

Last-In First-Out Oligopoly Dynamics*

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Abstract

This paper extends the static analysis of oligopoly structure into an infinite-horizon setting with sunk costs and demand uncertainty. The observation that exit rates decline with firm age motivates the assumption of last-in first-out dynamics: An entrant expects to produce no longer than any incumbent. This selects an essentially unique Markov-perfect equilibrium. With mild restrictions on the demand shocks, a sequence of thresholds describes firms' equilibrium entry and survival decisions. [Bresnahan and Reiss's \(1993\)](#) empirical analysis of oligopolists' entry and exit assumes that such thresholds govern the evolution of the number of competitors. Our analysis provides an infinite-horizon game-theoretic foundation for that structure.

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1 Introduction

This paper develops and presents a simple and tractable model of oligopoly dynamics based on the static entry game used by [Bresnahan and Reiss \(1990\)](#). A random number of consumers demands the industry’s services, and this state evolves stochastically. Entry possibly requires paying a sunk cost, and continued operation incurs fixed costs. Incumbents who wish to avoid these per-period fixed costs in markets that are no longer profitable exit. [Bresnahan and Reiss’s \(1993\)](#) empirical analysis of oligopolists’ entry and exit assumes that *thresholds* govern the evolution of the number of competitors. That is, entry occurs whenever demand passes above one in a sequence of entry thresholds, and exit occurs if it subsequently passes below a corresponding exit threshold. A monopolist uses a threshold-based rule for entry and exit if raising current demand stochastically increases tomorrow’s demand. We show that this condition alone does not guarantee that oligopolists use threshold rules, because a larger current market might make future entry more likely and consequently *reduce* an oligopolist’s value. Nevertheless, we provide mild conditions on the demand process that guarantee that thresholds govern all firms’ equilibrium entry and exit choices. In this way, our analysis provides an infinite-horizon game-theoretic foundation for [Bresnahan and Reiss’s \(1993\)](#) empirical framework, which can be applied to extend their earlier structural estimation of static oligopoly models to a fully dynamic setting. The model makes a unique equilibrium prediction which can be calculated very quickly, so it can be used for policy experiments. This paper’s companion ([Abbring and Campbell, 2007](#)) exemplifies this with an examination of how raising a barrier to entry for a second firm changes duopoly dynamics.

[Bresnahan and Reiss \(1991a\)](#) noted that the static oligopoly entry game can have multiple equilibria, which obviously complicates prediction. To select a unique equilibrium, both [Bresnahan and Reiss \(1990\)](#) and [Berry \(1992\)](#) assume that firms move sequentially. We take a similar approach by allowing older firms to commit to continuation before their younger counterparts. We also restrict attention to equilibria in which firms correctly believe that no firm will produce after an older rival exits. That is, the equilibria have a last-in first-out (LIFO) structure. Three considerations motivate this focus. First, it is consistent with the widespread observation that young firms exit more frequently than their older counterparts. Second, the equilibrium approximates the “natural” Markov-perfect equilibrium in an extension of the model in which firms’ costs decrease with age and the most efficient firms survive. Third and perhaps most importantly, this restriction vastly simplifies the equilibrium analysis. We prove that, irrespectively of whether the conditions for threshold rules are imposed or not, there always exists such an equilibrium and that it is (essentially) unique.

This simplicity comes at one substantial cost: All shocks occur at the market level, so there is no reason for simultaneous entry and exit.

The model’s theoretical simplicity makes it well-suited for exploring how parameter changes impact equilibrium dynamics and long-run market structure. To show this, we calculate the effects of increasing demand uncertainty on firms’ equilibrium entry and exit thresholds. Non-strategic analysis of the firm life cycle suggests that additional uncertainty should raise the value of the option to exit and thereby substantially lower both entry and exit thresholds. The oligopolistic exit thresholds do indeed fall with uncertainty, but the entry thresholds do not. Their relative invariance reflects an offsetting effect that a monopolist does not face: Increasing demand uncertainty raises the probability of further entry and thereby reduces a new firm’s value. We also calculate the population “estimates” of oligopoly profit margins using the static ordered Probit procedure of [Bresnahan and Reiss \(1990\)](#) and data generated from the model’s ergodic distribution. We find that the delay in exit arising from uncertainty (familiar from [Dixit and Pindyck, 1994](#)) biases these entry threshold estimates downwards and that this leads to a downward bias in the estimated rate that profits fall with additional competition. That is, a long-run procedure that abstracts from relevant dynamic considerations can find “evidence” that profit margins decline with entry when in fact they are constant.

The sequential nature of firms’ entry and exit decisions and the assumption that firms rationally expect LIFO dynamics substantially structures our analysis. In some previous work, the assumption that firms move sequentially commits early movers to their actions. Examples are [Dixit’s \(1980\)](#) two-period Stackelberg investment game and [Maskin and Tirole’s \(1988\)](#) infinite-horizon alternating-moves quantity game. In a finite-horizon game, ordering players’ moves usually selects a unique subgame-perfect Nash equilibrium. Such sequencing need not select a single Markov-perfect equilibrium in an infinite-horizon setting like ours, so researchers sometimes structure expectations with assumptions— such as LIFO—to select a “natural” equilibrium. For example, [Cabral \(1993\)](#) assumes that younger firms which have not yet exploited the learning curve and therefore have high costs exit before their older low-cost counterparts. Modifying our model to make fixed costs decline deterministically with a firm’s age is straightforward, and we obtain the LIFO equilibrium as a limit of the sequence of “natural” (in [Cabral’s](#) sense) equilibria to our model as we send the learning curve’s slope to zero. Thus, the LIFO equilibrium analysis seems useful for industries where incumbents enjoy small technological advantages over entrants.

[Jovanovic \(1982\)](#) and [Hopenhayn \(1992\)](#) provide analytic results for industry dynamics with many firms, but we know few similarly useful results for oligopolies with more than two

firms. This apparent intractability has led researchers to approach questions of oligopoly dynamics computationally within [Ericson and Pakes's \(1995\)](#) framework for the empirical analysis of Markov-perfect oligopoly dynamics. [Pakes and McGuire's \(1994\)](#) algorithm for its equilibrium calculation iterates on a Bellman-like operator for the firms' value functions. It is well known that Bellman iteration converges slowly for conventional dynamic programming problems with high-dimensional states, so it should be no surprise that applying it to the oligopoly problem is computationally expensive.

The proofs of equilibrium existence and uniqueness for our model proceed constructively, beginning with the survival decision for the youngest firm when no further entry can be profitable even at the highest possible demand realization. Because no incumbents exit during its lifetime, this survival decision corresponds to a standard dynamic programming problem with only the current demand as a state. With this in hand, we can determine the entry choice for any firm that would occupy this "final" position. Given these entry and survival decisions, the survival problem for a firm expecting at most one more entrant corresponds a dynamic programming problem with two states (current demand and the presence or absence of a younger competitor). Proceeding recursively in this way yields the unique Markov-perfect equilibrium decision rules and firm values with a LIFO structure. In this paper's companion ([Abbring and Campbell, 2007](#)), we have used this construction to analytically characterize the effects of raising late entrants' sunk costs and to calculate equilibria for *hundreds* of parameter values. Because the involved dynamic programming problems have a small state space, these calculations together take only a few minutes.

The remainder of this paper proceeds as follows. The next section presents the model's primitives and demonstrates the uniqueness of a Markov-perfect equilibrium with a LIFO structure. To clarify how the model's moving parts fit together, that section closes with an examination of a particular specification for the demand shocks that yields a pencil-and-paper solution. [Section 3](#) gives sufficient conditions for firms to use threshold rules for their equilibrium entry and exit decisions, and [Section 4](#) illustrates the model's application with the investigations of the dependence of oligopolists' entry and exit thresholds on demand uncertainty and of [Bresnahan and Reiss's \(1990\)](#) static estimation procedure. [Section 5](#) contains some concluding remarks regarding the model's estimation.

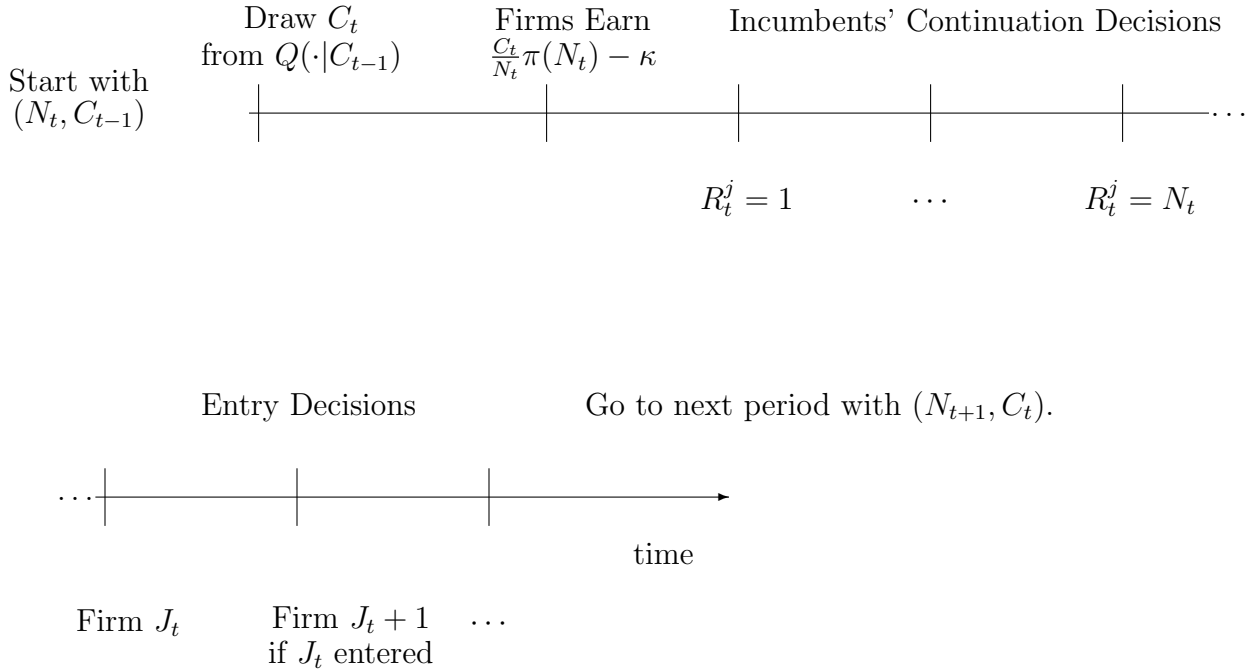


Figure 1: The Sequence of Actions within a Period

2 The Model

The model consists of a single oligopolistic market in discrete time $t \in \{0, 1, \dots\}$. There is a countably infinite number of firms that are potentially active in the market. We index these firms by $j \in \mathbb{N}$, and below we refer to j as the firm's *name*. At time 0, $N_0 = 0$ firms are active. Entry and subsequent exit determine the number of active firms in each later period, N_t . The number of consumers in the market, C_t , evolves exogenously according to a first-order Markov process bounded between $\hat{C} \geq 0$ and $\check{C} < \infty$. We denote the conditional distribution of C_t with $Q(c|C_{t-1}) \equiv \Pr[C_t \leq c|C_{t-1}]$.

Figure 1 illustrates the sequence of events and actions within a period. It begins with the inherited values of N_t and C_{t-1} . All participants observe the realization of C_t , and all active firms receive profits equal to $(C_t/N_t) \times \pi(N_t) - \kappa$. Here, each firm serves C_t/N_t consumers, and $\pi(N_t)$ is the producer surplus earned from each one. Increasing N_t weakly decreases $\pi(N_t)$. The term $\kappa > 0$ represents fixed costs of production.

After serving the market, active firms decide whether they will remain so. If firm j is

active in this period, denote the rank of its name in the set of all active firms' names with R_t^j . This equals one for the firm with the lowest name, and it equals N_t for the firm with the highest name. The active firms' continuation decisions proceed sequentially in increasing order of this rank. Exit is irreversible but otherwise costless. It allows the firm to avoid future periods' fixed production costs.

After active firms' continuation decisions, those firms that have not yet had an opportunity to enter make entry decisions in the order of their names, starting with J_t , where $J_0 = 0$. These continue until one potential entrant chooses to remain out of the industry. The first potential entrant for the next period, J_{t+1} , has this firm's name plus one. Because entry decisions proceed sequentially in increasing order of the firms' names, any entrant will have a rank greater than that of any incumbent. We denote this prospective rank with R' . The cost of entry is $\varphi(R')$. We assume that $\varphi(R') \geq 0$ and is weakly increasing in R' . This allows for, but does not require, later entrants to face a "barrier to entry" in the form of elevated sunk costs.¹ The payoff to staying out of the industry is always zero, because a firm with an entry opportunity cannot delay its choice. Both active firms' and potential entrants' decisions maximize their expected stream of profits discounted with a factor $\beta < 1$.

Before proceeding, we wish to highlight how two features of the environment interact to give older firms' priority in committing to continuation. Entry decisions proceed sequentially in increasing order of the firms' names. So if all active firms entered in different periods, their values of R_t^j rank their ages as well as their names. In this case, the ordering of continuation decisions by R_t^j puts the oldest firm first and the youngest firm last. We employ the convention of assigning R_t^j based on firms' names to break any ties that occur when two or more firms enter the market in the same period.

2.1 Markov-Perfect Equilibrium

We choose as our equilibrium concept symmetric Markov-perfect equilibrium. When firm j decides whether to stay or exit, $N_t - R_t^j$ (the number of active firms following it in the sequence), C_t , and R_{t+1}^j (its rank in the next period's sequence of active firms) are available and payoff-relevant. Collect these into $H_{jt} \equiv (N_t - R_t^j, C_t, R_{t+1}^j)$. Similarly, the payoff-relevant state to a potential entrant is $H_{jt} \equiv (C_t, R_{t+1}^j)$. Note that H_{jt} takes its values in $\mathcal{H}_S \equiv \mathbb{Z}_+ \times [\hat{C}, \check{C}] \times \mathbb{N}$ for firms active in period t and in $\mathcal{H}_E \equiv [\hat{C}, \check{C}] \times \mathbb{N}$ for potential entrants. Here and below, we use S and E to denote survivors and entrants.

¹This feature of the model is the focus of the companion paper, [Abbring and Campbell \(2007\)](#).

A Markov strategy for firm j is a pair $(A_S^j(H_S), A_E^j(H_E))$ for each $H_S \in \mathcal{H}_S$ and $H_E \in \mathcal{H}_E$. These represent the probability of being active in the next period given that the firm is currently active ($A_S^j(\cdot)$) and given that the firm has an entry opportunity ($A_E^j(\cdot)$). A symmetric Markov-perfect equilibrium is a subgame-perfect equilibrium in which all firms follow the same Markov strategy.

When firms use Markov strategies, the payoff-relevant state variables determine an active firm's expected discounted profits, which we denote with $v(H_S)$. In a Markov-perfect equilibrium, this satisfies the Bellman equation

$$v(H_S) = \max_{a \in [0,1]} a\beta \mathbb{E} \left[\frac{C'}{N'} \pi(N') - \kappa + v(H'_S) \mid H_S \right], \quad (1)$$

Here and throughout, we adopt conventional notation and denote the variable corresponding to X in the next period with X' . In Equation (1), the expectation of N' is calculated using all firms' strategies conditional on the particular firm of interest choosing to be active.

It is well known that multiple Markov-perfect equilibria can exist in similar models.² To overcome this standard difficulty, we restrict attention to equilibria in which firms' entry and exit policies arise from a last-in first-out (LIFO) strategy.

Definition 1. *A LIFO strategy is a strategy (A_S, A_E) such that $A_S(H_S) \in \{0, 1\}$, $A_E(H_E) \in \{0, 1\}$, and $A_S(N - R, C, R')$ is weakly decreasing in R .*

If all firms adopt a common LIFO strategy (A_S, A_E) , then an active firm with rank $R \geq 2$ never stays if the predecessor in the sequence of active firms exits, because

$$A_S(N - R, C, R') = 0 \Rightarrow A_S(N - R - 1, C, R') = 0.$$

As a consequence, if firms adopt a common LIFO strategy, they exit in the reverse order of their entry. Conversely, if firms use a common strategy and always exit in the reverse order of their entry, then the common strategy is a LIFO strategy.

Empirical studies of industry dynamics consistently find that young firms exit more frequently than their older rivals. For example, [Dunne, Roberts, and Samuelson \(1988\)](#) examined the survival of firms that entered manufacturing between the 1963 and 1967 Economic Censuses and lived to be sampled in 1967. Sixty four percent of these exited between 1967 and 1972. Their exit rate for the next five years was still substantial but considerably lower, 42 percent. This cohort's exit rate for the five years after that equalled 40 percent. [Jarmin,](#)

²See [Doraszelski and Satterthwaite \(2005\)](#).

Klimek, and Miranda (2003) calculated analogous five-year exit rates for firms that entered retail trade between 1977 and 1982 and lived until 1982. Of these, 60 percent exited by 1987. Their exit rates for 1987-1992 and 1992-1997 were 50 percent and 40 percent, so the decline of exit rates occurred more gradually for that cohort. The restriction to LIFO strategies mimics these declining exit rates in an extreme way, because the youngest firm always exits first. We do not doubt that firms sometimes outlast their older competitors, but the pervasive finding that exit rates decline with an entry cohort's age leads us to believe that the restriction to LIFO strategies provides a useful point of departure for examining dynamic oligopolies.

The payoff from restricting attention to LIFO strategies begins with the following proposition.

Proposition 1. *There exists a symmetric Markov-perfect equilibrium in a LIFO strategy (A_S, A_E) such that $A_S(N - R, C, R')$ is invariant in $N - R$ and weakly decreasing in R' .*

The equilibrium survival probability in Proposition 1 decreases with the firm's rank in the next period and is invariant to the number of firms with unresolved continuation decisions.

This paper's appendix contains the proposition's constructive proof, which has two critical steps. First, we note that the upper bound on C implies that the number of firms that ever produce in a Markov-perfect equilibrium cannot exceed some bound, which we call \check{N} . Because a firm with rank \check{N} expects none of its older competitors to cease production before it does, this firm's optimal exit rule corresponds to that from a simple dynamic programming problem. Second, we solve exit decision problems for firms with ranks $\check{N} - 1, \check{N} - 2, \dots, 1$ that embody the assumption that other firms follow a LIFO strategy. A firm with rank R forms its expectations about the behavior of firms with higher ranks using the solutions of those firms' decision problems. With the solutions to these standard dynamic programming problems in hand, we construct a candidate LIFO strategy and then verify that it satisfies the proposition's conditions and forms a Markov-perfect equilibrium.³

The existence proof strongly suggests that the Markov-perfect equilibrium in a LIFO strategy is unique, because the decision problems used in its construction have unique solutions to their Bellman equations. However, we might be able to construct multiple LIFO equilibria by varying a firm's actions in states of indifference between activity and inactivity.

³Amir and Lambson (2003) prove existence of a subgame-perfect equilibrium in an infinite-horizon model that is similar to ours, but in which firms move simultaneously in each stage game. They do so by constructing an equilibrium that is the limit of a sequence of LIFO equilibria in the finite-horizon versions of their model as the horizon grows to infinity. This suggests an alternative interpretation of our LIFO equilibrium as the limit of the sequence of equilibria from our model's finite-horizon analogues.

We sidestep this difficulty by concentrating on equilibria in which a firm defaults to inactivity.

Definition 2. A symmetric Markov-perfect equilibrium strategy (A_S, A_E) defaults to inactivity if $A_S(H_S) = 0$ whenever $v(H_S) = 0$ and $A_E(C, R') = 0$ whenever $v(0, C, R') = \phi(R')$.

Proposition 2. There exists a unique symmetric Markov-perfect equilibrium in a LIFO strategy that defaults to inactivity. This equilibrium's survival rule A_S is such that $A_S(N - R, C, R')$ is invariant in $N - R$ and weakly decreasing in R' .

Other symmetric Markov-perfect equilibria that default to inactivity might exist, but in them the apparent advantage of early entrants to commit to continuation does not translate into longevity. Henceforth, we constrain our attention to the unique symmetric Markov-perfect equilibrium in a LIFO strategy that defaults to inactivity.⁴

2.2 A Pencil-and-Paper Example

If we assume that $C_t = C_{t-1}$ with probability $1 - \lambda$ and that it equals a draw from a uniform distribution on $[\hat{C}, \check{C}]$ with the complementary probability, then we can calculate the model's equilibrium value functions and decision rules with pencil and paper. Before proceeding, we examine this special case to illustrate the model's moving parts. For further simplification, suppose that $\pi(N) = 0$ for $N \geq 3$, so at most two firms serve the industry. To ensure that the equilibrium dynamics are not trivial, we also assume that no firm will serve the industry if demand is low enough and that two firms will serve the industry if it is sufficiently high.⁵

To begin, consider an incumbent firm with rank 2. In an equilibrium in a LIFO strategy, its profit equals $(C/2)\pi(2) - \kappa$. It will earn this until the next time that C_t changes, at which point the new demand value will be statistically independent of its current value. It

⁴In the model, $\pi(N)$ and κ depend neither on the firm's name nor on its history. The previous literature on industry dynamics suggests three ways of relaxing this. The profitability of firms could be heterogeneous *ex ante*, drawn from some distribution upon entry, or improve with experience in the market. We can extend the model in these directions if the modifications never make a younger firm more profitable than any older rival. For example, we can incorporate irreproducible firm-specific capabilities by assigning higher fixed costs to firms with higher names. Drawing, upon entry, a firm's fixed cost from a two-point distribution with a very large higher value would also leave the model's LIFO structure intact. Such shocks can create simultaneous entry and exit as potential entrants sequentially try to draw a low fixed cost. The learning curve mentioned in this paper's introduction exemplifies firm-specific changes in profits after entry.

⁵A sufficient condition for these two properties is that $(1 - \lambda) \left[\hat{C}\pi(1) - \kappa \right] + \lambda \frac{\hat{C} + \check{C}}{2} \pi(1) - \kappa < 0$ and $\beta \left[\frac{(1 - \lambda) \left(\frac{\check{C}}{2} \pi(2) - \kappa \right) + \frac{\lambda}{1 - \beta} \left(\frac{1}{2} \frac{\hat{C} + \check{C}}{2} \pi(2) - \kappa \right)}{1 - \beta(1 - \lambda)} \right] > \varphi(2)$.

is straightforward to use these facts to show that this firm's value function is the following piecewise linear function of C :

$$v(0, C, 2) = \begin{cases} 0 & \text{if } C \leq \underline{C}_2, \\ \beta \frac{(1-\lambda)(\frac{C}{2}\pi(2)-\kappa)+\lambda\tilde{v}(0,2)}{1-\beta(1-\lambda)} & \text{if } C > \underline{C}_2, \end{cases}$$

where

$$\tilde{v}(0, 2) = \frac{1}{2} \left(\frac{\hat{C} + \check{C}}{2} \right) \pi(2) - \kappa + \int_{\hat{C}}^{\check{C}} \frac{v(0, C', 2)}{(\check{C} - \hat{C})} dC'.$$

Here, $\tilde{v}(0, 2)$ is the firm's average continuation value given a new draw of C_t and \underline{C}_2 is the largest value of C that satisfies $v(0, C, 2) = 0$. Optimality requires the firm to exit if $C < \underline{C}_2$. This value function is monotonic in C , so there is a unique entry threshold \bar{C}_2 that equates the continuation value with the entry cost. Thus, a second duopolist enters whenever C_t exceeds \bar{C}_2 and exits if it subsequently falls at or below \underline{C}_2 .

Next, consider the problem of an incumbent with rank 1. If this firm is currently a monopolist, it expects to remain so until $C_t > \bar{C}_2$; and if it is currently a duopolist, it expects to become a monopolist when C_t falls below \underline{C}_2 . This firm's value function is also piecewise linear. If the firm begins the period as the sole incumbent, it is

$$v(0, C, 1) = \begin{cases} 0 & \text{if } C \leq \underline{C}_1, \\ \beta \frac{(1-\lambda)(C\pi(1)-\kappa)+\lambda\tilde{v}(0,1)}{1-\beta(1-\lambda)} & \text{if } \underline{C}_1 < C \leq \bar{C}_2, \\ \beta \frac{(1-\lambda)(\frac{C}{2}\pi(2)-\kappa)+\lambda\tilde{v}(1,1)}{1-\beta(1-\lambda)} & \text{if } C > \bar{C}_2; \end{cases}$$

and if it begins as one of two incumbents it equals

$$v(1, C, 1) = \begin{cases} 0 & \text{if } C \leq \underline{C}_1, \\ \beta \frac{(1-\lambda)(C\pi(1)-\kappa)+\lambda\tilde{v}(0,1)}{1-\beta(1-\lambda)} & \text{if } \underline{C}_1 < C \leq \underline{C}_2, \\ \beta \frac{(1-\lambda)(\frac{C}{2}\pi(2)-\kappa)+\lambda\tilde{v}(1,1)}{1-\beta(1-\lambda)} & \text{if } C > \underline{C}_2. \end{cases}$$

The exit threshold \underline{C}_1 is the greatest value of C such that $v(0, C, 1) = 0$, and the average continuation values following a change in C_t for a monopolist and a duopolist are

$$\begin{aligned} \tilde{v}(0, 1) &= \left(\frac{\hat{C} + \check{C}}{2} \right) \pi(1) - \kappa + \int_{\hat{C}}^{\check{C}} \frac{v(0, C', 1)}{(\check{C} - \hat{C})} dC', \\ \tilde{v}(1, 1) &= \frac{1}{2} \left(\frac{\hat{C} + \check{C}}{2} \right) \pi(2) - \kappa + \int_{\hat{C}}^{\check{C}} \frac{v(1, C', 1)}{(\check{C} - \hat{C})} dC'. \end{aligned}$$

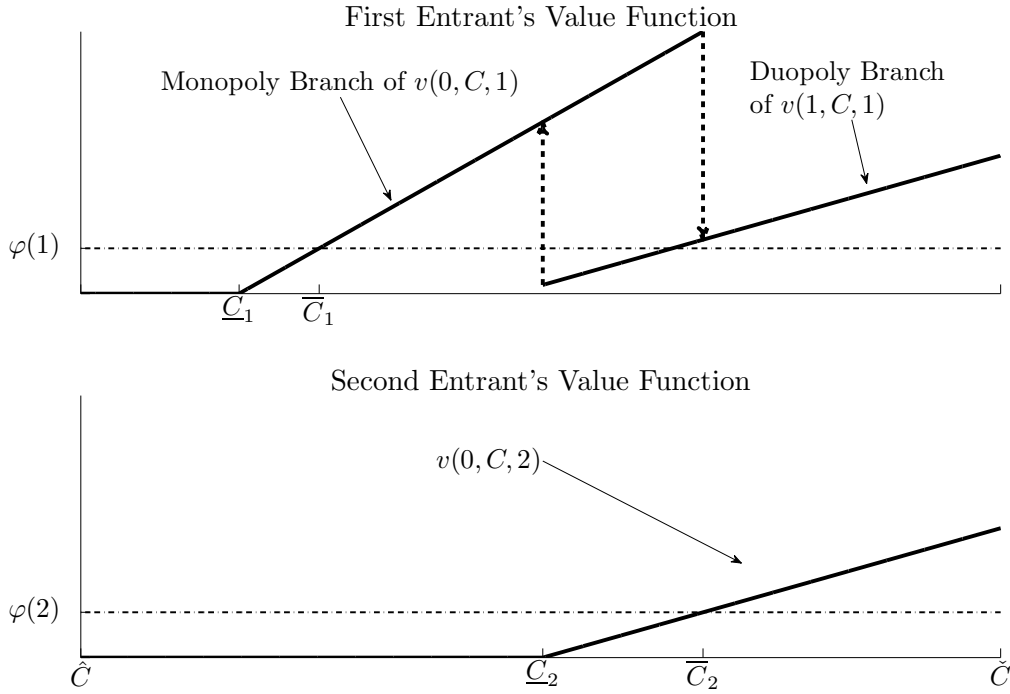


Figure 2: Equilibrium in the Pencil-and-Paper Example

The value function of a firm with rank 1 does not always increase with C , because slightly raising C from \bar{C}_2 induces entry by the second firm and causes both current profits and the continuation value to discretely drop. Nevertheless we know that they drop to a value above $\varphi(1)$, because at this point the second firm chooses to enter. Hence, it is still possible to find a unique entry threshold \bar{C}_1 that equates the value of entering with rank 1 to the cost of doing so.

Figure 2 visually represents the equilibrium. In each panel, C runs along the horizontal axis. The vertical axis gives the value of a firm when entry and exit decisions are made. The top panel plots the value of a firm with rank 1, while the bottom plots the value for a competitor with rank 2. The value of a duopolist with rank 2 equals zero for $C < \underline{C}_2$, and thereafter increases linearly with C . The entry threshold \bar{C}_2 equates this value with $\varphi(2)$. The value of an older firm with rank 1 has two branches. The monopoly branch gives the value of a monopolist expecting no further entry. If C increases above \bar{C}_2 and thus induces entry, the firm's value drops to the duopoly branch. This has the same slope as the

value function in the lower panel. Its intercept is higher, because the incumbent expects to eventually become a monopolist the first time that C passes below \underline{C}_2 . When this occurs, the firm’s value returns to the monopoly branch. The entry and exit thresholds for this firm occur where the monopoly branch intersects $\varphi(1)$ and 0.

The paper-and-pencil example provides a useful basic framework for analytically characterizing the effects of policy interventions in a dynamic duopoly. This paper’s companion (Abbring and Campbell, 2007) uses this to determine the effects of raising a barrier to entry in a monopoly by increasing a second entrant’s sunk costs, and to explore the consequences of replacing the LIFO assumption with a first-in first-out (FIFO) assumption.

3 Threshold Entry and Exit Rules

In the paper-and-pencil example, all firms follow *threshold rules* for their entry and continuation decisions.

Definition 3. *A firm with rank R' follows a threshold rule if there exist real numbers $\underline{C}_{R'}$ and $\overline{C}_{R'}$ such that $A_S(N - R, C, R') = I\{C > \underline{C}_{R'}\}$ and $A_E(C, R') = I\{C > \overline{C}_{R'}\}$.*

With such a rule, a potential entrant into a market with $R' - 1$ incumbents actually enters if and only if $C > \overline{C}_{R'}$, and this firm exits the first time thereafter that $C \leq \underline{C}_{R'}$.

There are three reasons to care about whether or not firms follow threshold rules. First, they pervade theoretical and empirical industrial organization. Second, they simplify the model’s analysis, as the pencil-and-paper example illustrated. Third, as the next proposition shows, higher realizations of demand always result in more active firms in our model if and only if all firms use threshold rules.

Proposition 3. *Consider an initial condition (N_t, C_t) , a sequence of subsequent demand realizations $C_{t+1}, \dots, C_{t+i-1}$, and the corresponding numbers of operating firms from the equilibrium of Proposition 2, N_{t+1}, \dots, N_{t+i} . Increasing C_t weakly increases N_{t+i} for positive i and any possible initial condition and sequence of subsequent demand realizations if and only if firms of all ranks follow threshold rules.*

Appendix B presents the proposition’s proof.

A monotonic influence of C_t on N_{t+i} appeals to us as “natural”. It is straightforward to show that a firm with the highest possible rank always follows a threshold rule given stochastic monotonicity ($Q(\cdot|C)$ decreases with C). Hopenhayn (1992) imposes this condition

on competitive firms' productivity shocks to demonstrate the existence of an optimal exit threshold. However, stochastic monotonicity does not guarantee that firms of all ranks use threshold rules in the LIFO equilibrium. Figure 3 illustrates this using a particular numerical example with $\check{N} = 2$. For this, we suppose that $\ln C_t \in [-1.5, 1.5]$ and that

$$Q(c|C) = \begin{cases} 0 & \text{if } \ln c < \max\{\ln C - 0.3, -1.5\} \\ 1/4 & \text{if } \max\{\ln C - 0.3, -1.5\} \leq \ln c < \ln C \\ 3/4 & \text{if } \ln C \leq \ln c < \min\{\ln C + 0.3, 1.5\} \\ 1 & \text{otherwise} \end{cases}$$

With this stochastic process, the probability of $\ln C_t$ remaining unchanged is $1/2$. With probability $1/4$ it falls to the maximum of $\ln C_t - 0.30$ and $\ln \hat{C}$, and with the same probability it rises to the minimum of $\ln C_t + 0.30$ and $\ln \check{C}$. The model's other parameters in this example are $\varphi(1) = \varphi(2) = 10$, $\pi(N) = 2 \times I\{N \leq 2\}$, $\kappa = 1$, and $\beta = 1.05^{-1}$.⁶

The example's stochastic process displays stochastic monotonicity, so the value function for the second entrant increases with C_t . As with the paper and pencil example, the first firm's value function decreases to a point above $\varphi(1)$ at the second entrant's entry threshold. However, the value function also decreases at several points to the left of this threshold. The drops occur when increasing $\ln C_t$ moves one of the two extreme points in the support of $\ln C_{t+1}$ over another drop. The implication of this non-monotonicity is that this firm's value function crosses $\varphi(1)$ thrice. As a result, a firm with this entry opportunity would take it if $\ln C_t$ is in either of the disconnected sets labeled A and B but it would stay out of the market if $\ln C_t$ fell in the region between them. Intuitively, moving $\ln C_t$ from A into the region between A and B decreases the value of entry by increasing the probability of further entry without a compensating gain from increasing $\ln C_{t+1}$.

The above example illustrates that firms do not generically use threshold rules in equilibrium. In it, increasing the current value of C can discontinuously increase the likelihood of crossing \bar{C}_2 and thereby discontinuously decrease the incumbent's value. In contrast, increasing C in the pencil-and-paper example leaves the probability of future entry unchanged. Together, these examples suggest that firms will use threshold rules if the stochastic process limits the negative "potential entry" effect of increasing C on expected future profits. Here we present sufficient conditions for this to be so.

To this end, define the conditional expectation $\mu(C) \equiv \int c dQ(c|C)$, the innovation error $U' \equiv C' - \mu(C)$, its marginal distribution $\tilde{Q}(u) \equiv \Pr(U' \leq u)$, and $\underline{C} \equiv \inf_C \{C - \mu(C)\}$. Consider the following assumptions on the transition function $Q(\cdot|C)$.

⁶For the computation, we used the algorithm described in Section 4.1.

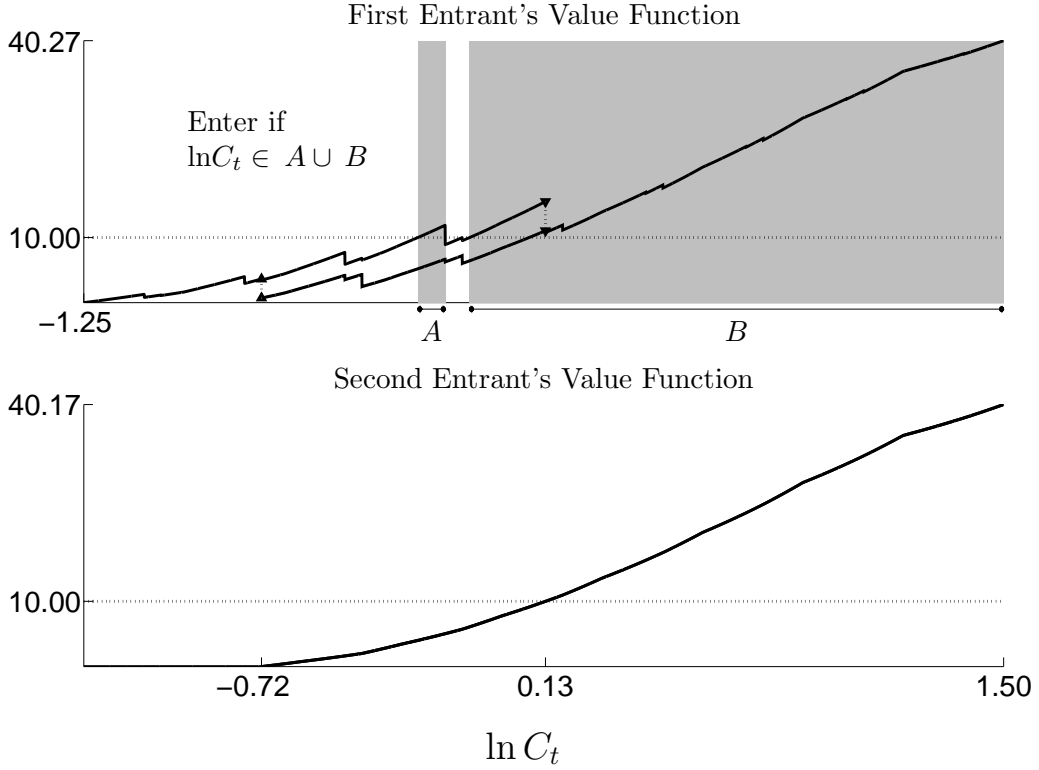


Figure 3: Example of LIFO Equilibrium with a Non-Monotonic Entry Rule

Assumption 1 (Monotonicity). $\mu(C)$ is weakly increasing in C .

Assumption 2 (Independence). U' is independent of C .

Assumption 3 (Concavity). \tilde{Q} is concave on $[\underline{U}, \infty)$.

Assumption 1 requires that higher demand now implies weakly higher expected demand tomorrow. Assumption 2 ensures that C' only depends on C through its first conditional moment. Taken together, Assumptions 1 and 2 imply stochastic monotonicity. Assumption 3 is satisfied if and only if \tilde{Q} has a nonincreasing Lebesgue density on (\underline{U}, ∞) . Note that $Q(c|C) = \Pr(\mu(C) + U' \leq c|C) = \tilde{Q}(c - \mu(C))$ under Assumption 2, and that $c - \mu(C) \geq C - \mu(C) \geq \underline{U}$ if $c \geq C$. Therefore, Assumptions 2 and 3 together imply that $Q(\cdot|C)$ is concave on $[C, \check{C}]$ and has a nonincreasing Lebesgue density on $(C, \check{C}]$.

With these assumptions in place, we can state this section's central result.

Proposition 4. *Let (A_S, A_E) be the unique symmetric Markov-perfect equilibrium in a LIFO strategy that defaults to inactivity. If $Q(\cdot|C)$ satisfies Assumptions 1–3, then firms with all ranks follow threshold policies.*

The proposition’s proof is given in Appendix B. The key step in this proof is the demonstration that an increase in C that makes further entry more likely does not reduce the expected continuation value below the firm’s cost of entry. To appreciate the contribution of Assumption 3, note that it ensures that the distribution of C' given C has no mass points to the right of C , and that its density on $(C, \check{C}]$ has no modes to the right of C . Thus, increasing C cannot move a “substantial” probability mass over another firm’s entry threshold as in our example of a non-monotonic exit rule.

The following representation theorem aids the construction of demand processes that satisfy Assumptions 1–3.

Proposition 5. *The transition function $Q(\cdot|C)$ satisfies Assumptions 1–3 if and only if there exist a sequence $Q^1(\cdot|C), Q^2(\cdot|C), \dots$ of transition functions such that*

$$\lim_{K \rightarrow \infty} \sup_{c, C} |Q^K(c|C) - Q(c|C)| = 0$$

and, for all $K \in \mathbb{N}$,

$$Q^K(\cdot|C) = \sum_{k=1}^K p_k^K Q_k^K(\cdot|C), \quad p_1^K, \dots, p_K^K \geq 0, \quad \sum_{k=1}^K p_k^K = 1,$$

where

$$Q_k^K(c|C) = \begin{cases} 1 & \text{if } c \geq \mu_k^K(C) + \sigma_k^K/2 \\ (c - \mu_k^K(C) + \sigma_k^K/2)/\sigma_k^K & \text{if } \mu_k^K(C) - \sigma_k^K/2 \leq c < \mu_k^K(C) + \sigma_k^K/2 \\ 0 & \text{otherwise,} \end{cases}$$

$\hat{C} + \sigma_k^K/2 \leq \mu_k^K(C) \leq \check{C} - \sigma_k^K/2$, $\sigma_k^K \geq 0$, $\mu_k^K(C)$ weakly increasing in C , and $\mu_k^K(C) \leq C + \sigma_k^K/2$, $k = 1, \dots, K$.

This proposition’s proof, in Appendix B, relies on the well-known fact that concave distributions can be represented by mixtures of uniform distributions with common lower ends of their supports.

Proposition 5 implies that each demand process that satisfies Assumptions 1–3 can be constructed by mixing uniform autoregressions that each satisfy monotonicity (as in Assumption

1) and a support condition. In particular, each of the mixed processes is a possibly nonlinear autoregression with conditional mean $\mu_k^K(C)$ and uniform innovations with standard deviation $\sigma_k^K/\sqrt{12}$. The coefficients p_k^K give the mixing probabilities. The support condition that $\mu_k^K(C) \leq C + \sigma_k^K/2$ ensures that the current state is always in or above the support of each mixing distribution. With uniformity, this implies that $Q(\cdot|C)$ has a nonincreasing Lebesgue density on $(C, \check{C}]$ (Assumption 3).

A wide variety of demand processes satisfy Assumptions 1–3 or, equivalently, are mixtures of the type displayed in Proposition 5. The stochastic process from the pencil-and-paper example is such a mixture, with α and $1 - \alpha$ serving as the mixing probabilities. In this case, one of the uniform distributions is degenerate at $\mu_1(C_t) = C_t$. To construct another example, consider a random walk reflected at \hat{C} and \check{C} . That is, set

$$\mu(c) = \begin{cases} \hat{C} + \frac{\sigma}{2} & c < \hat{C} + \frac{\sigma}{2}, \\ c & \text{if } \hat{C} + \frac{\sigma}{2} \leq c \leq \check{C} - \frac{\sigma}{2}, \text{ and} \\ \check{C} - \frac{\sigma}{2} & c > \check{C} - \frac{\sigma}{2}, \end{cases}$$

for some $0 < \sigma < \check{C} - \hat{C}$. By mixing over such reflected random walks, we can approximate any symmetric and continuous distribution for the growth rate of demand in the region away from the boundaries of $[\hat{C}, \check{C}]$.

4 Entry and Exit with Uncertainty

This section applies our analysis to two related questions: How does adding uncertainty impact oligopolists' entry and exit thresholds? How do estimates of oligopolists' producer surplus per consumer based on static models of the "long-run" without uncertainty or without sunk costs differ from their actual values?

[Dixit and Pindyck \(1994\)](#) review a large literature that characterizes competitive firms' entry and exit decisions with sunk costs and uncertain profits. Such a firm's value equals its *fundamental value*, the expected discounted profits from perpetual operation, plus the value of an option to sell this stream of (potentially negative) profits at a strike price of zero. This literature's key insight is that uncertainty about future profits raises the value of this put option and thereby decreases the frequency of exit. [Abbring and Campbell \(2006\)](#) estimated that this option value accounted for the majority of firm value in a particular competitive service industry. Our model allows us to investigate how the insights from this well-studied decision-theoretic problem apply to oligopolistic dynamics.

Our analysis of the second question follows a large literature based on static free-entry models of oligopoly structure, exemplified by [Bresnahan and Reiss \(1990, 1991b\)](#) and [Berry \(1992\)](#). They determined empirically how changing market size influenced the number of competitors using observations from cross-sections of local retail ([Bresnahan and Reiss](#)) and airline ([Berry](#)) markets. The models they used to structure their analysis can be viewed as versions of ours in which either demand remains unchanged over time or firms incur no sunk costs. These papers point to current demand as the key determinant of the number of firms: A market will attain N firms if N entrants can recover their fixed costs but $N + 1$ entrants cannot. These authors emphasize that the observed relationship between C and N depends on the rate at which $\pi(N)$ decreases (which [Sutton, 1991](#), labeled the “toughness of competition”) and the rate at which $\varphi(N)$ increases (which [McAfee, Mialon, and Williams, 2004](#), define to be an economic barrier to entry). If both of these functions are constant, then the number of active firms is roughly proportional to demand, $\bar{C}_i = i \times \bar{C}_1$. However, if either $\pi(N)$ decreases or $\varphi(N)$ increases, then N/C declines with C . In this sense, increasing the toughness of competition or imposing a sunk barrier to entry increases concentration.

Our approach to answering these questions is computational. Accordingly, we begin this section with an explicit presentation of the algorithm for equilibrium computation. We then show how demand uncertainty impacts equilibrium entry and exit thresholds for a particular model parameterization. The section concludes with a presentation of the entry thresholds and the producer surplus per consumer calculated from feeding data generated by our model’s ergodic distribution through a static Probit model of long-run equilibrium like that of [Bresnahan and Reiss \(1990, 1991b\)](#).

4.1 Equilibrium Computation

The proof of [Proposition 1](#) outlines a simple algorithm for computing the Markov-perfect equilibrium of interest:

- (i). Given values for the model’s primitives, we choose an evenly spaced grid of values for C in the interval $[\hat{C}, \check{C}]$ and a Markov chain over this grid to approximate $Q(\cdot|C)$.
- (ii). We set \check{N} equal to the largest value of N such that

$$\frac{\check{C}}{N} \pi(N) - \kappa \geq 0.$$

- (iii). We consider the entry and survival decision problem of a firm with rank \check{N} . This firm rationally expects no further entry and sets N' equal to \check{N} in all states $(N - R, C)$. Under

this supposition, we can solve the firm's dynamic programming problem by beginning with a trial value for its value function $v(0, \cdot, \check{N})$ and iterating on the Bellman equation (1) for $N = R = \check{N}$. This gives the firm's expected discounted profits $v(0, C, \check{N})$ for all C on the chosen grid. In practice, this takes very, very little computer time. From $v(0, \cdot, \check{N})$, we can calculate the sets of demand states C in which the firm chooses to enter and survive. We refer to these as the entry and survival sets

$$\mathcal{E}_{\check{N}} \equiv \{C | v(0, C, \check{N}) > \varphi(\check{N})\} \quad \text{and} \quad \mathcal{S}_{\check{N}} \equiv \{C | v(0, C, \check{N}) > 0\}.$$

- (iv). The rest of the computation proceeds recursively for $R = \check{N} - 1, \dots, 1$. Suppose that we have computed entry sets $\mathcal{E}_{R+1}, \dots, \mathcal{E}_{\check{N}}$ and survival sets $\mathcal{S}_{R+1}, \dots, \mathcal{S}_{\check{N}}$. A firm with rank R rationally expects that these sets govern younger firms' entry and survival decisions, and that no firm will enter with rank larger than \check{N} . Hence, it expects that

$$N'_R(N - R, C) \equiv R + \sum_{\tilde{R}=R+1}^{\check{N}} \left[I \left\{ \tilde{R} \leq N, C \in \mathcal{S}_{\tilde{R}} \right\} + I \left\{ \tilde{R} > N, C \in \mathcal{E}_{\tilde{R}} \right\} \right]$$

governs the evolution of the number of firms. With this specification for N' in place, we can solve this firm's dynamic programming problem by iterating on the Bellman equation (1) for given R , starting with *e.g.* the value function for a firm with rank $R + 1$. This produces the expected discounted profits $v(N - R, \cdot, R)$, $N = R, \dots, \check{N}$, and the entry and survival sets

$$\mathcal{E}_R \equiv \{C | v(0, C, R) > \varphi(R)\} \quad \text{and} \quad \mathcal{S}_R \equiv \{C | v(0, C, R) > 0\}$$

for a firm with rank R .

With the equilibrium entry and survival sets for all \check{N} possible ranks in place, calculating observable aspects of industry dynamics (such as the ergodic distribution of N_t) is straightforward. Matlab programs and documentation are available in this project's [replication file](#).

4.2 Equilibrium Entry and Exit Thresholds

With this algorithm, we have calculated the equilibrium entry and exit thresholds for a particular specification of the model that satisfies the sufficient conditions for firms to use threshold-based entry and exit policies. We set one model period to equal one year and chose β to replicate a 5% annual rate of interest. We set $\kappa = 1.75$ and $\varphi = 0.25\beta/(1 - \beta)$, so

the fixed costs of a continuing establishment equal seven times sunk costs' rental equivalent value. We also set $\pi(N) = 4$ for all N . With these parameter values and no demand uncertainty, the entry thresholds are $8/7$ of the corresponding exit thresholds and the entry threshold for a second firm equals one. We set $\hat{C} = e^{-1.5}$, $\check{C} = e^{1.5}$, and $Q(\cdot|C)$ to equal a mixture over 51 reflected random walks in the *logarithm* of C with uniformly distributed innovations. The mixture approximates a normally distributed innovation. We denote the standard deviation of the normal distribution we seek to approximate with σ . Proposition 4 can be easily extended to the case where Assumptions 1, 2, and 3 apply to a monotonic transformation of C_t , so the logarithmic specification for demand has no direct theoretical consequences. We choose it because population and income measures typically require a logarithmic transformation to display homoskedasticity across time.

The first two panels of Table 1 report the equilibrium entry and exit thresholds for this specification for four values of σ , 0, 0.10, 0.20, and 0.30. Given the support of C_t , up to eight firms could populate the industry if $\sigma = 0$. Because C_t is reflected off of \check{C} , demand displays mean reversion. Thus, such high states of demand are somewhat temporary if $\sigma > 0$, and the maximum number of firms observed in the ergodic distribution accordingly decreases with σ . The two panels' cells for those missing firms' entry and exit thresholds are blank.

Consider first the impact of increasing σ on the entry thresholds. At least one firm enters an empty market with no demand uncertainty if $C_t > 0.50$. This threshold hardly changes as σ increases. Likewise, the entry threshold for a second firm remains very close to 1.00 as σ rises. The thresholds for higher-ranked entrants all rise with σ with one exception (to be discussed further below). Apparently, increasing demand uncertainty makes entry into an oligopoly less likely for a given value of C_t . Demand uncertainty has exactly the opposite impact on the entry of a potential monopolist. For such a firm, increasing uncertainty increases the value of the put option associated with exit, thereby raising profitability and *lowering* the firm's entry threshold.

This difference between oligopolists' and monopolists' entry decisions arises from the threat of potential entry. A monopolist captures all of the increased profit from a favorable demand shock. For an oligopolist, further entry chops this right tail off of the profit distribution and thereby reduces the firm's option value. This explanation squares with the single exception to the rule that increasing σ increases the entry threshold. Increasing σ from 0.20 to 0.30 simultaneously eliminates the possibility that a sixth firm enters and reduces \bar{C}_5 from 2.72 to 2.56.⁷ The third panel of Table 1 further illustrates this effect. It reports the

⁷Entry by an eighth firm does not occur when there is demand uncertainty, so this discussion begs the

equilibrium entry thresholds for the case where $\pi(N) = 4 \times I\{N < 5\}$, so that no more than four firms will populate the market. The entry thresholds for the first, second, and third firms are nearly identical to their values in the first panel. However, the entry thresholds for the fourth firm (facing no further entry) decline with σ .

Next examine the exit thresholds in the table’s second panel. Without demand uncertainty, these form a line out of the origin with a slope approximately equal to 0.44. As expected, raising σ decreases all of the exit thresholds. This mimics the well-known effect of increased uncertainty on monopolists’ exit decisions: Uncertainty raises the value of the firm’s put option, and exit requires this option’s exercise. For completeness, Table 1 reports the equilibrium exit thresholds when $\pi(N) = 4 \times I\{N < 5\}$. As expected, this change has almost no impact on the exit thresholds for firms with ranks less than four. For the fourth firm, eliminating the possibility of further entry makes survival more attractive and thereby lowers the exit threshold even further.

To summarize, adding uncertainty either leaves the equilibrium entry thresholds unchanged or raises them somewhat. This result embodies two effects: Increasing uncertainty alone would make entry more attractive, but the accompanying increase in the probability of future entry reduces expected future profits. On the other hand, adding uncertainty substantially reduces equilibrium exit thresholds.

4.3 Static Analysis of Market Size and Entry

We now characterize how a static “long-run” analysis of market size and entry interprets data generated by our dynamic model. For this, we modify our model to obtain a framework observationally equivalent to that estimated by [Bresnahan and Reiss \(1990, 1991b\)](#). We begin by eliminating all but trivial dynamic considerations with the assumption that $C_t = C_{t-1}$ always. All markets begin with zero competitors, so removing demand uncertainty from the model eliminates exit. It also removes any meaningful distinction between fixed and sunk costs, so we set the latter to zero. This version of the model is econometrically degenerate because C_{t-1} determines N_t nonstochastically. We follow [Bresnahan and Reiss](#) and solve this problem by adding a random component to firms’ fixed costs. Specifically, the fixed costs of any firm serving the market are $e^\varepsilon \kappa$, where ε is a normally distributed across markets with mean 0 and variance ς^2 .

question of why increasing σ from 0 to 0.10 raises \bar{C}_7 from 3.49 to 3.60. This change reflects the mean reversion noted above. The same principle explains the rise of \bar{C}_6 from 3.13 to 3.19 when σ goes from 0.10 to 0.20.

Table 1: Equilibrium Entry and Exit Thresholds^(i,ii)

$\pi(N) = 4$								
Entry Thresholds								
σ	1	2	3	4	5	6	7	8
0.00	0.50	1.00	1.49	1.99	2.48	2.97	3.49	3.97
0.10	0.48	0.98	1.51	2.03	2.56	3.13	3.60	
0.20	0.46	1.00	1.55	2.12	2.72	3.19		
0.30	0.46	1.02	1.62	2.20	2.56			

Exit Thresholds								
σ	1	2	3	4	5	6	7	8
0.00	0.44	0.87	1.31	1.73	2.18	2.61	3.03	3.49
0.10	0.38	0.76	1.15	1.54	1.93	2.32	2.69	
0.20	0.33	0.69	1.05	1.40	1.77	2.12		
0.30	0.28	0.63	0.97	1.30	1.58			

$\pi(N) = 4 \times I\{N < 5\}$								
Entry Thresholds								
σ	1	2	3	4	5	6	7	8
0.00	0.50	1.00	1.49	1.99				
0.10	0.48	0.98	1.52	1.90				
0.20	0.46	0.99	1.57	1.77				
0.30	0.45	1.03	1.57	1.75				

Exit Thresholds								
σ	1	2	3	4	5	6	7	8
0.00	0.44	0.87	1.31	1.73				
0.10	0.38	0.76	1.15	1.51				
0.20	0.33	0.69	1.05	1.31				
0.30	0.28	0.64	0.96	1.19				

(i) The parameter values used were $\kappa = 1.75$, $\beta = 1.05^{-1}$, $\varphi = 0.25 \times \beta / (1 - \beta)$, $\hat{C} = e^{-1.5}$, $\check{C} = e^{1.5}$, and $Q(\cdot|C)$ a mixture over reflected random walks in the logarithm of C with uniformly distributed innovations and approximate innovation variance σ^2 . (ii) An empty cell indicates that the ergodic distribution of N_t puts zero probability on the given value of N . Please see the text for details.

Free entry requires that all active firms earn a positive profit and that an additional firm would earn a non-positive profit. That is

$$\frac{C}{N} \times \pi(N) > e^\varepsilon \kappa \quad \text{and} \quad \frac{C}{N+1} \times \pi(N+1) \leq e^\varepsilon \kappa.$$

For each $N = 1, \dots, \check{N}$; define the deterministic entry threshold C_N^* to be the unique solution to $(C/N)\pi(N) - \kappa = 0$. Exactly N firms will serve the industry if $\ln C > \ln C_N^* + \varepsilon$ and $\ln C \leq \ln C_{N+1}^* + \varepsilon$. The probability that this occurs is $\Phi\left(\frac{\ln(C/C_N^*)}{\varsigma}\right) - \Phi\left(\frac{\ln(C/C_{N+1}^*)}{\varsigma}\right)$. In this expression, we set $\ln C_0^* = -\infty$ and $\ln C_{\check{N}+1}^* = \infty$.⁸

Given observations of C and N from a cross section of markets, ordered Probit estimation immediately yields estimates for $C_1^*, \dots, C_{\check{N}}^*$ and ς . With these, [Bresnahan and Reiss](#) estimate how the producer surplus per consumer falls with additional competitors. Specifically, the definition of C_N^* implies that $\pi(N)/\pi(1) = C_1^* \times N/C_N^*$. If the level of demand required to support N firms equals N times the level required for a monopolist, then the producer surplus per consumer does not fall with additional entry. On the other hand, if demand must exceed $N \times C_1^*$ to induce N firms to enter, then the surplus per consumer must decline with N . In this way, [Bresnahan and Reiss](#) infer the toughness of competition from the relationship between market size and the number of competitors.

For *any* given joint distribution of C and N , we can define the population counterparts to the estimated thresholds by minimizing the population analogue of the ordered Probit's log-likelihood function.

$$\mathcal{L}(C_1^*, \dots, C_{\check{N}}^*, \varsigma) \equiv \mathbb{E} \left[\sum_{R=0}^{\check{N}} I\{N = R\} \ln \left(\Phi \left(\frac{\ln(C/C_R^*)}{\varsigma} \right) - \Phi \left(\frac{\ln(C/C_{R+1}^*)}{\varsigma} \right) \right) \right]$$

Because the ordered Probit likelihood function is always concave, even if it does not represent the true data generating process, this function has a unique minimizer. Population “estimates” of $\pi(N)/\pi(1)$ correspond to these minimizing values for $C_1^*, \dots, C_{\check{N}}^*$. Calculating these using the joint distribution of C_t and N_t from our model's ergodic distribution, mimicking [Bresnahan and Reiss](#) by using these to estimate $\pi(N)/\pi(1)$, and comparing these estimates to the true values of $\pi(N)/\pi(1)$ from the dynamic model indicates whether static estimates of the toughness of competition suffer from substantial bias.

⁸The free entry conditions equal those from a model without dynamic considerations, because we have set sunk costs to zero. If we had assumed instead that the per period fixed costs equal zero and that the sunk costs equal $e^\varepsilon \varphi$, then the free entry conditions would be identical with $\varphi(1 - \beta)/\beta$ replacing κ everywhere.

Table 2: Static Probit Analysis of Market Structure⁽ⁱ⁾

Implied Static Entry Thresholds							
σ	1	2	3	4	5	6	7
0.10	0.43	0.87	1.32	1.78	2.24	2.71	3.13
0.20	0.39	0.84	1.29	1.74	2.21	2.62	
0.30	0.34	0.81	1.26	1.70	2.04		

Implied $\pi(N)/\pi(1)$							
σ	1	2	3	4	5	6	7
0.10	1.00	0.98	0.97	0.96	0.95	0.95	0.96
0.20	1.00	0.94	0.92	0.90	0.89	0.90	
0.30	1.00	0.85	0.81	0.80	0.84		

(i) The table’s top panel reports population values of Probit-based entry thresholds from the static model of [Bresnahan and Reiss](#) calculated using the ergodic distribution of the dynamic model specification of Section 4.2 and Table 1. The bottom panel reports the implied values of $\pi(N)$ normalized by $\pi(1)$. An empty cell indicates that the ergodic distribution of N_t puts zero probability on the given value of N . Please see the text for further details.

The top panel of Table 2 reports ordered Probit estimates of C_1^*, \dots, C_N^* from the ergodic distribution of the dynamic model specification examined in Section 4.2, and its bottom panel gives the implied estimates of $\pi(N)/\pi(1)$. For all three of values of σ used, the static entry thresholds almost exactly equal the average of the dynamic model’s corresponding entry and exit thresholds. That is, the static analysis “splits the difference” between them.

Recall that the true values of $\pi(N)/\pi(1)$ all equal one. That is, an additional competitor steals business from incumbents but does not lower the producer surplus earned per consumer. For the case with $\sigma = 0.10$, the implied values deviate little from the truth. However, raising σ further substantially lowers these “estimates”. When $\sigma = 0.3$, the implied value of $\pi(2)/\pi(1)$ equals 0.85. Further increases in N change this little.

Apparently, the static Probit analysis can find evidence that $\pi(N)$ falls with data generated from a dynamic model in which $\pi(N)$ is constant. To gain some insight in the way sunk costs and uncertainty affect the static analysis of the toughness of competition, recall that the static Probit’s thresholds are roughly the average of the dynamic entry and exit thresholds.

Hence, without uncertainty both the static and the dynamic thresholds are evenly spaced if $\pi(N)$ is constant. For example, from the static analysis we may find that it takes 1000 consumers for one firm and 2000 consumers for two firms to be active in the market. With uncertainty, however, option value considerations will make firms of all ranks more reluctant to exit. Because this effect is not offset by a change in the entry thresholds (see the previous section and Table 1), the static thresholds will decrease as well. We may now “observe” that that it takes only 500 consumers for one firm and 1500 consumers for two firms to be active. The fact that the number of consumers needs to triple to entice a second firm to join the first suggests that the producer surplus per consumer served falls substantially when a second firm enters. However, this inference arises from uncertainty; the actual producer surplus per consumer served is constant.⁹

In our analysis, the delay in exit arising from option-value considerations imparts a substantial downward bias to each estimated threshold. This bias is large in the specification under consideration. Determining its importance for empirical work must proceed on a case-by-case basis, but we expect option-value considerations to pervade oligopolists’ exit decisions. A comparison of the results of [Bresnahan and Reiss \(1991b\)](#) with “estimates” in Table 2 supports this view. Their abstract reports

Our empirical results suggest that competitive conduct changes quickly as the number of incumbents increases. In markets with five or fewer incumbents, almost all variation in competitive conduct occurs with the entry of the second or third firm.

This is exactly the pattern displayed in Table 2.

5 Conclusion: Estimation

Since static estimation can give misleading estimates when applied to a dynamic world, we wish to extend the exercise of [Bresnahan and Reiss](#) into the truly dynamic setting of our model. We have successfully reconstructed the data from those authors’ attempt at dynamic estimation ([Bresnahan and Reiss, 1993](#)) from the original sources. It contains the number of dentists serving each of 152 isolated rural markets in each year from 1980 through 1988

⁹Another logical possibility is that the fall in the implied static thresholds reflects mean reversion: Because C cannot fall below \hat{C} , a potential entrant does not expect extremely low values of C to persist. We examined whether this contributes to our results by changing \hat{C} and \check{C} from $e^{-1.5}$ and $e^{1.5}$ to e^{-2} and e^2 . The results differ only minimally from those in Table 2.

as well as demographic and population measures relevant for measuring market size and cross-market cost differences.¹⁰ In these concluding remarks, we outline the possibilities for proceeding with our model’s estimation using these observations.

As we noted in the simulation of the static estimation procedure, our model is stochastically singular: In general, (C_{t-1}, N_{t-1}) determine N_t exactly. One could use this as the basis for estimation by choosing parameters to minimize the number of times N_t differs from its predicted value in a given data set, but we believe that there are advantages to the original econometric program of explicitly incorporating unobservable sources of variation into the structural model.

We can think of two distinct approaches for doing so. The first follows [Rust \(1987\)](#) and adds an error to firms’ common fixed cost that is *i.i.d.* over time and revealed to all active and potentially active firms one period in advance. This expands each firm’s state space by exactly one dimension, so the model remains tractable. Given a trial vector of parameters, we can calculate the model’s equilibrium. With a normally-distributed error, the resulting distribution of N_t given (N_{t-1}, C_{t-1}) has an ordered probit structure. Parameter estimates come from maximizing the likelihood function so constructed.

The second approach builds on our earlier work estimating decision-theoretic models of firm exit with unobserved heterogeneity and persistent, privately-observed shocks to profitability ([Abbring and Campbell, 2006](#)). This adds a measurement error to C_t which sums a permanent random effect with *i.i.d.* shocks. The potential and actual market participants base their decisions on an error-free measure of C_t , so they have superior information about the market’s evolution. If we restrict the stochastic process for C_t to satisfy Assumptions 1, 2 and 3, then the identification analysis we developed for the case of a single firm following a threshold-based exit rule extends immediately to this more complicated environment with multiple entry and exit thresholds. The non-linear Kalman filter procedure we used to calculate the likelihood function for the decision-theoretic model can be applied in this game-theoretic environment with only minimal extensions. In principle, this approach has the advantage of relaxing the strong assumption that an econometrician with the correct parameter estimates in hand suffers no informational disadvantage relative to the market participants. The extent to which this translates into practical differences in inference remains to be seen.

¹⁰This is the time frame from [Bresnahan and Reiss \(1993\)](#). Extending the observation period using publicly-available American Dental Association directories is straightforward.

References

- Abbring, J. H. and J. R. Campbell (2006). A firm's first year. Federal Reserve Bank of Chicago Working Paper 2003-11. [15](#), [24](#)
- Abbring, J. H. and J. R. Campbell (2007). Duopoly dynamics with a barrier to entry. Federal Reserve Bank of Chicago Working Paper 2006-29. [1](#), [3](#), [5](#), [11](#)
- Amir, R. and V. E. Lambson (2003). Entry, exit, and imperfect information in the long run. *Journal of Economic Theory* 110, 191–203. [7](#)
- Berry, S. (1992). Estimation of a model of entry in the airline industry. *Econometrica* 60(4), 889–917. [1](#), [16](#)
- Bresnahan, T. F. and P. C. Reiss (1990). Entry in monopoly markets. *Review of Economic Studies* 57(4), 531–553. [1](#), [2](#), [3](#), [16](#), [19](#), [23](#)
- Bresnahan, T. F. and P. C. Reiss (1991a). Empirical models of discrete games. *Journal of Econometrics* 48(1-2), 57–81. [1](#)
- Bresnahan, T. F. and P. C. Reiss (1991b, October). Entry and competition in concentrated markets. *Journal of Political Economy* 99(5), 977–1009. [16](#), [19](#), [21](#), [22](#), [23](#)
- Bresnahan, T. F. and P. C. Reiss (1993). Measuring the importance of sunk costs. *Annales d'Economie et de Statistique* 31, 181–217. [0](#), [1](#), [23](#), [24](#)
- Cabral, L. M. (1993). Experience advantages and entry dynamics. *Journal of Economic Theory* 59, 403–416. [2](#)
- Dixit, A. K. (1980, March). The role of investment in entry-deterrence. *Economic Journal* 90(357), 95–106. [2](#)
- Dixit, A. K. and R. S. Pindyck (1994). *Investment under Uncertainty*. Princeton University Press. [2](#), [15](#)
- Doraszelski, U. and M. Satterthwaite (2005). Foundations of Markov-perfect industry dynamics: Existence, purification, and multiplicity. Technical report, Harvard University and Northwestern University. [6](#)
- Dunne, T., M. J. Roberts, and L. Samuelson (1988). Patterns of firm entry and exit in U.S. manufacturing industries. *Rand Journal of Economics* 19, 495–515. [6](#)

- Ericson, R. and A. Pakes (1995). Markov-perfect industry dynamics: A framework for empirical work. *Review of Economic Studies* 62, 53–82. [3](#)
- Feller, W. (1971). *An Introduction to Probability Theory and Its Applications* (2nd ed.), Volume II. New York: Wiley. [33](#)
- Fudenberg, D. and J. Tirole (1991). *Game Theory*. Cambridge, MA: MIT Press. [30](#)
- Hopenhayn, H. A. (1992). Entry, exit, and firm dynamics in long run equilibrium. *Econometrica* 60(5), 1127–1150. [2](#), [11](#)
- Jarmin, R. S., S. D. Klimek, and J. Miranda (2003, October). Firm entry and exit in the U.S. Retail sector: 1977–1997. Technical Report 04-17, Center for Economic Studies, U.S. Bureau of the Census, Washington, DC. [6](#)
- Jovanovic, B. (1982). Selection and the evolution of industry. *Econometrica* 50(3), 649–670. [2](#)
- Maskin, E. and J. Tirole (1988). A theory of dynamic oligopoly, I: Overview and quantity competition with large fixed costs. *Econometrica* 56(3), 549–569. [2](#)
- McAfee, R. P., H. M. Mialon, and M. A. Williams (2004). What is a barrier to entry? *American Economic Review* 94(2), 461–465. [16](#)
- Pakes, A. and P. McGuire (1994). Computing Markov-perfect Nash equilibria: Numerical implications of a dynamic differentiated product model. *The RAND Journal of Economics* 25(4), 555–589. [3](#)
- Rust, J. (1987). Optimal replacement of GMC bus engines: An empirical model of Harold Zurcher. *Econometrica* 55, 999–1033. [24](#)
- Sutton, J. (1991). *Sunk Costs and Market Structure*. Cambridge, MA: MIT Press. [16](#)

Appendix

A Proofs of Results in Section 2

Proof of Proposition 1. The proof proceeds by first constructing a candidate equilibrium strategy and then verifying that it is a LIFO strategy that satisfies the conditions of Proposition 1 and forms an equilibrium.

To construct the candidate strategy, define \check{N} as in Section 4.1. Because \check{C} is finite, $\pi(R)$ is weakly decreasing in R , and $\kappa > 0$; $\check{N} < \infty$. This is an upper bound on the number of firms that would ever produce in a LIFO equilibrium.

Next, consider the exit decision problem of a firm that entered with rank $R \leq \check{N}$ and expects the number of firms to evolve according to the deterministic transition rule $N'_R : \mathbb{Z}_+ \times [\hat{C}, \check{C}] \rightarrow \{R, R+1, \dots, \check{N}\}$. Here, $N'_R(X, C)$ is the number of firms that the firm with rank R expects to be active next period given a decision to continue, X younger firms are active this period, and the number of consumers equals C . The expected number of active firms next period is defined for the event that $R + X > \check{N}$, but it never exceeds \check{N} . Define \mathcal{W} to be the space of all functions

$$w : \{0, \dots, \check{N} - 1\} \times [\hat{C}, \check{C}] \rightarrow \left[0, \frac{\beta\pi(1)\check{C}}{1-\beta}\right]$$

and define the Bellman operator $T_R : \mathcal{W} \rightarrow \mathcal{W}$ with

$$T_R(w)(X, C) = \max_{a \in [0,1]} a\beta\mathbb{E} \left[\frac{\pi(N'_R(X, C))C'}{N'_R(X, C)} - \kappa + w(N'_R(X, C) - R, C') \right]. \quad (2)$$

Note that T_R depends on the specification for N'_R . It satisfies Blackwell's sufficient conditions for a contraction mapping, and \mathcal{W} is a complete metric space. Hence, T_R has a unique fixed point, the value function w_R that gives this firm's expected discounted profits at each state

$$(X, C) \in \{0, \dots, \check{N} - 1\} \times [\hat{C}, \check{C}]$$

To construct the candidate equilibrium, begin with the decision problem for a firm with rank \check{N} and the transition rule $N'_{\check{N}}(X, C) = \check{N}$ for all (X, C) . This transition rule reflects the firm's expectations that it produces no longer than any earlier entrant, any younger active firms will exit, and no firms will enter. The fixed point $w_{\check{N}}$ of $T_{\check{N}}$ can be uniquely extended to a value function on the entire state space $\mathbb{Z}_+ \times [\hat{C}, \check{C}]$ by assigning $w_{\check{N}}(X, C) = w_{\check{N}}(0, C)$

for all (X, C) . Denote the set $\{C|w_{\check{N}}(0, C) > 0\}$ with $\mathcal{S}_{\check{N}}$ and the set $\{C|w_{\check{N}}(0, C) > \varphi(\check{N})\}$ with $\mathcal{E}_{\check{N}}$. Under the maintained hypotheses of this maximization problem, this firm chooses to remain active if and only if $C \in \mathcal{S}_{\check{N}}$ and it chooses to enter the industry if and only if $C \in \mathcal{E}_{\check{N}}$.¹¹

Next, iterate the following argument for $R = \check{N} - 1, \dots, 1$. Suppose that we have determined value functions $w_{R+1}, \dots, w_{\check{N}}$, entry sets $\mathcal{E}_{R+1}, \dots, \mathcal{E}_{\check{N}}$, and survival sets $\mathcal{S}_{R+1}, \dots, \mathcal{S}_{\check{N}}$. Suppose that we have established that

- (i). $\mathcal{E}_{R+1} \supseteq \dots \supseteq \mathcal{E}_{\check{N}}$,
- (ii). $\mathcal{S}_{R+1} \supseteq \dots \supseteq \mathcal{S}_{\check{N}}$,
- (iii). for all $\tilde{R} \geq R + 1$, $w_{\tilde{R}}(X, C) = w_{\tilde{R}}(\check{N} - \tilde{R}, C)$ if $X > \check{N} - \tilde{R}$, and
- (iv). for all $\tilde{R} \geq R + 1$, $w_{\tilde{R}}(X, C) > 0$ if and only if $C \in \mathcal{S}_{\tilde{R}}$.

Consider the decision problem for a firm with rank R and transition rule

$$N'_R(X, C) = R + \sum_{i=1}^{\infty} [I\{i \leq X, C \in \mathcal{S}_{R+i}\} + I\{i > X, C \in \mathcal{E}_{R+i}\}], \quad (3)$$

where $\mathcal{E}_{\tilde{R}} = \mathcal{S}_{\tilde{R}} = \emptyset$ for $\tilde{R} > \check{N}$. This transition rule reflects the firm's expectations that it produces no longer than any earlier entrant and that \mathcal{E}_{R+i} and \mathcal{S}_{R+i} , $i \in \mathbb{N}$, govern younger firms' entry and survival. The specification for N'_R implies that $N'_R(X, C) = N'_R(\check{N} - R, C)$ if $X > \check{N} - R$. Therefore, we can uniquely extend the fixed point w_R of T_R to a value function on the entire state space $\mathbb{Z}_+ \times [\hat{C}, \check{C}]$ by assigning $w_R(X, C) = w_R(\check{N} - R, C)$ for all $X > \check{N} - R$.

We first prove some properties of this value function. Consider the complete subspace $\mathcal{W}_R \subseteq \mathcal{W}$ of functions w such that $w(X + 1, C) \geq w_{R+1}(X, C)$, $X = 0, \dots, \check{N} - 2$, and $w(X, C)$ is weakly decreasing in X , for all C . To show that the Bellman operator T_R maps \mathcal{W}_R into itself, note that

- (i). $N'_R(X, C)$ is weakly increasing in X , so that $T_R(w)(X, C)$ is weakly decreasing in X if $w \in \mathcal{W}_R$;
- (ii). we have that

¹¹This specification of $\mathcal{S}_{\check{N}}$ and $\mathcal{E}_{\check{N}}$ ensures that the firm defaults to inactivity in the case of indifference.

(a) $0 \leq N'_{R+1}(X, C) - N'_R(X + 1, C) = I\{C \notin \mathcal{S}_{R+1}\} \leq 1$, so that

$$\frac{\pi(N'_R(X + 1, C))C'}{N'_R(X + 1, C)} \geq \frac{\pi(N'_{R+1}(X, C))C'}{N'_{R+1}(X, C)}; \text{ and}$$

(b) for $w \in \mathcal{W}_R$,

$$\begin{aligned} w(N'_R(X + 1, C) - R, C') &\geq w(N'_{R+1}(X, C) - (R + 1) + 1, C') \\ &\geq w_{R+1}(N'_{R+1}(X, C) - (R + 1), C'), \end{aligned}$$

so that we can write

$$\begin{aligned} T_R(w)(X + 1, C) &= \max_{a \in [0,1]} a\beta \mathbb{E} \left[\frac{\pi(N'_R(X + 1, C))C'}{N'_R(X + 1, C)} - \kappa + w(N'_R(X + 1, C) - R, C') \right] \\ &\geq \max_{a \in [0,1]} a\beta \mathbb{E} \left[\frac{\pi(N'_{R+1}(X, C))C'}{N'_{R+1}(X, C)} - \kappa + w_{R+1}(N'_{R+1}(X, C) - (R + 1), C') \right] \\ &= w_{R+1}(X, C). \end{aligned}$$

Since T_R maps \mathcal{W}_R into itself, $w_R \in \mathcal{W}_R$. That is,

- (i). $w_R(X + 1, C) \geq w_{R+1}(X, C)$ for all $X = 0, \dots, \check{N} - 2$ and all C , and
- (ii). $w_R(X, C)$ is weakly decreasing in X for all C .

These properties extend to the entire state space $\mathbb{Z}_+ \times [\hat{C}, \check{C}]$, because, for $X \geq \check{N} - R$,

- (i). $w_R(X + 1, C) = w_R(\check{N} - R, C) \geq w_{R+1}(\check{N} - R - 1, C) = w_{R+1}(X, C)$ and
- (ii). $w_R(X, C) = w_R(\check{N} - R, C)$.

The firm chooses to enter the industry if and only if $C \in \mathcal{E}_R \equiv \{C | w_R(0, C) > \varphi(R)\}$. If the firm is active and $X = 0$, it stays in the industry if and only if $C \in \mathcal{S}_R \equiv \{C | w_R(0, C) > 0\} \supseteq \mathcal{E}_R$. To show that it is also optimal for an active firm with $X \geq 1$ to stay in the industry if and only if $C \in \mathcal{S}_R$, note that

- (i). if $C \in \mathcal{S}_R$ then survival is optimal because either
 - (a) $C \notin \mathcal{S}_{R+1}$, so that $w_R(X, C) = w_R(0, C) > 0$, or
 - (b) $C \in \mathcal{S}_{R+1}$, so that $w_R(X, C) \geq w_{R+1}(X - 1, C) > 0$;

(ii). if $C \notin \mathcal{S}_R$ then exit is optimal because $w_R(X, C) \leq w_R(0, C) \leq 0$.

Finally, $w_R(0, C) \geq w_R(1, C) \geq w_{R+1}(0, C)$ for all C , so that $\mathcal{E}_R \supseteq \mathcal{E}_{R+1}$ and $\mathcal{S}_R \supseteq \mathcal{S}_{R+1}$.

With the value functions in hand and their properties established, consider the strategy

$$A_S(X, C, R) = \begin{cases} 1 & \text{if } C \in \mathcal{S}_R \text{ and} \\ 0 & \text{otherwise,} \end{cases}$$

and

$$A_E(C, R) = \begin{cases} 1 & \text{if } C \in \mathcal{E}_R \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

By construction, this strategy is a LIFO strategy that satisfies the conditions of Proposition 1. It forms a symmetric Markov-perfect equilibrium if no firm can gain by a one-shot deviation from the strategy (e.g. Fudenberg and Tirole, 1991, Theorem 4.2). By construction, the strategy prescribes the optimal action in each state if all other firms follow the same strategy. Hence, no firm can profit from a one-shot deviation and the strategy forms an equilibrium. \square

Proof of Proposition 2. The LIFO strategy constructed in the proof of Proposition 1 defaults to inactivity. Thus, a symmetric Markov-perfect equilibrium in a LIFO strategy that defaults to inactivity exists.

Uniqueness can be proven recursively, following the recursive construction of a candidate equilibrium strategy in the proof of Proposition 1.

- (i). In any equilibrium in a LIFO strategy that defaults to inactivity, the expected discounted profits $v(X, C, R)$ equal 0 and the entry and survival sets equal $\mathcal{E}_R = \mathcal{S}_R = \emptyset$ in all states (X, C, R) such that $R > \check{N}$.
- (ii). Consequently, in any such equilibrium, $N'_{\check{N}}(X, C)$ gives the number of firms in the next period in all states (X, C, \check{N}) . Therefore, the expected discounted profits $v(X, C, \check{N})$ equal the unique solution $w_{\check{N}}(X, C)$ to Proposition 1's decision problem of a firm with rank \check{N} , the entry set equals this decision problem's entry set $\mathcal{E}_{\check{N}}$, and the survival set equals its survival set $\mathcal{S}_{\check{N}}$, in all states (X, C, \check{N}) .
- (iii). Next, iterate the following argument for $R = \check{N} - 1, \dots, 1$. Suppose that, in any equilibrium in a LIFO strategy that defaults to inactivity, the expected discounted profits $v(X, C, \check{R})$ equal $w_{\check{R}}(X, C)$, the entry set equals $\mathcal{E}_{\check{R}}$, and the survival set equals

$\mathcal{S}_{\tilde{R}}$ in all states (X, C, \tilde{R}) such that $\tilde{R} > R$. Then, $N'_R(X, C)$ defined by equation (3) gives the number of firms in the next period in state (X, C, R) . Hence, in all such equilibria, the expected discounted profits $v(X, C, R)$ equal the solution $w_R(X, C)$ to Proposition 1's decision problem of a firm with rank R , the entry set equals this decision problem's entry set \mathcal{E}_R , and the survival set equals its survival set \mathcal{S}_R , in all states (X, C, R) .

Finally, note that the corresponding survival rule A_S is such that $A_S(N - R, C, R')$ is invariant in $N - R$ and weakly decreasing in R' . \square

B Proofs of Results in Section 3

Proof of Proposition 3. The strategy of Proposition 2's equilibrium can be written as $A_S(N - R, C, R') = I\{C \in \mathcal{S}_{R'}\}$ and $A_E(C, R') = I\{C \in \mathcal{E}_{R'}\}$, where the sets $\mathcal{S}_{R'}$ and $\mathcal{E}_{R'}$ are those defined in the proof of Proposition 1. From that proof, we know that $\mathcal{E}_1 \supseteq \dots \supseteq \mathcal{E}_{\check{N}}$, $\mathcal{S}_1 \supseteq \dots \supseteq \mathcal{S}_{\check{N}}$, and $\mathcal{S}_R \supseteq \mathcal{E}_R$ for $R = 1, \dots, \check{N}$.

First, suppose that firms of all ranks follow threshold rules with entry and exit thresholds $\bar{C}_1, \bar{C}_2, \dots, \bar{C}_{\check{N}}$ and $\underline{C}_1, \underline{C}_2, \dots, \underline{C}_{\check{N}}$. From the above properties on the equilibrium strategy, we know that $\bar{C}_1 \leq \dots \leq \bar{C}_{\check{N}}$, $\underline{C}_1 \leq \dots \leq \underline{C}_{\check{N}}$, and $\underline{C}_R \leq \bar{C}_R$ for $R = 1, \dots, \check{N}$. This allows us to write the transition rule for N_t as

$$N_{t+1} = \begin{cases} i < N_t & \text{if } C_t \in (\underline{C}_i, \underline{C}_{i+1}], \\ N_t & \text{if } C_t \in (\underline{C}_{N_t}, \bar{C}_{N_t+1}], \\ k > N_t & \text{if } C_t \in (\bar{C}_k, \bar{C}_{k+1}]. \end{cases} \quad (4)$$

In this, we set $\underline{C}_0 = -\infty$ and $\bar{C}_{\check{N}+1} = \check{C}$. This is obviously increasing in C_t , so the proposition is true for the case with $i = 1$. To show that this is true for higher i , proceed inductively. Suppose that C_t weakly increases N_{t+i-1} . Inspection of (4) shows that increasing N_{t+i-1} weakly increases N_{t+i} , so increasing C_t weakly increases N_{t+i} .

Second, suppose that all firms' entry and exit rules are such that increasing C_t weakly increases N_{t+i} for positive i and any possible initial condition (N_t, C_t) and sequence of subsequent demand realizations $C_{t+1}, \dots, C_{t+i-1}$. To show that this implies that all firms' entry and exit decisions follow threshold policies, suppose to the contrary that the survival decision of a firm with some rank R does not. That is, there exist two values of C_t , $C_L < C_H$ such that $C_L \in \mathcal{S}_R$ and $C_H \notin \mathcal{S}_R$. Suppose that $N_t = R$ and increase C_t from C_L to C_H . Then this change decreases N_{t+1} by at least one firm. (The decrease is two or more if $C_L \in \mathcal{E}_{R+1}$ or

$C_H \notin \mathcal{S}_{R-1}$.) This contradicts the original supposition, so all firms' survival decisions must follow threshold rules. Next, suppose that the entry of a firm with potential rank R does not follow a threshold rule. Again there must exist $C_L < C_H$ such that $C_L \in \mathcal{E}_R$ and $C_H \notin \mathcal{E}_R$. Suppose that $N_t = R - 1$. Because $\mathcal{S}_{R-1} \supseteq \mathcal{S}_R \supseteq \mathcal{E}_R$, all $R - 1$ incumbent firms continue if $C_t = C_L$. Thus, increasing C_t from C_L to C_H reduces N_{t+1} by at least one firm (the decrease is two or more if $C_L \in \mathcal{E}_{R+1}$ or $C_H \notin \mathcal{S}_{R-1}$), which again contradicts the original supposition. Hence threshold policies govern the survival and entry decisions of firms of all ranks. \square

The proof of Proposition 4 relies on Proposition 5, so we first present a proof of the latter.

Proof of Proposition 5. First, suppose that the stochastic process for C_t has Proposition 5's mixture representation. It follows that C_t satisfies Assumptions 1–3:

(i). Assumption 1 is satisfied, because $\mu_k^K(C)$ is nondecreasing with C for all k, K , and $\mu(C) = \lim_{K \rightarrow \infty} \sum_{k=1}^K p_k^K \mu_k^K(C)$.

(ii). Define

$$\tilde{Q}_k^K(u|C) \equiv Q_k^K(u + \mu_k^K(C)|C)$$

and denote the distribution of $U'|C$ with

$$\tilde{Q}(u|C) \equiv Q(u + \mu(C)|C).$$

Assumption 2 is satisfied, because the functions $Q_k^K(\cdot|C)$, $k = 1, \dots, K$; $K \in \mathbb{N}$, do not depend of C , and $\tilde{Q}(\cdot|C) = \lim_{K \rightarrow \infty} \sum_{k=1}^K p_k^K \tilde{Q}_k^K(\cdot|C)$.

(iii). Define $\hat{Q}_k^K(u|C) \equiv Q_k^K(u + \mu(C)|C)$. Because $\hat{Q}_k^K(\cdot|C)$ is a uniform distribution with support including or below $C - \mu(C)$ for all k, K , and

$$\tilde{Q}(\cdot|C) = \lim_{K \rightarrow \infty} \sum_{k=1}^K p_k^K \hat{Q}_k^K(u|C),$$

$\tilde{Q}(\cdot|C)$ is concave on $[C - \mu(C), \infty)$. Because $\tilde{Q}(\cdot|C)$ is independent of C — note that Assumption 2 was already shown to hold true— and equal to $\tilde{Q}(\cdot)$, this implies Assumption 3.

Second, suppose that the transition function $Q(\cdot|C)$ satisfies Assumptions 1–3. We complete the proof by showing that $Q(\cdot|C)$ can be constructed as a mixture of the type displayed in Proposition 5. The construction proceeds in two steps:

- (i). We write the distribution \tilde{Q} corresponding to $Q(\cdot|C)$ as a mixture of uniform distributions that are independent of C , distinguishing between its general part on $(-\infty, \underline{U})$ and its concave part on $[\underline{U}, \infty)$.
- (ii). Subsequently, we write $Q(\cdot|C)$ as this same mixture, with $\mu(C)$ added to the mean of each uniform distribution, and verify that the construction satisfies the requirements of Proposition 5.

Let $G_{\mu, \sigma}$ denote a uniform cumulative distribution function on $[\mu - \sigma/2, \mu + \sigma/2]$.

- (i). Consider the construction of \tilde{Q} as a mixture of uniform distributions.
- (a) Because \tilde{Q} is nondecreasing and bounded on $(-\infty, \underline{U})$, there exist $p_k^K \geq 0$, $\nu_k^K < \underline{U}$, and $\sigma_k^K \geq 0$, $k = 1, \dots, K/2$, such that $\sum_{k=1}^{K/2} p_k^K = \tilde{Q}(\underline{U}^-) \equiv \lim_{u \uparrow \underline{U}} \tilde{Q}(u)$, for even $K \in \mathbb{N}$, and

$$\lim_{K \rightarrow \infty} \sup_{u < \underline{U}, C} \left| Q(u|C) - \sum_{k=1}^{K/2} p_k^K G_{\nu_k^K, \sigma_k^K}(u) \right| = 0.$$

- (b) Because \tilde{Q} is concave on $[\underline{U}, \infty)$ (Assumption 3), by a result of e.g. Feller (1971, p. 158), there exist $p_k^K \geq 0$ and $\sigma_k^K \geq 0$, $k = K/2 + 1, \dots, K$, such that $\sum_{k=K/2+1}^K p_k^K = 1 - \tilde{Q}(\underline{U}^-)$, for even $K \in \mathbb{N}$, and

$$\lim_{K \rightarrow \infty} \sup_{u \geq \underline{U}, C} \left| \tilde{Q}(u|C) - \tilde{Q}(\underline{U}^-|C) - \sum_{k=K/2+1}^K p_k^K G_{\underline{U} + \sigma_k^K/2, \sigma_k^K}(u) \right| = 0.$$

- (ii). Subsequently, for even $K \in \mathbb{N}$, set

$$\mu_k^K(C) = \begin{cases} \nu_k^K + \mu(C), & k = 1, \dots, K/2; \\ \underline{U} + \sigma_k^K/2 + \mu(C), & k = K/2 + 1, \dots, K; \end{cases}$$

and $Q_k^K(\cdot|C) = G_{\mu_k^K(C), \sigma_k^K}(\cdot)$. Then, $\lim_{K \rightarrow \infty} \sup_{c, C} \left| \sum_{k=1}^K p_k^K Q_k^K(c|C) - Q(c|C) \right| = 0$. Moreover, each $\mu_k^K(C)$ is weakly increasing with C and $Q_k^K(\cdot|C)$ is a uniform distribution with the lower end of its support equal to $\nu_k^K + \mu(C) - \sigma_k^K/2 < \underline{U} + \mu(C) - \sigma_k^K/2 \leq C - \sigma_k^K/2 \leq C$ if $k \leq K/2$, and equal to $\underline{U} + \mu(C) \leq C$ if $k > K/2$.

□

Proposition 5 allows us to replace Assumptions 1–3 with the representation of $Q(\cdot|C)$ as a mixture of uniform autoregressions. Before we present a proof of Proposition 4 that exploits this result, we first develop three auxiliary results.

Definition 4. A function $f : [\hat{C}, \check{C}] \rightarrow \mathbb{R}$ is \tilde{C} -separable, $\tilde{C} \in [\hat{C}, \check{C}]$, if (i) $f(C) \geq f(\tilde{C})$ for all $C > \tilde{C}$ and (ii) $f(C) \leq f(\tilde{C})$ for all $C < \tilde{C}$.

Lemma 1. Let $f : [\hat{C}, \check{C}] \rightarrow \mathbb{R}$ be integrable with respect to a uniform measure over its domain, \tilde{C} -separable, and non-decreasing on $[\hat{C}, \tilde{C}]$, for some $\tilde{C} \in [\hat{C}, \check{C}]$. Given a conditional probability distribution $Q(\cdot|C)$ for C' with non-decreasing expectation $\mu(C)$ that satisfies either

(i). $Q(\cdot|C)$ is degenerate at $\mu(C) \leq C$ for all $C \in [\hat{C}, \check{C}]$, or

(ii). $Q(\cdot|C)$ is uniform on $[\mu(C) - \frac{\sigma}{2}, \mu(C) + \frac{\sigma}{2}] \subseteq [\hat{C}, \check{C}]$ with $\sigma > 0$ and $\mu(C) - \frac{\sigma}{2} \leq C$ for all $C \in [\hat{C}, \check{C}]$

then $g(C) \equiv \int_{\hat{C}}^{\check{C}} f(c)dQ(c|C)$ is non-decreasing in C on $[\hat{C}, \check{C}]$.

Proof. In Case (i), the result follows immediately from $g(C) = f(\mu(C))$. Now consider Case (ii). First, note that $g(C) = \sigma^{-1} \int_{\mu(C)-\sigma/2}^{\mu(C)+\sigma/2} f(c)dc$. Because f is non-decreasing on $[\hat{C}, \tilde{C}]$, it immediately follows that g is non-decreasing on $\{C \in [\hat{C}, \check{C}] | \mu(C) + \sigma/2 \leq \tilde{C}\}$. Next, for $C^* \leq C \leq \tilde{C}$ such that $\mu(C^*) + \sigma/2 \geq \tilde{C}$, we have that

$$\begin{aligned} \sigma(g(C) - g(C^*)) &= \int_{\mu(C^*)+\sigma/2}^{\mu(C)+\sigma/2} f(c)dc - \int_{\mu(C^*)-\sigma/2}^{\mu(C)-\sigma/2} f(c)dc \\ &\geq \int_{\mu(C^*)+\sigma/2}^{\mu(C)+\sigma/2} f(\tilde{C})dc - \int_{\mu(C^*)-\sigma/2}^{\mu(C)-\sigma/2} f(c)dc \\ &\geq \int_{\mu(C^*)+\sigma/2}^{\mu(C)+\sigma/2} f(\tilde{C})dc - \int_{\mu(C^*)-\sigma/2}^{\mu(C)-\sigma/2} f(\tilde{C})dc \\ &= 0. \end{aligned}$$

Here, the first inequality uses that the first integral is over an interval to the right of \tilde{C} and that f is \tilde{C} -separable. The second inequality uses that the second integral is over an interval to the left of \tilde{C} and that f is non-decreasing on $[\hat{C}, \tilde{C}]$. Taken together, this implies that g is non-decreasing on $[\hat{C}, \check{C}]$. \square

Lemma 2. Let $f : [\hat{C}, \check{C}] \rightarrow \mathbb{R}$ and \tilde{C} satisfy the conditions of Lemma 1. If $Q^K(\cdot|C) = \sum_{k=1}^K p_k^K Q_k^K(\cdot|C)$ for some positive p_1^K, \dots, p_K^K and $Q_1^K(\cdot|C), \dots, Q_K^K(\cdot|C)$ that each individually satisfy the conditions of Lemma 1, then $g^K(C) \equiv \int_{\hat{C}}^{\check{C}} f(c)dQ^K(c|C)$ is non-decreasing in C on $[\hat{C}, \check{C}]$.

Proof. Lemma 1 implies that $g_k(C) \equiv \int_{\hat{C}}^{\check{C}} f(c) dQ_k(c|C)$ is non-decreasing on $[\hat{C}, \check{C}]$, $k = 1, \dots, K$. In turn, because $g^K(C) = \sum_{k=1}^K p_k g_k(C)$, this implies that $g^K(C)$ is non-decreasing on $[\hat{C}, \check{C}]$. \square

Lemma 3. *Let $f : [\hat{C}, \check{C}] \rightarrow \mathbb{R}$ be bounded, \check{C} -separable, and non-decreasing on $[\hat{C}, \check{C}]$, for some $\check{C} \in (\hat{C}, \check{C}]$. Let Q^1, Q^2, \dots be a sequence of mixture Markov transition functions satisfying the conditions of Lemma 2 such that $\sup |Q^K - Q| \rightarrow 0$ for some Markov transition distribution function Q as $K \rightarrow \infty$. Then, $g(C) \equiv \int_{\hat{C}}^{\check{C}} f(c) dQ(c|C)$ is non-decreasing in C on $[\hat{C}, \check{C}]$.*

Proof. Lemma 2 implies that the function g^K corresponding to each Q^K , $K = 1, 2, \dots$, is non-decreasing on $[\hat{C}, \check{C}]$. Because f is bounded, $g^K \rightarrow g$ as $K \rightarrow \infty$ and g is non-decreasing on $[\hat{C}, \check{C}]$. \square

We are now prepared to present the proof of Proposition 4, using Proposition 5's mixture representation of $Q(\cdot|C)$.

Proof of Proposition 4. The proof begins with a characterization of $\mathcal{S}_{\check{N}} = \{C | v(0, C, \check{N}) > 0\}$ and $\mathcal{E}_{\check{N}} = \{C | v(0, C, \check{N}) > \varphi(\check{N})\}$. Recall from the proof of Proposition 2 that $v(0, C, \check{N}) = w_{\check{N}}(0, C)$, with $w_{\check{N}}$ the unique fixed point of the Bellman operator $T_{\check{N}}$ defined by Equation (2). This operator maps the space of functions in \mathcal{W} that are non-decreasing in C into itself, so the value function $v(0, C, \check{N})$ is non-decreasing in C . It immediately follows that there exist thresholds $\underline{C}_{\check{N}}$ and $\overline{C}_{\check{N}}$ such that $\mathcal{S}_{\check{N}} = \{C | C > \underline{C}_{\check{N}}\}$ and $\mathcal{E}_{\check{N}} = \{C | C > \overline{C}_{\check{N}}\}$. Note that either of these thresholds might equal \hat{C}^- or \check{C} , where \hat{C}^- is an arbitrary number strictly below \hat{C} representing the rule to be active in all demand states (including \hat{C}).

Next, iterate the following argument for $R = \check{N} - 1, \dots, 1$. Suppose that, for all $\tilde{R} = R + 1, \dots, \check{N}$, there exist thresholds $\overline{C}_{\tilde{R}}$ and $\underline{C}_{\tilde{R}}$ such that $\mathcal{S}_{\tilde{R}} = \{C | C > \underline{C}_{\tilde{R}}\}$ and $\mathcal{E}_{\tilde{R}} = \{C | C > \overline{C}_{\tilde{R}}\}$ and $v(0, C, \tilde{R})$ is non-decreasing in C for all $C < \overline{C}_{\tilde{R}}$. Consider the characterization of $\mathcal{S}_R = \{C | v(0, C, R) > 0\}$ and $\mathcal{E}_R = \{C | v(0, C, R) > \varphi(R)\}$. There are two cases to consider.

- (i). In the first, $\overline{C}_{\tilde{R}} = \check{C}$ for all $\tilde{R} = R + 1, \dots, \check{N}$, so that a firm entering with rank R expects no further entry to occur during its lifetime. This case is identical to the case where $R = \check{N}$, so there exist thresholds \underline{C}_R and \overline{C}_R such that $\mathcal{S}_R = \{C | C > \underline{C}_R\}$ and $\mathcal{E}_R = \{C | C > \overline{C}_R\}$.
- (ii). In the second case, $\overline{C}_{R+1} < \check{C}$. Here, there are two sub-cases to consider.
 - (a) In the first, $v(0, C, R) > \varphi(R)$ for all C , so we can set $\underline{C}_R = \overline{C}_R = \hat{C}^-$, for any $\hat{C}^- < \hat{C}$.

- (b) In the second sub-case, $v(0, C, R) \leq \varphi(R)$ for some C . The argument for this sub-case requires the construction of an auxiliary sequence of value functions by iterating on the Bellman operator T_R . To this end, recall that $v(0, C, R+1) = w_{R+1}(0, C)$ (where w_{R+1} is the unique fixed point of T_{R+1}), and initialize $w_R^1 \equiv w_{R+1}$. Then, for $i = 2, 3, \dots$, set $w_R^i \equiv T_R(w_R^{i-1})$. From Equation (3), it follows that

$$\begin{aligned} N'_R(X, C) - R - [N'_{R+1}(X, C) - (R+1)] \\ = I\{C \in \mathcal{S}_{R+1}\} + I\{C \in \mathcal{E}_{R+X+1}\} - I\{C \in \mathcal{S}_{R+X+1}\}. \end{aligned}$$

Because $\mathcal{S}_{R+1} \supseteq \mathcal{S}_{R+X+1} \supseteq \mathcal{E}_{R+X+1}$, this implies that

$$0 \leq N'_R(X, C) - R - [N'_{R+1}(X, C) - (R+1)] \leq 1.$$

From this, $w_{R+1} \in \mathcal{W}_{R+1}$, and $N_R(X, C) \leq N_{R+1}(X, C)$; it follows that $w_R^2 = T_R(w_R^1) \geq w_R^1 = w_{R+1}$. Because T_R is monotonic, this implies that $w_R^i \geq w_R^{i-1}$ for all $i \geq 2$.

Define $v^i(0, C, R) \equiv w_R^i(0, C)$ for all i and C . We first show with induction that $v^i(0, C, R)$ and $\bar{C}_R^i \equiv \inf\{C | v^i(0, C, R) > \varphi(R)\} \leq \bar{C}_R^{i-1}$ together satisfy the conditions for $f(C)$ and \tilde{C} in Lemma 3. By assumption, this is the case for $v^1(0, C, R)$ and \bar{C}_R^1 , because $\bar{C}_R^1 \leq \bar{C}_{R+1}$. Next, suppose that $v^{i-1}(0, C, R)$ and \bar{C}_R^{i-1} satisfy Lemma 3's requirements for $f(C)$ and \tilde{C} . Then this Lemma implies that $\mathbb{E}[v^{i-1}(0, C', R) | C]$ is non-decreasing in C on $[\hat{C}, \bar{C}_R^{i-1}]$. Therefore, inspection of Equation (2) determines that $v^i(0, C, R)$ is non-decreasing in C on the same interval. Because $v^i(0, C, R) \geq v^{i-1}(0, C, R)$, we have that $\bar{C}_R^i \leq \bar{C}_R^{i-1}$. Thus, $v^i(0, C, R)$ and \bar{C}_R^i satisfy Lemma 3's requirements of $f(C)$ and \tilde{C} .

Define $\bar{C}_R = \lim_{i \rightarrow \infty} \bar{C}_R^i$. We wish to show that

- (A) $v(0, C, R) \leq \varphi(R)$ and non-decreasing in C for all $C \in [\hat{C}, \bar{C}_R]$ and
- (B) $v(0, C, R) > \varphi(R)$ for all $C \in (\bar{C}_R, \check{C}]$.

To show (A), first note that it holds trivially if $\bar{C}_R = \hat{C}$ and focus on the case that $\bar{C}_R > \hat{C}$. Note that $v^i(0, C, R)$ is non-decreasing in C and weakly less than $\varphi(R)$ on $[\hat{C}, \bar{C}_R^i)$ for all i . Because $\bar{C}_R \leq \bar{C}_R^i$, it must be that for all $C^* \leq C \leq \bar{C}_R$ that $\lim_{i \rightarrow \infty} v^i(0, C, R) \leq \varphi(R)$ and $\lim_{i \rightarrow \infty} v^i(0, C^*, R) \leq \lim_{i \rightarrow \infty} v^i(0, C, R)$.

We demonstrate (B) inductively. Because $\varphi(R) \leq \varphi(R+1)$ and $v^1(0, C, R) = w_R(0, C)$ is non-decreasing in C on $[\bar{C}_R^1, \bar{C}_{R+1}]$, we know that $v^1(0, C, R) > \varphi(R)$

for $C \in (\bar{C}_R^1, \check{C}]$. Suppose that $v^{i-1}(0, C, R) > \varphi(R)$ for all $C \in (\bar{C}_R^{i-1}, \check{C}]$. Then, $v^i(0, C, R) \geq v^{i-1}(0, C, R) > \varphi(R)$ for all $C \in (\bar{C}_R^{i-1}, \check{C}]$ as well. Furthermore, because $v^i(0, C, R)$ is non-decreasing in C on $[\hat{C}, \bar{C}_R^{i-1}]$, the definition of \bar{C}_R^i implies that $v^i(0, C, R) > \varphi(R)$ for all $C \in (\bar{C}_R^i, \bar{C}_R^{i-1}]$. Because the sequence $\{v^i(0, C, R)\}$ is non-decreasing, $v^i(0, C, R) > \varphi(R)$ for all $C \in (\bar{C}_R^i, \check{C}]$.

With this established, clearly $\mathcal{E}_R = \{C | C > \bar{C}_R\}$. Define

$$\underline{C}_R \equiv \sup\{C | v(0, C, R) \leq 0\}$$

if $\{C | v(0, C, R) \leq 0\} \neq \emptyset$, and $\underline{C}_R \equiv \hat{C}^-$ otherwise. By construction, $\underline{C}_R \leq \bar{C}_R$. Because $v(0, C, R)$ is non-decreasing for $C \leq \bar{C}_R$, we can write $\mathcal{S}_R = \{C | C > \underline{C}_R\}$.

□