+\gamma) - \gamma(1-a)^\rho) / (1+\gamma).$$

IV: Turning to the simple banking model, utility starting in the second period is the  $\gamma = 0$  solution

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<sup>46</sup> We do not know if the first-order condition has a unique solution, except in the logarithmic case.

$$\begin{aligned}
U_2(y_2) &= \frac{y_2^{1-\rho}}{1-\rho} \frac{(1-a^*)^{1-\rho}}{1-\delta(Ra^*)^{1-\rho}} - \frac{1}{(1-\delta)(1-\rho)} \\
&= \frac{y_2^{1-\rho}}{1-\rho} \frac{1}{(1-a^*)^\rho} - \frac{1}{(1-\delta)(1-\rho)} \\
&= \frac{y_2^{1-\rho}}{1-\rho} \frac{1}{(1-\delta^{1/\rho}R^{(1-\rho)/\rho})^\rho} - \frac{1}{(1-\delta)(1-\rho)}
\end{aligned}$$

The utility of both selves in the first period is

$$(1+\gamma) \frac{(c_1)^{1-\rho} - 1}{1-\rho} - \gamma \frac{(x_1 + z_1)^{1-\rho} - 1}{1-\rho},$$

and so the overall objective of the long-run self is to maximize

$$\begin{aligned}
&(1+\gamma) \frac{(c_1)^{1-\rho} - 1}{1-\rho} - \gamma \frac{(x_1 + z_1)^{1-\rho} - 1}{1-\rho} \\
&+ \frac{(R(y_1 + z_1 - c_1))^{1-\rho}}{1-\rho} \frac{\delta}{(1-\delta^{1/\rho}R^{(1-\rho)/\rho})^\rho} - \frac{\delta}{(1-\delta)(1-\rho)}
\end{aligned}$$

The first order condition for optimal consumption is

$$\frac{c_1}{R(y_1 + z_1 - c_1)} = \frac{(1+\gamma)^{1/\rho} (1-\delta^{1/\rho}R^{(1-\rho)/\rho})}{(\delta R)^{1/\rho}}.$$

If there are one or more solutions that satisfy the constraint  $c_1 \leq x_1 + z_1$  then one of them represents the optimum; otherwise the optimum is to consume all pocket cash,  $c_1 = x_1 + z_1$ .

Note that  $x_1$  is the solution for  $\gamma = 0$ , so it satisfies

$$\frac{x_1}{R(y_1 - x_1)} = \frac{(1-\delta^{1/\rho}R^{(1-\rho)/\rho})}{R^{1/\rho}}.$$

Thus we can write the first order condition as

$$\frac{c_1}{y_1 + z_1 - c_1} = (1+\gamma)^{1/\rho} \frac{x_1}{y_1 - x_1}$$

or

$$\begin{aligned}
c_1 &= \frac{(1 + \gamma)^{1/\rho} x_1}{y_1 - x_1 + (1 + \gamma)^{1/\rho} x_1} (y_1 + z_1) \\
&= \frac{1 + [(1 + \gamma)^{1/\rho} - 1]}{1 + [(1 + \gamma)^{1/\rho} - 1](1 - a^*)} (1 - a^*) (y_1 + z_1) \\
&= B(1 - a^*) (y_1 + z_1)
\end{aligned}$$

### Appendix B: Procrastination and Delay

**Theorem 3** (i)  $\underline{x} \leq x^* < \bar{x}$ ; if  $\delta v > \delta \mu + (1 - \delta)(1 + \gamma)\underline{x}$  then  $x^* > \underline{x}$ .

(ii) When  $\underline{x} < x^* < \bar{x}$ ,

$$-\gamma x^* + \delta V = x^* + \delta W(x^*, \gamma) \quad (9).$$

*Proof:*

i) Suppose that the optimum is at  $x^* = \bar{x}$ , and that  $x_t = \bar{x}$ . Then doing it now gives payoff of  $\delta V - \gamma \bar{x}$ , while waiting one period and then conforming to the presumed optimal rule gives  $\bar{x} + \delta^2 V - \delta \gamma \mu$ , because the agent is certain to act next period. Thus waiting is optimal unless  $\delta(1 - \delta)V \geq \bar{x} + \gamma(\bar{x} - \delta \mu)$ , but this contradicts  $\delta v < \bar{x}$ . Now suppose  $x^* = \underline{x}$  and  $x_t = \underline{x}$ . Conforming to the strategy yields payoff  $\underline{x} + \delta \mu / (1 - \delta)$ , while acting yields

$$\delta V - \gamma \underline{x} = \frac{\delta v}{(1 - \delta)} - \gamma \underline{x},$$

so acting is better if  $(1 - \delta)\underline{x} + \delta \mu < \delta v - (1 - \delta)\gamma \underline{x}$  or  $\delta v > (1 - \delta)(1 + \gamma)\underline{x} + \delta \mu$ .

(ii) If the optimal cutoff  $x^*$  is in the interior of  $[\underline{x}, \bar{x}]$ , optimality implies that the agent is indifferent between waiting and acting when  $x = x^*$ . Thus  $-\gamma x^* + \delta V = x^* + \delta W(x^*, \gamma)$ , which establishes

$$x^* (1 + \gamma) = \delta V - \delta W(x^*, \gamma). \quad (9)$$

Because  $x^*$  maximizes steady state payoff,  $W_x|_{x=x^*} = 0$ , Thus

$$(1 + \gamma) \frac{dx^*}{d\gamma} = -x^* - W_\gamma(x^*(\gamma), \gamma),$$

so 
$$\frac{dx^*}{d\gamma} < 0 \text{ if } x^* > -W_\gamma(x^*(\gamma), \gamma)$$

Now from (8),

$$\begin{aligned} -W_\gamma &= \frac{P(x^*)}{1 - \delta + \delta P(x^*)} E(x | x < x^*) \\ &< \frac{P(x^*)}{(1 - \delta)1 + \delta P(x^*)} x^* < x^*. \end{aligned} ,$$

So  $\frac{dx^*}{d\gamma} < 0$ .

☑

**Theorem 4:**

a) If  $v > \mu$  and  $\bar{x} > \delta\beta v$ , the “sophisticated quasi-hyperbolic model” has a stationary equilibrium with  $\underline{x} < x^{**} < \bar{x}$ .

b) There is an open set of parameters that satisfies the restrictions of part a) and for which there are other equilibria.

*proof:* Substituting (10) into (11) we see that

$$\begin{aligned}
x^{**} &= \delta\beta \left( \frac{V - \delta V + \delta P(x^{**})V - P(x^{**})(\delta V) - (1 - P(x^{**}))E(x \mid x > x^{**})}{1 - \delta + \delta P(x^{**})} \right) \\
&= \\
&\delta\beta \left( \frac{v - (1 - P(x^{**}))E(x \mid x > x^{**})}{1 - \delta + \delta P(x^{**})} \right).
\end{aligned}$$

$$\text{Let } F(x^{**}) = x^{**} - \delta\beta \left( \frac{v - (1 - P(x^{**}))E(x \mid x > x^{**})}{1 - \delta + \delta P(x^{**})} \right).$$

To prove the theorem it suffices to show that there is an  $x^{**}$  where  $F(x^{**}) = 0$ . The assumption that  $v > \mu$  implies that  $F(0) = 0 - \delta\beta \left( \frac{v - \mu}{1 - \delta} \right) < 0$ , and the assumption that  $\bar{x} > \delta\beta v$  implies that  $F(\bar{x}) = \bar{x} - \delta\beta v > 0$ .

b) Suppose that  $x^1 = \underline{x}$  and  $x^2 = \bar{x}$  so that the odd-numbered agents never act and even ones always act. Then the equilibrium payoff of an even-numbered agent is  $\delta\beta V$ , and the payoff of an even-numbered agent with cost  $x_t$  who chooses to wait is  $x_t + \delta\beta\mu + \delta^3\beta V$ , so the even agents' strategy is a best response for all  $x_t$  if  $\delta\beta V(1 - \delta^2) > \bar{x} + \delta\beta\mu$ , or equivalently

$$\delta\beta(v(1 + \delta) - \mu) > \bar{x}. \quad (12)$$

The payoff of the odd-numbered agents who wait is  $x_t + \delta^2\beta V$ , and the payoff to acting is  $\delta\beta V$ , so waiting is better for all  $x_t$  if

$$\underline{x} > \delta\beta(1 - \delta)V = \delta\beta v. \quad (13)$$

To complete the proof we must show that there is an open set of parameters such that (12) and (13) and the restrictions  $v > \mu$  and  $\bar{x} > \delta\beta v$ . To do this, fix  $v > 0$  and  $\delta, \beta \in (0, 1)$ , and some  $\varepsilon > 0$ . Set  $\underline{x} = (1 + \varepsilon)\delta\beta v$ ,  $\mu = (1 + 2\varepsilon)\delta\beta v$ , and  $\bar{x} = (1 + 3\varepsilon)\delta\beta v$ . (Note that these conditions are consistent with a range of distributions, including the uniform.) By construction this satisfies  $\bar{x} > \delta\beta v$  and (13).

If  $\varepsilon < \frac{1 - \delta\beta}{2\delta\beta}$  then  $\mu = (1 + 2\varepsilon)\delta\beta v < (1 + \frac{1 - \delta\beta}{\delta\beta})\delta\beta v = v$ , and for

$\varepsilon < \frac{\delta(1 - \beta)}{(3 + 2\delta\beta)}$  we compute that

$$\begin{aligned}
 \delta\beta(v(1 + \delta) - \mu) &= \delta\beta(v(1 + \delta) - (1 + 2\varepsilon)\delta\beta v) \\
 &= \delta\beta v(1 + \delta(1 - \beta) - 2\varepsilon\delta\beta) \\
 &> \\
 &\delta\beta v(1 + 3\varepsilon) \\
 &= \bar{x}.
 \end{aligned}$$

This shows that there is a ‘2-cycle equilibrium’ whenever  $\varepsilon$  is sufficiently small. Since the inequalities in (12) and (13) hold strictly for the specified relationship between the parameters, they hold for an open set of  $\beta, \delta, V, \mu$ ; the inequalities also hold for a range of distributions with the given mean and endpoints.

☑