Stabilizing, Pareto Improving Policies[1].

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by

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Abstract

One of the objectives of the current policies of many central banks arguably is to stabilize economic activity. One possible justification for such a policy is that there is volatility in macro variables that individual agents cannot insure against. We study the simplest possible extension of the stochastic 2-period, one agent and one commodity OLG model, where we have added 1 more period, with only one potential activity, namely trading of contingent commodities. We assume, however, that markets are incomplete. In this case the monetary equilibrium is not Pareto Optimal and for an open set of economies an allocation where fluctuations in realized savings are removed, Pareto dominates the monetary equilibrium. This allocation may be implemented by means of a monetary/fiscal policy. The policy considered has a simple rationale, namely that it removes some of the uncertainty that agents face by reducing price, i.e interest rate volatility.

We consider two fundamental sources of such volatility, namely respectively an objective and a subjective signal about the distribution of future endowments. The first case is when agents have Rational Expectations while the second case is studied in the context of agents having Rational Beliefs, beliefs which are consistent with empirical observations but not (necessarily) correct. In the context of rational beliefs, the uncertainty is about what beliefs future agents will hold, and we interpret this as being a consequence of diverse beliefs.

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1 Introduction

In a Pareto optimal equilibrium for a stochastic economy, price volatility, although having a negative impact on risk averse agents, is simply reflecting the randomness of underlying fundamentals. In such an equilibrium a policy that seeks to curb price volatility will only inhibit the well functioning of the market economy and is thus mistaken. None the less one can interpret the contemporary policies of many central banks as trying to stabilize economic activity using the interest rate. If one believes that the rationale of such policies is that they are Pareto improving, then some of the assumptions of the classical general equilibrium model has to be reconsidered.

Here we consider a simple 3-period OLG model with incomplete markets and show how economic stabilization may be welfare improving. Volatility of economic variables is created by volatility in demand and supply. Thus curbing price volatility must be done by controlling (directly or indirectly) the actions of agents. There is then a trade-off, since from the individual agent's viewpoint, the ideal policy would control the actions of all other agents, and thus control the (price) volatility he is facing, but would allow himself to act freely. In this paper this trade-off is exposed and some conditions for when a stabilizing policy is considered beneficial for the individual (representative) agents are provided.

This study was motivated by an interest in understanding the relationship between rational beliefs and optimality. Elsewhere (Nielsen, 1998, Nielsen 2009a, Nielsen 2009b) we have adressed this question using the concept of ex-post optimality (see f.i. Hammond, 1981), but here we are interested in Pareto optimality. As is well known, diversity of beliefs plays no role in the definition of Pareto optimality. This is, we argued in the just cited studies, problematic, but it does not rule out it also being of interest to understand if there is any connection between Pareto suboptimality and rational beliefs. Diversity of beliefs is the central aspect of rational beliefs and also play the central role in understanding why rational belief equilibria may not be ex-post optimal. It is conceivable that pure diversity of belief could lead to Pareto suboptimality, however here we have chosen to focus instead on another important aspect of rational beliefs, namely that if the subjective rational beliefs are correlated the result may be increased fluctuations in endogenous variables beyond what is created by changes in fundamentals (so called endogenous volatility). When markets are incomplete such fluctuations may lead to Pareto suboptimality.

Sunspots in the OLG model have been used to explain why unwarranted economic volatility unrelated to fundamentals may be present in the economy (se for instance Azariadis, 1981).[2] But there are some open questions regarding this explanation. Firstly, removing the price instability caused by sunspot is not improving when one uses Conditional Pareto Optimality as welfare concept (as was already noted by Cass and Shell(1983). If on the other hand one wishes to use Equal Treatment Optimality as a criterion then the stationary stochastic allocation may no longer be optimal in a model with stochastic fundamentals. Stabilization is then no longer an obvious choice and just describing the optimal allocations becomes difficult even in the two-period model that is usually being used. Thirdly, one may doubt whether a realistic version of the OLG model (where agents live for many periods) can explain the magnitude and form of real and nominal fluctuations observed in the economy. The sunspots effects in the OLG model are the result of random reallocations from/to newborn generations, via an inflationary tax/subsidy. Apart from the issue whether this is a realistic description of what happens during the business cycle is the question, whether the magnitude of such transfers in a monetary equilibrium are realistic. In the case where agents live for 2 periods a relatively high proportion of the economy's resources can be shifted from one generation to another in equilibrium. But how much can be shifted between the first generation and the 77 remaining in a monetary equilibrium? This is an issue that seems to call for further analysis.

In our model incompleteness of markets are not derived from fundamentals. This seems to be the rule in macroeconomic studies seeking to explain the effectiveness of monetary or fiscal policies. Calvet(2001), for instance provides an explanation of "excess" price volatility based on incompleteness of markets. The New Keynesian macroeconomics literature (see for instance Woodford, 2003) is assuming that individual suppliers who are unable to change their prices in the short run cannot insure against shocks that would make them want to change prices. Lorenzoni(2007) studies the effectiveness of monetary policy. His model belongs to a class of models (originating with Lucas, 1972) where a continuum of agents have asymmetric information (here about productivity shocks), which is what makes monetary policy relevant. The question that naturally arises in such models is why each informationally small agent does not simply announce his private information, in which case there is no need for intervention by the monetary authorities. Note that all agents doing this constitutes a Nash equilibrium in this model.

The standard two-period one commodity OLG model with a single (representative) agent trivially has (sequentially) complete markets, in the sense that agents who live at the same time can freely trade in all spot and contingent commodities, but will not do so. It is only by assuming that (representative)

[2] As is well known, sunspots can only have an effect when markets are incomplete.

agents live for more than two periods, that there are more than one commodity or that there are heterogenous agents that the issue of completeness becomes nontrivial. Except for the last possibility such models are unfortunately in general difficult to study if one are interested in equilibria which are ergodic and stationary or, more generally, from which an empirical distribution of prices can be extracted (see Duffie, Geanakoplos, Mas-Colell, and McLennan, 1994 and Gottardi, 1996). Therefore, we have chosen to study only the simplest possible extension of the representative agent model, namely where agents live for 3 periods and their potential activity in the first period is only to trade contingent commodities. Markets are however assumed to be incomplete and such trade cannot take place. We show how stabilizing economic activity may improve welfare. Of course this means that active monetary/fiscal policies matter, something which has already been shown in many contexts (see for instance Gottardi, 1995 for the case of an OLG model, with heterogeneous agents who live for two periods). But the explicit study of the effects of economic stabilization found in this paper appears to be new.

The paper is organized as follows. In the next section the OLG model is described. Furthermore we introduce two interpretations of the model, one assuming that agents hold Rational Expectations the other assuming that they hold Rational Beliefs. We also consider two possible concepts of Pareto Optimality known from the literature and argue in favor of one of them. In the following section we study the features of the monetary equilibria and provide the policy results. The appendices deals with more technical issues. In the first we show some of the results for a more general version of the OLG model. In the other we provide a brief introduction to the theory of Rational Beliefs.

2 The one-commodity OLG model; Rational Expectations and Rational Beliefs

2.1 Model and Monetary Equilibrium

We consider an overlapping generations model with one commodity where agents (representative or a continuum, indexed by $[0, 1]$) are born in the first period , receive endowments e_a in the second period and a random endowment, e_t either e_b or e_c in the third and final period of their lives. Furthermore, in the second period of their lives the agents receive a signal, $z_t \in \{1, 2\}$ about the (objective or perceived) distribution, $\pi^z = (\pi_b^z, \pi_c^z)$ (with $\pi^1 \neq \pi^2$) of the endowments in the last period of their lives[3]. We assume $e_c < e_b$ and $\pi_c^1 < \pi_c^2$. The stochastic sequence $\{z_t\}$ is i.i.d. and independent of all past information, with probability vector (q_1, q_2).

[3] In appendix 4.1 we study the case where the number of signals and second period endowments is arbitrary but finite.

Agents, which are all ex-ante identical, only have utility, u over consumption in the second (C_1) and third (C_2) period of their lifes. u is defined on $\Re^2_{++}$, C^2, strictly increasing, strictly concave, and with indifference curves whose closures are contained in $\Re^2_{++}$.[4] So we essentially have a classical OLG model, except that agents are born before they know what "type" they are, i.e. before they know the signal about the distribution of the endowments in the last period of their lives. We consider a monetary equilibrium for this economy, assuming that the amount of outside fiat money is $\overline{M}$ unit. In such an equilibrium there will be two possible prices (of money in terms of the commodity good), p_1 and p_2, at each date t, depending on the signal of the then middle-aged.

DEFINITION 1 *Monetary Equilibrium. Price vector* $(p_1, p_2) \in \Re^2_{++}$ *such that when an agent with signal* $z = k$ *solves:*

$$Max_{M \geq 0} \sum_{i=2}^{2} \sum_{s \in \{b,c\}} u(e_a - p_k M, e_s + p_i M) q_i \pi_s^k \tag{1}$$

the solution is $M = \overline{M}$ ■

Such an equilibrium (where money is valued) may or may not exist, depending on preferences, endowments and beliefs. In appendix 4.1 a sufficient condition for existence is provided. Using the First Order Conditions we then have that a monetary equilibrium is uniquely characterized by

$$\sum_{i=2}^{2} \sum_{s \in \{a,b\}} [-\frac{\partial u}{\partial C_1}(e_a - p_k\overline{M}, e_s + p_i\overline{M})p_k + \frac{\partial u}{\partial C_2}(e_a - p_k\overline{M}, e_s + p_i\overline{M})p_i] q_i \pi_s^k = 0, k = 1, 2 \tag{2}$$

Until Section 3.2 we shall assume that $\overline{M} = 1$. In what follows it will be useful to employ the following notation (we suppress reference to the prices).

$$Y_{ki} = \sum_{s \in \{b,c\}} \frac{\partial u}{\partial c_1}[e_a - p_k, e_s + p_i] p_k \pi_s^k q_i, k = 1, 2, i = 1, 2$$

and

$$X_{ki} = \sum_{s \in \{b,c\}} \frac{\partial u}{\partial c_2}[e_a - p_k, e_s + p_i] p_i \pi_s^k q_i, k = 1, 2, i = 1, 2$$

With this notation we can rewrite the requirements for a monetary equilibrium as

$$-\sum_{i=1}^{2} Y_{ki} + \sum_{i=1}^{2} X_{ki} = 0, k = 1, 2 \tag{3}$$

REMARK 1

[4] When making genericity statements, the topology of C^2 uniform convergence on compacta (MasColell(1985),p.50) is used.

We will sometimes refer to the two-period version of the model considered here. In this version agents live for two periods corresponding to the two last periods of the three-period version of the model, i.e. when they are born the signal about the distribution of endowments in the last period of their life has already been realized. Despite this difference, for the two-period version of the model the definition of monetary equilibrium is exactly as above. ■

2.2 Interpretation of z_t

We provide two different not necessarily exclusive interpretations of the signal z_t. According to the first, z_t is an objective signal about the distribution of the endowments the next period, and as such can be considered to be a supply shock, about which agents hold Rational Expectations.

In the other explanation, z_t is a signal which coordinates the subjective expectations of the agents, expectations which may not necessarily be correct. That subjective beliefs are indeed present and significant even among major actors on the financial markets is convincingly demonstrated in Kurz and Motolese (2007). For this case the signals are guiding subjective beliefs they can better be interpreted as a demand shock in that they affect the beliefs i.e. preferences of agents.

In the context of this second explanation agents are supposed to hold Rational Beliefs about the distribution of the endowments[5]. The empirical distribution of the endowments is assumed to be known. The Rational Beliefs theory suggests that agents may think that more can be known than just this empirical distribution. Specifically, agents may form statistical models or theories according to which the endowment process, $\{e_t\}$ is correlated with a process of signals $\{z_t\}$. We do not assume that agents know the empirical distribution of the joint process $\{e_t, z_t\}$, only that they know the two marginals of the empirical distribution, each assumed to be i.i.d.. We denote the empirical distribution of $\{e_t\}$ by $\overline{B} \equiv (\overline{\pi}_b, \overline{\pi}_c)$. The type of statistical model we consider in the context of this paper is as follows. When agents at date t observe $z_t = i, i = 1, 2$ they pick the belief $B_t = B^i = (\pi_b^i, \pi_c^i)$, which they hold to be the distribution of the the endowments in the following period. To simplify and unify the exposition of the model z_t is assumed to be known to be i.i.d. Assuming, as we did, that both the empirical distribution of $\{e_t\}$ and of $\{z_t\}$ is known, rational agents can only adopt a statistical model, or belief, if it generates the same empirical distribution of the endowments as the known one. The model generates the empirical distribution $q_1 B^1 + q_2 B^2$ and consequently, what we call the rationality

[5]See Appendix 4.2 for a brief introduction to the theory of Rational Beliefs, and Kurz(1994) and Nielsen(2008) for more comprehensive expositions.

requirement on individual subjective beliefs is

$$q_1 B^1 + q_2 B^2 = \overline{B} \tag{4}$$

Under this condition, the stochastic sequence $\{B_t\}_t$ generates the same empirical diistribution as the model in which agents live.

REMARK 2 *Diversity of beliefs*

Each agent is born knowing his (generation's) belief, but not knowing the beliefs of future generations. In this sense beliefs are not common knowledge, hence diverse. To further strenghten this interpretation of the model, we could postulate that young and old agents hold beliefs that are diverse from those held by middle aged agents, although in a monetary equilibrium, only beliefs of middle aged matter in equilibrium. Note also that under this interpretation, the need of middle aged agents to forecast the future beliefs of future middle aged agents, has the flavor of the beauty contest story of Keynes. In Nielsen(2007) stochastic sunspots are causing correlated changes in the distribution of (subjective) beliefs in the market and we think of the presents model as a simplified way of achieving the same effect.

Let us note that there is a problem of interpretation of the model of the beliefs as presented here. Presumably agents would know the empirical distribution of the joint process of prices and endowments. But since prices are determined by beliefs, i.e. z_t, implicitly they know the empirical distribution of the joint process $\{z_t, e_t\}$, something we assumed that they did not. In fact this consistency problem can be remedied, but at the expensive of using the considerably more complicated model we alluded to before ■

In the context of the present model, we do not need to distinguish between the two types of sources of volatility, signals about endowments (technology) or subjective signals. This is due to two factors. Firstly, for the Rational Beliefs version of the model we have assumed perfect correlation between beliefs. Some correlation is needed for subjective changes in expectations to show up in aggregated like prices, but the correlation need certainly not be perfect[6]. As the heterogeneity of beliefs increases the analysis below becomes more complicated and this is one reason why we have chosen the simpler case. Incidently, one way to empirically separate the Rational Expectations case from the Rational Beliefs case is by looking for heterogeneity of beliefs among equally informed market participants. Such heterogeneity has been observed in the markets for foreign exchange (see Taylor, 1995) and in

[6]In Kurz(1998) it is demonstrated that even without correlation between beliefs excess volatility may be generated.

financial markets (see Odean, 1998 and Daniel, Hirschleifer, and Subrahmanyam, 1998) for reviews of the literature and further references. See also Kurz(1997) for a discussion of endogenous fluctuations and diversity of beliefs from the viewpoint of the theory of Rational Beliefs. Secondly, we are here concerned with Pareto optimality and improvements and in that context it is irrelevant whether beliefs are correct or not.

2.3 Some concepts of Optimality

Several concepts of Pareto Optimality for stochastic OLG models have been proposed in the literature, two notable being that of Conditional Pareto Optimality (also called Dynamical Pareto Optimality in Cass and Shell, 1983) and that of Equal Treatment-Pareto Optimality (defined in Muench, 1977). In Peled(1982) these two notions are compared and it is argued that Conditional Pareto Optimality is the right criterion. When using the Conditional Pareto Optimality criterion the definition of an agent includes the stochastic state in which he is born. Thus Pareto improving transfers cannot make an agent born in some state worse off, even though the "same" agent born in another state might be made better off. The notion of Conditional Pareto Optimality is then weaker than the notion of Equal Treatment-Pareto Optimality, according to which such transfers may be considered Pareto improving if they make all agents better off in an expected sense. Peled's argument in favor of this optimality criterion is that according to it the set of feasible allocations is equal to the set of allocations that can be achieved by trade among agents. In contrast, for Equal Treatment Pareto optimality the first set is strictly larger than the second one. It is easy to show that all (sun-spot) equilibria in the model of Azariadis(1981) are conditionally Pareto Optimal (this observation is the same as Proposition 6 of Cass and Shell, 1983). This is our motivation for studying a three period model, where we have added a period, right after birth, to the life of (representative) agents. The only effect of this period is that is allows for an addition of utility across states of beliefs, as it would allow for an addition of utility across (sun-spot) states in the Azariadis model. In this sense we move closer to the concept of Equal Treatment-Pareto Optimality .

To define a conditionally Pareto optimal allocation, we first define an event at date t:

$$I_t = \{z_s, e_s\}_{s=-\infty}^{t} \text{ a member of } \mathcal{I}_t = \times_{s=-\infty}^{t}[\{1,2\} \times \{e_b, e_c\}]$$

- the set of events at date t. For the definition we assume that the beliefs of the young agents at date t depend on the signal z_t and this signal as well as e_t (the endowment of the old at date t) are realized before they are born. An allocation is then a sequence of functions $\{C_{mt}, C_{ot}\}_{t=1}^{\infty}$ where $C_{it} : \mathcal{I}_t \rightarrow$

$\Re_+, i = m, o$. Such an allocation is said to be feasible if, $\forall t, \forall I_t \in \mathcal{I}_t$: $C_{mt}(I_t) + C_{ot}(I_t) = e_a + e_t$. The conditional expected utility of an agent born at date t at event I_t is

$$\sum_{i=1}^{2} \sum_{j=1}^{2} \sum_{s \in \{b,c\}} \sum_{s' \in \{b,c\}} u[C_{mt+1}(I_t, z_i, e_s), C_{ot+2}(I_t, z_i, e_s, z_j, e_{s'})] \pi_s^{z_t} \pi_{s'}^{j} q_i q_j$$

A feasible allocation is $\{C_{mt}, C_{ot}\}_{t=1}^{\infty}$ is then conditionally Pareto optimal if there is no other feasible allocation $\{C'_{mt}, C'_{ot}\}_{t=1}^{\infty}$ that for no t and I_t decreases the conditional utility and for some t and I_t strictly increases conditional expected utility. In this definition we let time go from $-\infty$ to ∞ which is similar to ignoring the initial middle aged and old when there is an initial date.

The stabilizing policies that we shall consider do not necessarily lead to a conditionally Pareto optimal allocation, but only to improvements. Such improvements are possible because with agents living for three periods there are in our model missing markets - in particular young agents do not sign contract with middle aged agents for contingent delivery in the following period. We now turn to defining the stabilizing policies.

3 Stabilizing Policies

The following proposition is, under a rank condition for beliefs, generalized to generic utility functions in Appendix 4.1.

PROPOSITION 1 *Suppose that $u(C_1, C_2) = u_1(C_1) + u_2(C_2)$. Then p_1 and p_2 are different in a monetary equilibrium.*

Proof: Suppose not, i.e. $p_1 = p_2 = p$. $\sum_{i=1}^{2} \sum_{s \in \{a,b\}} [-\frac{\partial u}{\partial C_1}(e_a - p, e_s + p)p] q_i \pi_s^k$ does not depend on k, but $\sum_{i=2}^{2} \sum_{s \in \{a,b\}} [\frac{\partial u}{\partial C_2}(e_a - p, e_s + p)p] q_i \pi_s^k$ does, since $\frac{\partial u}{\partial C_2}(e_a - p, e_1 + p) \neq \frac{\partial u}{\partial C_2}(e_a - p, e_2 + p)$, a contradiction with (2) ■

The proposition says that under the stated conditions there will be price volatility in a monetary equilibrium. Note that without the signal $\{z_t\}$ no such volatility would be present.

REMARK 3

Suppose we were only considering a two period model, i.e. where each agent is born with his beliefs and endowment, e_a and have random endowments, e_b or e_c when old. For this model, any vector $(p_1, p_2) \in \Re_+^2$ which for some weights, (w_1, w_2) solves the problem,

$$Max \sum_{k=1}^{2} w_k \sum_{i=1}^{2} \sum_{s \in \{b,c\}} u(e_a - p_k, e_s + p_i) q_i \pi_s^k \tag{5}$$

implements a stationary conditionally Pareto Optimal allocation (there are other such allocations). Notice that with $w_k = q_k$ (5) is formally identical to (9) below. If interior, the First Order Conditions for a solution to this problem are

$$w_k \sum_{i=1}^{2} \sum_{s\in\{b,c\}} -\frac{\partial u}{\partial C_1}(e_a - p_k, e_s + p_i) q_i \pi_s^k + \sum_{i=1}^{2} w_j \sum_{s\in\{b,c\}} \frac{\partial u}{\partial C_2}(e_a - p_i, e_s + p_k) q_k \pi_s^i = 0, k = 1, 2 \quad (6)$$

which by conavity are also sufficient. Using the previously introduced notation these requirements may be rewritten (after multiplying through with p_k) as

$$-w_k \sum_{i=1}^{2} Y_{ki} + \sum_{i=1}^{2} w_i X_{ik} = 0, k = 1, 2 \quad (7)$$

Suppose the equilibrium conditions (3) for a monetary equilibrium hold. Set $w_1 = p_1 \sum_{s\in\{b,c\}} \frac{\partial u}{\partial C_2}(e_a - p_2, e_s + p_1) q_1 \pi_s^2$ and $w_2 = p_2 \sum_{s\in\{b,c\}} \frac{\partial u}{\partial C_2}(e_a - p_1, e_s + p_2) q_2 \pi_s^1$ and the conditions (7) will also hold so that the monetary equilibrium is indeed Conditionally Pareto optimal. This remark (which is generalized in Appendix 4.1) serves as a motivation for considering a 3-period model. ■

3.1 Pareto improving policies

In a monetary equilibrium, (p_1, p_2) the expected utility of a young agent is

$$\sum_{k=1}^{2} q_k \sum_{i=1}^{2} \sum_{s\in\{b,c\}} u(e_a - p_k, e_s + p_i) q_i \pi_s^k \quad (8)$$

From this expression one sees intuitively that the agent would like to be able to react on his own belief (i.e. have his savings depend on the signal he receives) but would prefer that others were not able to (i.e. would prefer that the return on his savings does not depend on other agents' signals).

In a monetary equilibrium the prices are transfers between generations. To see if we can improve on the allocation associated with a monetary equilibrium it is natural to consider the following problem:

$$Max_{p_1\geq 0, p_2 \geq 0} \sum_{k=1}^{2} q_k \sum_{i=1}^{2} \sum_{s\in\{b,c\}} u(e_a - p_k, e_s + p_i) q_i \pi_s^k \quad (9)$$

Thus we are asking if there are stationary allocations which improve on the allocation that is the outcome of the monetary equilibrium.

The First Order Conditions for an interior solution to this problem are

$$\sum_{i=1}^{2} \sum_{s\in\{b,c\}} -q_1 \frac{\partial u}{\partial C_1}(e_a - p_1, e_s + p_i) q_i \pi_s^1 + \sum_{j=1}^{2} q_j \sum_{s\in\{b,c\}} \frac{\partial u}{\partial C_2}(e_a - p_j, e_s + p_1) q_1 \pi_s^j = 0 \quad (10)$$

$$\sum_{i=1}^{2} \sum_{s\in\{b,c\}} -q_2 \frac{\partial u}{\partial C_1}(e_a - p_2, e_s + p_i) q_i \pi_s^2 + \sum_{k=1}^{2} q_j \sum_{s\in\{b,c\}} \frac{\partial u}{\partial C_2}(e_a - p_j, e_s + p_2) q_2 \pi_s^j = 0 \quad (11)$$

Proposition 2 *Generically, the monetary equilibrium is not conditionally Pareto Optimal.*

Proof: (10), multiplied through by p_1/q_1 and (11), multiplied through by p_2/q_2 may be rewritten as

$$-\sum_{i=1}^{2} Y_{1i} + \sum_{j=1}^{2} X_{j1} = 0 \tag{12}$$

and

$$-\sum_{i=1}^{2} Y_{2i} + \sum_{j=1}^{2} X_{j2} = 0 \tag{13}$$

Combining these requirements with (3) we get

$$\sum_{j=1}^{2} X_{j1} = \sum_{i=1}^{2} X_{1i} \text{ and } \sum_{j=1}^{2} X_{j2} = \sum_{i=1}^{2} X_{2i}$$

i.e. $X_{21} = X_{12}$ or written out:

$$p_1 \sum_{s\in\{b,c\}} \frac{\partial u}{\partial C_2}(e_a - p_2, e_s + p_1) q_1 \pi_s^2 = p_2 \sum_{s\in\{b,c\}} \frac{\partial u}{\partial C_2}(e_a - p_1, e_s + p_2) q_2 \pi_s^1 \tag{14}$$

Without making the statement precise it is intuitively clear, that generically (10), (11), and (14) (3 equations with 2 unknowns) cannot hold, meaning that a monetary equilibrium can be improved upon. This is formally shown in Appendix 4.1

The proposition is saying that the implied weigths w_1 and w_2 are unlikely to be equal to the probabilities q_1 and q_2.

The transfers in the solution to (9) are in general still random. Let us now consider another problem. Suppose we restrict the young agents to making a transfer between the date when they are middle aged and the date where they are old, independently of the beliefs they have, when middle aged, i .e. suppose we consider the following problem:

$$Max_{p\geq 0} \sum_{k=1}^{2} q_k \sum_{i=1}^{2} \sum_{s\in\{b,c\}} u(e_a - p, e_s + p) q_i \pi_s^k = \tag{15}$$

$$Max_{p\geq 0} \sum_{s\in\{b,c\}} u(e_a - p, e_s + p) \overline{\pi}_s^k$$

In this problem the emphasis is on the transfer of consumption among periods, not on the insurance motive stemming from uncertainty about endowments and beliefs. It is interesting to notice the connection between this objective and the notion of ex-post optimality as it has been employed in Nielsen(2003), Nielsen(2009a) and Nielsen(2009b). The objective ignores the subjective beliefs of agents and uses instead the stationary measure. However, the objective does respect the subjective need to save for later consumption. If we were employing ex-post welfare optimality as welfare criterion, (15) would be a natural objective in the two-period model.

Letting $\hat{p}$ be the solution to this problem, the first order condition for it is

$$\sum_{s\in\{b,c\}} \overline{\pi}_s[-\frac{\partial u}{\partial C_1}(e_a-\hat{p},e_s+\hat{p})+\frac{\partial u}{\partial C_2}(e_a-\hat{p},e_s+\hat{p})]=0$$

which for the special case where u is separable, $u(C_1,C_2)=u_1(C_1)+u_2(C_2)$, can be rewritten as

$$-u_1'(e_a-\hat{p})+\sum_{s\in\{b,c\}} \overline{\pi}_s u_2'(e_s+\hat{p})=0 \tag{16}$$

It is now easy to see, that if preferences are time separable, a solution to (9) has $p_1=p_2$, i.e. is a solution to (15):

Proposition 3 *Suppose $u=u_1+u_2$. Then a solution to (9) has $p_1=p_2$ and is a solution to (15).*

Proof: The assumption implies that the formats of (10) and (11) (after dividing through by q_1 and q_2 respectively) become identical to that of (16) ■

Under the stated conditions the result implies that in the monetary equilibria studied here, if it is possible for the government to pursue a policy which results in the fixed price $\hat{p}$ then such a policy is better than the laissez faire policy. In other words it is desirable to remove all volatility in economic activity related to the signal. Note that for utility functions in an open neighborhood of time separable utility functions the solution to (15) still Pareto dominates the Monetary Equilibrium. Before turning to how to implement the constant price let us briefly consider local changes in the two prices.

Suppose we are in a monetary equilibrium, so that (2) holds but that (14) does not hold. If in stead

$$p_1 \sum_{s\in\{b,c\}} \frac{\partial u}{\partial C_2}(e_a-p_2,e_s+p_1)q_1\pi_s^2 > p_2 \sum_{s\in\{b,c\}} \frac{\partial u}{\partial C_2}(e_a-p_1,e_s+p_2)q_2\pi_s^1 \tag{17}$$

then the LHS of (10) is >0, while the LHS of (11) is <0:

$$-\sum_{i=1}^{2} X_{1i}+\sum_{j=1}^{2} X_{j1}=-X_{12}+X_{21}>0$$

and

$$-\sum_{i=1}^{2} X_{2i}+\sum_{j=1}^{2} X_{j2}=-X_{21}+X_{12}<0$$

This means that if we slightly increase the price, p_1 associated with signal 1 and slightly decrease p_2 then we increase utility for all agents. To interpret this observation note the following result:

Proposition 4 *If $\frac{\partial^2 u}{\partial x_1 \partial x_2}\geq 0$ then in a monetary equilibrium $p_1<p_2$.*

Proof: We assume the opposite, $p_1 \geq p_2$ and obtain a contradiction. We then have

$$(a) \quad \sum_{i=1}^{2} \sum_{s \in \{b,c\}} \frac{\partial u}{\partial C_1}(e_a - p_1, e_s + p_i) q_i \pi_s^1 \geq \sum_{i=1}^{2} \sum_{s \in \{b,c\}} \frac{\partial u}{\partial C_1}(e_a - p_1, e_s + p_i) q_i \pi_s^2$$

$$\geq \sum_{i=1}^{2} \sum_{s \in \{b,c\}} \frac{\partial u}{\partial C_1}(e_a - p_2, e_s + p_i) q_i \pi_s^2$$

and

$$(b) \quad \sum_{i=1}^{2} \sum_{s \in \{b,c\}} \frac{\partial u}{\partial C_2}(e_a - p_1, e_s + p_i) q_i \pi_s^1 < \sum_{i=1}^{2} \sum_{s \in \{b,c\}} \frac{\partial u}{\partial C_2}(e_a - p_1, e_s + p_i) q_i \pi_s^2$$

$$\leq \sum_{i=1}^{2} \sum_{s \in \{b,c\}} \frac{\partial u}{\partial C_2}(e_a - p_2, e_s + p_i) q_i \pi_s^2$$

which implies that the condition, (2) for a monetary equilibrium cannot hold ■

Thus if the assumptions of the result as well as (17) hold, then decreasing the "price volatility" slightly, by increasing the smaller price and decreasing the larger price increases welfare. Of course if the assumptions of the result still hold but instead (17) holds with reverse inequality, then increasing "price volatility" increases welfare.

REMARK 4

The conditions stated in Proposition 4 together with (17) in a monteray equilibrium are non-empty as the following example demonstrates.

Let $q_1 = q_2 = \frac{1}{2}$, $e_a = 2$, $e_b = 1$ and $e_c = 0$. Suppose $\pi^1 = (1, 0)$ and $\pi^2 = (0, 1)$ and that $u(C_1, C_2) = \ln C_1 + \ln C_2$.

We first find the monetary equilibrium for this economy. Using the first order conditions (2) we have

$$-\frac{p_1}{2 - p_1} + \sum_{i=1}^{2} \sum_{s \in \{b,c\}} \frac{p_i}{e_s + p_i} \frac{1}{2} \pi_s^1 = 0$$

$$-\frac{p_2}{2 - p_2} + \sum_{i=1}^{2} \sum_{s \in \{b,c\}} \frac{p_i}{e_s + p_i} \frac{1}{2} \pi_s^2 = 0$$

Solving this system one finds $p_2 = 1$ and $p_1 = \frac{1+\sqrt{57}}{14}$. Comparing the LHS and RHS in (17) one finds

$$p_1 \sum_{s \in \{b,c\}} \frac{\partial u}{\partial C_2}(e_a - p_2, e_s + p_1) q_1 \pi_s^2 = p_1 \frac{1}{0 + p_1} \frac{1}{2} = \frac{1}{2} > p_2 \sum_{s \in \{b,c\}} \frac{\partial u}{\partial C_2}(e_a - p_1, e_s + p_2) q_2 \pi_s^1 = p_2 \frac{1}{1 + p_2} \frac{1}{2} = \frac{1}{4} \blacksquare$$

3.2 Implementing the Pareto-improvement

We find a fiscal/monetary policy that results in the allocation associated with (15). Let M_t and p_t be given s.t.

$$M_t p_t = \hat{p} \tag{18}$$

Suppose the current signal, z_t has the value k. Then define next period's price to be

$$p_{t+1}^k = \frac{\sum_{s\in\{b,c\}} \frac{\partial u}{\partial C_1}(e_a - \hat{p}, e_s + \hat{p})\pi_s^k}{\sum_{s\in\{b,c\}} \frac{\partial u}{\partial C_2}(e_a - \hat{p}, e_s + \hat{p})\pi_s^k} p_t$$

Notice that this price is conditionally deterministic, i.e. conditionally there is no price volatility. Let real taxes (subsidies) imposed on next period's old be:

$$\tau_{t+1}^k = p_{t+1}^k M_t - \hat{p} \text{ i.e., nominally the tax (subsidy) is: } \frac{\tau_{t+1}^k}{p_{t+1}^k} = M_t - \frac{\hat{p}}{p_{t+1}^k} \quad (19)$$

Agents then solve the problem:

$$Max_{M\geq 0} \sum_{s\in\{b,c\}} u(e_a - p_t M, e_s + p_{t+1}^k M - \tau_{t+1}^k)\pi_s^k$$

with First Order Conditions

$$\sum_{s\in\{b,c\}} [-\frac{\partial u}{\partial C_1}(e_a - p_t M, e_s + p_{t+1}^k M - \tau_{t+1}^k)p_t + \frac{\partial u}{\partial C_2}(e_a - p_t M, e_s + p_{t+1}^k M - \tau_{t+1}^k)p_{t+1}^k]\pi_s^k = 0 \quad (20)$$

Using the definitions of prices and taxes, if we set $M = M_t$, the LHS of (20) becomes

$$\sum_{s\in\{b,c\}} -\frac{\partial u}{\partial C_1}(e_a - \hat{p}, e_s + \hat{p})p_t\pi_s^k +$$

$$\sum_{s\in\{b,c\}} \frac{\partial u}{\partial C_2}(e_a - \hat{p}, e_s + \hat{p})\pi_s^k \frac{\sum_{s\in\{b,c\}} \frac{\partial u}{\partial C_1}(e_a - \hat{p}, e_s + \hat{p})p_t\pi_s^k}{\sum_{s\in\{b,c\}} \frac{\partial u}{\partial C_2}(e_a - \hat{p}, e_s + \hat{p})\pi_s^k} = 0, k = 1, 2$$

Thus $M = M_t$ solves the problem of the agents with the future prices as defined. Finally, we define M_{t+1} via the equation

$$p_{t+1}^k M_{t+1} = \hat{p} \quad (21)$$

so that in this equilibrium the real value of the money in circulaiton is constant. Combining (21) and (19) be then get the government's budget constraint:

$$p_{t+1}^k (M_{t+1} - M_t) + \tau_{t+1}^k = 0$$

In other words, the government is financing any subsidy to the current old by selling money to the current middle aged and is using any tax on the current old to buy money from the current old. With this combination of monetary and fiscal policy in place all uncertainty relative to the price is removed, so that the only uncertainty left is the fundamental one about the resources of the economy, when the agents are old. It should be noted though, that the implicit assumptions about the information available to the governments are quite strong. Most importantly, it is assumed that at any date, the government is able to find out what signal agents have received, i.e. the state of beliefs for the rational beliefs interpretation of the model.

Consider the case where u is separable. We then have $p^k_{t+1} = \frac{u'_1(e_a - \hat{p})}{\sum_s u'_2(e_s - \hat{p})\pi^k_s} p_t$. From (21) and (19) we have

$$\tau^k_{t+1} = p^k_{t+1} M_t - \hat{p} = \frac{p^k_{t+1} p_t}{p_t} M_t - \hat{p} = (\frac{p^k_{t+1}}{p_t} - 1)\hat{p}$$

Since $-u'_1(e_a - \hat{p}) + q_1 \sum_{s\in\{b,c\}} u'_2(e_s + \hat{p})\pi^1_s + q_2 \sum_{s\in\{b,c\}} u'_2(e_s + \hat{p})\pi^2_s = 0$, it follows that when the signal is negative, $k = 2$, $p^k_{t+1} < p_t$ i.e. the net interest rate is positive and the tax is negative while the money supply increases (when the signal is positive, $k = 1$, the opposite is the case). Thus to countervail the tendency to save more and consume less, when the signal is negative the government promises a lumpsum transfer and via an expansionary money policy decreases the interest rate.

Evolution of prices for the time separable case

$p_{t+1} = \frac{p_{t+1}}{p_t} \frac{p_t}{p_{t-1}} \dots \frac{p_2}{p_1} \cdot p_1$ which means that $\ln p_{t+1} = \sum_{s=1}^t \ln \frac{p_{s+1}}{p_s} + \ln p_1$. By the law of large numbers,

$$\lim_{t\to\infty} \frac{1}{t} \sum_{s=1}^{t} \ln \frac{p_{s+1}}{p_s} = \lim_{t\to\infty} \frac{1}{t} \sum_{s=1}^{t} \ln \left[\frac{u'_1(e - \hat{p})}{\sum_{s\in\{b,c\}} u'_2(e_s + \hat{p})\pi^{z_s}_s} \right] =$$

$$q_1 \ln \left[\frac{u'_1(e - \hat{p})}{\sum_{s\in\{b,c\}} u'_2(e_s + \hat{p})\pi^1_s} \right] + q_2 \ln \left[\frac{u'_1(e - \hat{p})}{\sum_{s\in\{b,c\}} u'_2(e_s + \hat{p})\pi^2_s} \right]$$

with probability 1. Using that $q_1 \ln[\sum_{s\in\{b,c\}} u'_2(e_s+\hat{p})\pi^1_s] + q_2 \ln[\sum_{s\in\{b,c\}} u'_2(e_s+\hat{p})\pi^2_s] < \ln[q_1 \sum_{s\in\{b,c\}} u'_2(e_s+\hat{p})\pi^1_s + q_2 \sum_{s\in\{b,c\}} u'_2(e_s+\hat{p})\pi^2_s] = \ln[\sum_{s\in\{b,c\}} u'_2(e_s+\hat{p})\overline{\pi}_s]$, this limit is $> \ln[u'_1(e-\hat{p})] - \ln[\sum_{s\in\{b,c\}} u'_2(e_s+\hat{p})\overline{\pi}_s] = 0$. In other words,

$$\lim_{t\to\infty} \frac{1}{t} p_{t+1} > 0, \quad \text{w. probability 1 so } p_{t+1} \to \infty \text{ w. probability 1}$$

So with probability one we have deflation and the money stock tends to zero.

REMARK 5

For the case where preferences are time separable the allocation may be implemented in a different way. Let today's signal $z_t = k$ and define next period's price conditional on next period's fundamental state, s:

$$p^{z_t}_{t+1,s} = \frac{\overline{\pi}_s}{\pi^k_s} p_t, s = b, c$$

Thus, with this implementation there is (conditional) price uncertainty.

Inserting this and $M = M_t$ in (20) we get, using (18) and (19):

$$p_t \sum_{s\in\{b,c\}} [-u'_1(e_a - \hat{p}) + u'_2(, e_s + \hat{p})\overline{\pi}_s] = 0$$

since this is the First Order Conditions, (16) for solving (11). In this case, since $p_t = \frac{\hat{p}}{M_t}$ and $p_{t+1}M_{t+1} = \hat{p}$ we get

$$M_{t+1} = \frac{\pi^{z_t}_{s_{t+1}}}{\overline{\pi}_{s_t+1}} M_t \text{ in other words, } M_T = \times_{t=1}^{T} \frac{\pi^{z_t}_{s_{t+1}}}{\overline{\pi}_{s_t+1}} M_1$$

or

$$\frac{1}{T} \ln M_T = \frac{1}{T} \sum_{t=1}^{T} \ln \left[\frac{\pi^{z_t}_{s_{t+1}}}{\overline{\pi}_{s_{t+1}}} \right] + \frac{1}{T} \ln M_1 \tag{22}$$

What happens to the money stock under the monetary policy considered here? Under the assumption that $\{z_t\}_t$ and $\{e_t\}_t$ are independent, with probability 1, the RHS of (22) tends to

$$\sum_{k=1}^{2} \sum_{s \in \{b,c\}} q_k \overline{\pi}_s \ln \left[\frac{\pi^k_s}{\overline{\pi}_s} \right] < 0$$

as $T \to \infty$. The inequality follows from the fact that by concavity of ln, $\sum_{k=1}^{2} q_k \ln \pi^k_s < \ln \sum_{k=1}^{2} q_k \pi^k_s = \ln \overline{\pi}_s$.

This in turn means that with probability 1, $M_t \to 0$ as $T \to \infty$. ■

3.3 Concluding Remarks

Both the policies of the European Central Bank and the American Federal Reserve Board have, wholly or partially, been directed towards stabilizing inflation, that is to keep inflation inside a target zone. To achieve this goal interest rates are being adjusted in reaction to developments in the real economy. Stabilizing prices (or price movements) will reduce the macroeconomic part of the price uncertainty that agents face and as long as this price uncertainty is not completely insurable may have a positive impact on economic welfare. This was what the present study sought to formalize. However, if we take into account that agents typically have diverse and in particular wrong expectations such stabilization has another beneficial effect (as was demonstrated, in another context in Nielsen, 1998) namely in reducing the mistakes agents make in forecasting prices. Such an effect may have been intended since the policy of the Federal Reserve Board to some extent has been concerned with preventing (probably unsuccessfully) build-ups of bubbles (and their consequentive bursts) especially in the financial markets. While these two beneficial effects of price stabilizations are not exclusive they may, for clarity, best be studied separately as was done here and elsewhere.

4 Appendices

4.1 Generalization of some Results

<u>The Generalized Model</u> For the generalized model we assume that there are $K \geq 2$ signals, with signal k having probability $_k > 0$, and thus K conditional distributions $\pi^k, k = 1, ..., K$, Furthermore, that there are $S \geq 2$ states, $e_1 < e_2 < ... < e_s < ...e_S$ for the endowments in the last period of the life of the agents, such that $\sum_{s=1}^{S} e_s \pi_s^j < \sum_{s=1}^{S} e_s \pi_s^{j'}$, whenever $j < j'$. Let $\bar{e} = max\{e, e_1, \ldots, e_S\}$. The assumptions from 2.2 about u are maintained.

A Monetary Equilibrium is then a vector $(p_1, ..., p_K)$ s.t. for each k the solution to the problem

$$Max_M \sum_{i=1}^{K} \sum_{s=1}^{S} u(e - p_k M, e_s + p_i M) q_i \pi_s^k \tag{23}$$

is $M = 1$. Note that an equilibrium price must be in $(0, \bar{e})^K$

The formulation and proof of the following proposition follow Peled(1982).

PROPOSITION 5 *Suppose that* $\frac{\frac{\partial u}{\partial C_1}}{\frac{\partial u}{\partial C_2}}(e, e_s) < 1, \forall s$. *Then there exists a monetary equilibrium for the economy.*

Proof: In step 1 of the proof existence of an equilibrium is shown and in step 2 it is shown that in this equilibrium money has value. Consider normalized prices for consumption and money, $p = (p^c, p^m) \in \Delta^{2K}$. For $p \in int\Delta^{2K}$, let $M_k(p)$ be the solution to

$$Max_{M \geq 0} \sum_{s=1}^{S} \sum_{i=1}^{K} u(e - M\frac{p_k^m}{p_k^c}, e_s + M\frac{p_i^m}{p_i^c}) q_i \pi_s^k.$$

Let $C_k(p) = e - M\frac{p_k^m}{p_k^c} + \frac{p_k^m}{p_k^c}$ the demand for consumption by young and old agents, $(C, M)(p) = [C_1(p), ..., C_K(p), M_1(p), ..., M_K(p)]$, and $E(p) = (E^c(p), E^m(p)) = (C, M)(p) - (e, e, .., e, 1, ..1)$ - the excess demand at price p. We have

$$E(p)p = \sum_{k=1}^{K} p_k^c \left[-M_k(p)\frac{p_k^m}{p_k^c} + \frac{p_k^m}{p_k^c} \right] + \sum_{k=1}^{K} p_k^m \left[M_k(p) - 1 \right] = 0.$$

Consider a sequence $\Delta^n \subset int\Delta^{2K}$ s.t. $\Delta^n \uparrow \Delta^{2K}$, where Δ^n is non-empty, compact and convex. From Debreu's theorem it follows (by continuity of E) that for each n there is a $(\hat{p}^n, \hat{E}^n)$ s.t.

(i) $\hat{p}^n \in \Delta^n$

(ii) $\hat{E}^n = E(\hat{p}^n)$

(iii) $\hat{E}^n p \leq 0, \forall p \in \Delta^n$

$\hat{E}^n$ is bounded from below by $(e, e, .., e, 1, .., 1)$. It follows from (iii) that $\hat{E}^n$ is also bounded from above. We conclude that $(\hat{p}^n, \hat{E}^n)$ has a convergent subsequence $(\hat{p}^{n_q}, \hat{E}^{n_q}) \to (\bar{p}, \bar{E}) \in \Delta^{2K} \times \Re^{2K}$.

We have $\bar{p}\bar{E} = 0$ and $\bar{E}p \leq 0, \forall p \in \Delta^{2K}$. By (iii) we have that $\bar{E} \leq 0$ (for if for some j, $\bar{E}_j > 0$, choose $p_j = 1$, $p_r = 0, r \neq j$).

Define for all n,

$$\tilde{p}^n = (\frac{\hat{p}_1^{m,n}}{\hat{p}_1^{c,n}}, \ldots, \frac{\hat{p}_K^{m,n}}{\hat{p}_K^{c,n}})$$

Since $E^{n_q} \to \overline{E} \leq 0$ and since if $\frac{\hat{p}_k^{m,n}}{\hat{p}_k^{c,n}} \to \infty$, because $M_k(p^{n_q})\frac{\hat{p}_k^{m,n}}{\hat{p}_k^{c,n}}$ is bounded by e, $E_k^{n_q} \to \infty$, we have that $\tilde{p}^{n_q}$ is bounded, hence has a convergent subsequence, converging to some $(\tilde{p}_1, \ldots, \tilde{p}_K)$. To simplify notation, we assume this subsequence to be the original sequence $\hat{p}^n$.

From the format of the young consumer's problem it follows that if $\lim_{\to n\infty} \frac{\hat{p}_k^{m,n}}{\hat{p}_k^{c,n}} = 0$ for some k' then $\lim_{\to n\infty} \frac{\hat{p}_k^{m,n}}{\hat{p}_k^{c,n}} = 0, \forall k$, in other words, $\tilde{p} = (0, 0, \ldots, 0)$ or $\tilde{p} => 0$. We now rule out the first possibility.

Rewrite for any n, the FOC (which says that (24) below is $= 0$)for the young consumer as

$$\sum_{s=1}^{S}\sum_{i=1}^{K} \frac{\partial u}{\partial C_1}\left(e - M_k^n\frac{\hat{p}_k^{m,n}}{\hat{p}_k^{c,n}}, e_s + M_k^n\frac{\hat{p}_i^{m,n}}{\hat{p}_i^{c,n}}\right)\frac{\hat{p}_k^{m,n}}{\hat{p}_k^{c,n}}\left[\frac{\frac{\partial u}{\partial C_2}\left(e - M_k^n\frac{\hat{p}_k^{m,n}}{\hat{p}_k^{c,n}}, e_s + M_k^n\frac{\hat{p}_i^{m,n}}{\hat{p}_i^{c,n}}\right)}{\frac{\partial u}{\partial C_1}\left(e - M_k^n\frac{\hat{p}_k^{m,n}}{\hat{p}_k^{c,n}}, e_s + M_k^n\frac{\hat{p}_i^{m,n}}{\hat{p}_i^{c,n}}\right)}\frac{\hat{p}_i^{m,n}}{\hat{p}_i^{c,n}}\frac{\hat{p}_k^{c,n}}{\hat{p}_k^{m,n}} - 1\right]q_i\pi_s^k = 0$$

Since M_k^n is bounded,

$$\frac{\frac{\partial u}{\partial C_2}\left(e - M_k^n\frac{\hat{p}_k^{m,n}}{\hat{p}_k^{c,n}}, e_s + M_k^n\frac{\hat{p}_i^{m,n}}{\hat{p}_i^{c,n}}\right)}{\frac{\partial u}{\partial C_1}\left(e - M_k^n\frac{\hat{p}_k^{m,n}}{\hat{p}_k^{c,n}}, e_s + M_k^n\frac{\hat{p}_i^{m,n}}{\hat{p}_i^{c,n}}\right)} \to \frac{\frac{\partial u}{\partial C_2}(e, e_s)}{\frac{\partial u}{\partial C_1}(e, e_s)} > 1 \text{ as } n \to \infty$$

There is also some k' s.t. for some subsequence n_q

$$\frac{\hat{p}_i^{m,n_q}}{\hat{p}_i^{c,n_q}} \geq \frac{\hat{p}_{k'}^{m,n_q}}{\hat{p}_{k'}^{c,n_q}}, \forall q, \forall i$$

If we consider what is inside the bracket in the reformulated FOC, be see that for this subsequence and for $k = k'$ is is $> 0, \forall i$, which means that the equality cannot hold for all n for this k' ■

bigskip

<u>Regularity of the Monetary Equilibrium</u> Let $\mathcal{U}$ be the set of C^2 utility functions defined on $\Re^2_{++}$ which are strictly increasing, strictly concave, and with indifference curves who's closure is contained in $\Re^2_{++}$. Let $\mathcal{U}^* \subset \mathcal{U}$ be the set for which a monetary equilibrium exists - it has non-empty interior and finally define the open set $\mathcal{U}^{**} \subset \mathcal{U}^*$ (the inclusion is Proposition 5),

$$\mathcal{U}^{**} = \left\{u \in \mathcal{U} : \frac{\frac{\partial u}{\partial C_1}(e, e_s)}{\frac{\partial u}{\partial C_2}(e, e_s)} < 1, \text{ for } s = 1, 2, \ldots, S\right\}$$

Define F on $\mathcal{U} \times \Re^K_{++}$ with values in $\Re^K$ by

$$F_k(u,p) = -\sum_{s=1}^{S}\sum_{i=1}^{K}\left[\frac{\partial u}{\partial C_1}(e-p_k, e_s+p_i)\,p_k - \frac{\partial u}{\partial C_2}(e-p_k, e_s+p_i)\,p_i\right] q_i \pi_s^k \tag{24}$$

Thus $p > 0$ is a Monetary Equilibrium price for the economy u if and only if $F(u,p) = 0$. F is continuous.

PROPOSITION 6 *For an open and dense set of utility functions in $\mathcal{U}^{**}$ we have that if $F(u,p) = 0$ for some p then there is an open ball, B_u containing u and an $\underline{p} \in \Re^K_{++}$ s.t. for all $u' \in B_u$ we have that if $F(u',p) = 0$ then $p \geq \underline{p}$.*

Proof: The condition for inclusion in $\mathcal{U}^{**}$ implies that there is for any $u \in \mathcal{U}^{**}$, $\underline{p} > 0$ s.t. $\frac{\frac{\partial u}{\partial C_1}(e-\underline{p}, e_s+\underline{p})}{\frac{\partial u}{\partial C_2}(e, e_s)} < 1$, for $s = 1, 2, \ldots, S$ and as can be seen from the proof of Proposition 5 this means . there is $\underline{p} \in \Re^K_{++}$ s.t. $F(u,p) = 0 \Rightarrow p > \underline{p}$. We can then not have a sequence $u^n \to u$ s.t. for each n there is $p^n \leq \underline{p}$ with $F(u^n, p^n) = 0$ ■

Regularity for an economy u means that $F(u,p) = 0, p > 0 \Rightarrow \partial_p F(u,p)$ has full rank i.e. rank K. The set of regular u contains an open set. For suppose that u is regular and in $\mathcal{U}^{**}$. If there were a sequence $u^n \to u$ where u^n were not regular there would be a sequence p^n s.t. $F(u^n, p^n) = 0$ and $p^n > 0$ but $|\partial_p F(u^n, p^n)| = 0, \forall n$ (where $|\cdot|$ means determinant). Since p^n is bounded (u^n, p^n) has a cluster point $(u, \bar{p})$, implying by the continuity of F and $|\partial_p F|$ that $F(u, \bar{p}) = 0$ (so that by Proposition 6) $\bar{p} > 0$) and $|\partial_p F(u, \bar{p}| = 0$, a contradiction.

PROPOSITION 7 *Suppose that the rank of the matrix*

$$\Pi = \begin{pmatrix} \pi_1^1 & \pi_2^1 & \cdot & \cdot & \cdot & \pi_S^1 \\ \pi_1^2 & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ \pi_1^K & \cdot & \cdot & & & \pi_S^K \end{pmatrix}$$

is K. Then the set of regular u is dense in $\mathcal{U}^{**}$.

Proof: Pick any $u \in \mathcal{U}^{**}$. Then show that there is a sequence $\{u^n\} \to u$ s.t. $\forall n :$ "$F(u^n, p) = 0, p > 0 \Rightarrow |\partial_p F(u^n, p)| \neq 0$". First we find an open ball, B, around u and contained in $\mathcal{U}^{**}$ s.t. (using Proposition 6) there is a lower bound $\underline{p}$ such that any equilibrium price p for $u' \in B$ is $\geq (\underline{p}, \ldots, \underline{p})$. We consider the following form of parametrization:

$$u_{(\epsilon, r_1, \ldots, r_L)} = u + \epsilon \sum_{j=1}^{L} u_j$$

for $(\epsilon, r_1, \ldots, r_L) \in (0, \bar{\epsilon}) \times \prod_{j=1}^{L} R_j$ where the R_js are open intervals in $\Re_{++}$ and $u_j(C_1, C_2) = -e^{-r_j C_2}$. In the proof we choose L and $R_1, \ldots, R_L$ s.t. if we let $\hat{F}(\epsilon, r_1, \ldots, r_L, p) = F(u_{(\epsilon, r_1, \ldots, r_L)}, p)$ then $\partial \hat{F}(\epsilon, r_1, \ldots, r_L, p)$ has rank K for all $(\epsilon, r_1, \ldots, r_L, p) \in (0, \bar{\epsilon}) \times \prod_{j=1}^{L} R_j \times [\underline{p}, \bar{e})^K$. It follows from the transversality theorem that the set of $(\epsilon, r_1, \ldots, r_L)$ for which $F(u_{(\epsilon, r_1, \ldots, r_L)}, p) = 0 \Rightarrow |\partial_p F(u_{(\epsilon, r_1, \ldots, r_L)}| \neq 0$ has full Lebesgue measure in $(0, \bar{\epsilon}) \times \prod_{j=1}^{L} R_j$ which implies the result.

We have

$$\hat{F}(\epsilon, r, p)_k = F(u, p) - \epsilon \sum_{j=1}^{L} r_j \sum_i \sum_s e^{-r_j [e_s + p_i]} p_i q_i \pi_s^k = F(u, p) - \epsilon \sum_{j=1}^{L} r_j \left[\sum_i e^{-r_j p_i} p_i q_i \right] \left[\sum_s e^{-r_j e_s} \pi_s^k \right]$$

Then

$$\partial \hat{F}(\epsilon, r, p) = \begin{pmatrix} \partial_\epsilon \hat{F}(\epsilon, r, p) & -\epsilon A(r, p) & \partial_p \hat{F}(\epsilon, r, p) \end{pmatrix}$$

where $A(r, p)$ is an K by L matrix with elements

$$A_{kj}(r, p) = a_k(r_j, p) = \left[\sum_i e^{-r_j p_i} p_i q_i \right] \left[\sum_s e^{-r_j e_s} \pi_s^k \right] -$$
$$r_j \left\{ \left[\sum_i e^{-r_j p_i} p_i^2 q_i \right] \left[\sum_s e^{-r_j e_s} \pi_s^k \right] + \left[\sum_i e^{-r_j p_i} p_i q_i \right] \left[\sum_s e^{-r_j e_s} e_s \pi_s^k \right] \right\}$$

We now let $p \in [\underline{p}, \bar{e}]^K$ be given and show in Lemmatas 1 and 2 below that there are $r_1(p), \ldots, r_K(p)$ such that the K by K matrix $A(r_1, \ldots, r_L, p)$ with elements $a_{kj}(r_j(p), p)$ has full rank. This implies that there are open intervals, $R_j(p)$ containing $r_j(p)$ and an open ball, $B(p)$ containing p such that for any $(p, r_1, \ldots, r_K) \in B(p) \times R_1(p) \times \cdots \times R_K(p)$ we have full rank of $A(r_1, \ldots, r_L, p)$. Now $\{B(p) : p \in [\underline{p}, \bar{e}]^K\}$ forms an open covering of $[\underline{p}, \bar{e}]^K$ and consequently there is a finite subcover $\{B(p_h)\}_{h=1}^{H}$. We then let $A(p, r_1, \ldots, r_L)$ be the K by $KH = L$ matrix defined on $[\underline{p}, \bar{e}]^K \times \prod_{h=1}^{H} \prod_{k=1}^{K} R_k(p_h)$ and we see that $A(p, r_1, \ldots, r_L)$ has full rank. This completes the proof ■

Lemma. 1 *There are, for any $p \in [\underline{p}, \bar{e}]^K$, $r_1(p), \ldots, r_K(p)$ such that the K by K matrix $A(r_1, \ldots, r_L, p)$ with elements $a_{kj}(r_j(p), p)$ has full rank*

Proof: We start with some $r_K > 0$ and get the K'th row of the matrix:

$$A_K(r_K, p) = \begin{pmatrix} a_1(r_K, p) \\ \cdot \\ \cdot \\ \cdot \\ a_K(r_K, p) \end{pmatrix}$$

We then find $0 < r_{K-1}$ s.t. $A_{K-1}(r_{K-1,p})$ is not parallel to $A_K(r_K, p)$. In the n'th step we find $0 < r_{K-n}$ s.t. $A_{K-n}(r_{K-n}, p)$ is not in the subspace spanned by $[A_K(r_K, p), \cdots, A_{K-n+1}(r_{K-n+1}, p)]$. For this procedure it is sufficient to show the following: for all sets of K-vectors $[V_1, \ldots, V_{K-1}]$ there is $0 < r$ s.t. $A_K(r, p) \notin span[V_1, \ldots, V_{K-1}]$. Tt is enough to show that for all $N \in \Re^K$ there is $0 < r$ s.t. $A_K(r, p) \cdot N \neq 0$ because this inequality will then hold for any normal N to $span[V_1, \ldots, V_{K-1}]$.

If $A_k(0, p) \cdot N \neq 0$ then, by continuity, this also holds for some $0 < r$ and we are done. Consider then the other case where $A_k(0, p) \cdot N = 0$. If for all $r > 0$, $A_k(r, p) \cdot N = 0$ then, in particular, $\partial^l A_k(0, p) \cdot N = 0$ for $l = 1, 2, ...$ (where $\partial^l A_k(0)$ is the l'th derivative of $(a_1(r, p), a_2(r, p), \ldots, a_K(r, p))^T$ evaluated in $(0, p)$). So the result is proven by showing that the matrix

$$\left[\partial^{l_1} A_k(0) \middle| \, \partial^{l_2} A_k(0) \middle| \cdots \middle| \partial^{l_K} A_k(0)\right]$$

- a K by K matrix has full rank for some $l_1, \ldots, l_K$. The derivatives in this matrix are K-dmensional column vectors with elements which can be shown to be of the form $(\sum_{i=1}^{l_k} a_i \sum_s e_s^i \pi_s^k$. Because of this form of the derivatives and the lemma below, we need only prove the following: there are $l_1, \ldots, l_K$ such that

$$\begin{pmatrix} \sum_s e_s^{l_1} \pi_s^1 & \cdot & \cdot & \cdot \sum_s e_s^{l_K} \pi_s^1 \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \sum_s e_s^{l_1} \pi_s^K & \cdot & \cdot & \cdot \sum_s e_s^{l_K} \pi_s^K \end{pmatrix}$$

has full rank. Suppose that

$$\begin{pmatrix} \sum_s e_s^{l_1} \pi_s^1 & \cdot & \cdot & \cdot \sum_s e_s^{l_k} \pi_s^1 \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \sum_s e_s^{l_1} \pi_s^K & \cdot & \cdot & \cdot \sum_s e_s^{l_k} \pi_s^K \end{pmatrix}$$

has maximal rank and let $N \in \Re^K$ be a normal to the space spanned by the k columns in this matrix. By the assumption that Π has rank K we have $N \cdot \Pi \neq 0$. Consider

$$N \cdot \begin{bmatrix} \sum_s e_s^l \pi_s^1 \\ \cdot \\ \cdot \\ \cdot \ \sum_s e_s^l \pi_s^K \end{bmatrix} = \sum_{k=1}^K N_k \sum_s e_s^l \pi_s^k = \sum_s e_s^l \sum_k N_k \pi_s^k.$$

As noted, we have $\sum_k N_k \pi_s^k \neq 0$. To show that there is l s.t. $\sum_{k=1}^K N_k \sum_s e_s^l \pi_s^k \neq 0$ we only need to show that for $a \in \Re^S \setminus \{0\}$ there is l such that $\sum_s e_s^l a_s \neq 0$. Recalling that $e_s > e_{s'}$ when $s < s'$, let $\bar{s} = max\{s : a_s \neq 0\}$.

$$\sum_s e_s^l a_s = \sum_{s=1}^{\bar{s}} e_s^l a_s = e_{\bar{s}}^l \left[\sum_{s=1}^{\bar{s}-1} \left(\frac{e_s}{e_{\bar{s}}} \right)^l a_s + a_{\bar{s}} \right]$$

and as $l \to \infty$ $\sum_{s=1}^{\bar{s}-1} \left(\frac{e_s}{e_{\bar{s}}}\right)^l a_s \to 0$ and $e_{\bar{s}}^l a_{\bar{s}} \to sign(a_{\bar{s}})\infty$, giving us the result ■

LEMMA. 2 *Let $X_i \in \Re^K, i = 1, 2, \dots$ Let $L_1 = 1$ and for $j = 2, 3, .., K$ define L_j inductively s.t. L_j is the smallest number s.t. $[X_{L_1}, X_{L_2}, \dots, X_{L_j}]$ has rank j assuming that such L_j exists. Let for $i = 1, 2, \dots, K$, $V_{L_i} = a_{i1}X_{L_i} + a_{i2}X_{L_2} + \dots + a_{ii}X_{L_i}$, with $a_{ij} \neq 0, \forall j$. Then $[V_{L_1}, V_{L_2}, \dots, V_{L_K}]$ has maximal rank (i.e. rank K).*

Proof: For every j V_{L_j} is linearly independent of $[V_{L_1}, \dots, V_{L_{j-1}}]$. Suppose not. There would be $\alpha_1, \dots, \alpha_j$ s.t. $\sum_{i=1}^{j} \alpha_i V_{L_i} = 0$. But then

$$\sum_{h=1}^{j-1} \sum_{i=1}^{j} \alpha_i a_{i,h} X_{L_h} = \alpha_j a_{jj} X_{L_j}$$

in contradiction with that X_{L_j} is linearly independent of $[X_1, \dots, X_{L_j-1}]$ ■

Genericity of Price Volatility

Let $A = \{p \in \Re_{++}^K : p_1 = p_2 = \dots = p_K\}$ and $B = \{u \in \mathcal{U}^{**} : F(u,p) = 0 \Rightarrow p \notin A\}$. We show that B is an open set. So let $u \in B$, $\underline{p} \in \Re_{++}$ s.t. $f(u,p) = 0 \to p_k \gg \underline{p}, \forall k$ (using Proposition 6) and suppose $u^n \to u$. Suppose there were for each n a $p^n \in A \cap [\underline{P}, \bar{e}]^K$ s.t. $F(u^n, p^n) = 0$. Since $A \cap [\underline{p}, \bar{e}]^K$ is compact p^n has a cluster point, $\bar{p} \in A$, and, since F is continuous, $F(u, \bar{p}) = 0$, a contradiction ■

REMARK 6

If we assume that the $K \times K$ matrix

$$\begin{pmatrix} \sum_s e_s \pi_s^1 & \sum_s e_s^2 \pi_s^1 & \cdot & \cdot & \cdot & \sum_s e_s \pi_s^K \\ \sum_s e_s \pi_s^2 & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ \sum_s e_s \pi_s^K & & \cdot & \cdot & & \sum_s e_s^K \pi_s^K \end{pmatrix}$$

has maximal rank (a generic condition), Proposition 7 holds without the rank condition on Π ■

Denseness of price volatility

PROPOSITION 8 *Suppose the set $\{u \in \mathcal{U}^{**} : u$ is regular$\}$ is dense. Then B is dense.*

Proof: It is sufficient to show that for every $u \in \{u \in \mathcal{U}^{**} : u$ is regular$\}$ there is a sequence $u^n \to u$ s.t. $u^n \in B, \forall n$. Because of regularity, such a u has finitely many equilibria, say $p^1, ..., p^H$. Consider the following parametrization:

$$u_\epsilon = u - \epsilon \left[A_1 e^{-r_1 C_2} + A_2 e^{-r_2 C_2}\right]$$

wherek_1, k_2, r_1, and r_2 are to be chosen. The k'th coordinate of the parametrized F now is

$$F(\epsilon,p)_k = F(u,p)_k + \epsilon \left\{ k_1 r_1 \sum_i \sum_s e^{-r_1(e_s+p_i)} p_i q_i \pi_s^k + k_2 r_2 \sum_i \sum_s e^{-r_2(e_s+p_i)} p_i q_i \pi_s^k \right\}$$

By the implicit function theorem locally the equilibrium price $p^h, h = 1, 2, \ldots, H$ is a function of ϵ with derivative

$$-\left[\partial_p F(0,p^h)\right]^{-1} \partial_\epsilon F(0,p^h).$$

If this derivative is $\neq$

$$\begin{pmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ 1 \end{pmatrix} \equiv \mathbf{1}$$

then locally, $p^h(\epsilon) \notin A$ (for the case where $p^h \notin A$ this follows from continuity of $F(\cdot,\cdot)$, else it follows from the condition on the derivative).

$$\partial_\epsilon F(0,p^h) = \left[A_1 r_1 \sum_i \sum_s e^{-r_1(e_s+p_i^h)} p_i^h q_i \pi_s^k + A_2 r_2 \sum_i e^{-r_2 p_i^h} p_i^h q_i \sum_s e^{-r_2 e_s} p_i^h q_i \pi_s^k \right]_{k=1}^K =$$

$$\left[A_1 r_1 \sum_i e^{-r_1 p_i^h} p_i^h q_i \sum_s e^{-r_1 e_s} \pi_s^k + A_2 r_2 \sum_i e^{-r_2 p_i^h} p_i^h q_i \sum_s e^{-r_2 e_s} \pi_s^k \right]_{k=1}^K$$

We can find r_1 and r_2 s.t. $\{\sum_s e^{-r_1 e_s} \pi_s^k\}_{k=1}^K$ and $\{\sum_s e^{-r_2 e_s} \pi_s^k\}_{k=1}^K$ are linearly independent. To se why, note that the derivative w.r.t. r is at $r = 0$

$$\begin{pmatrix} \sum_s e_s \pi_s^1 \\ \sum_s e_s \pi_s^2 \\ \cdot \\ \cdot \\ \sum_s e_s \pi_s^K \end{pmatrix} \neq \mathbf{1}$$

Note that we have $\partial_p F(u_\epsilon, p^h)_{|\epsilon=0} = \partial_p F(u, p^h)$ the last having full rank. So there is a unique X^h s.t

$$-[\partial_p F(u,p^h)]^{-1} X^h = \mathbf{1}$$

We can then pick A_1 and A_2 such that

$$\left\{ A_1 r_1 \sum_i e^{-r_1 p_i^h} p_i^h q_i \sum_s e^{-r_1 e_s} \pi_s^k + A_2 r_2 \sum_i e^{-r_2 p_i^h} p_i^h q_i \sum_s e^{-r_2 e_s} \pi_s^k \right\}_{k=1}^K \neq \lambda X^h, \forall h = 1, 2, ..H, \forall \lambda \in \Re$$

In other words we have $\frac{\partial p^h}{\partial \epsilon}_{|\epsilon=0} \neq \lambda\, \mathbf{1}, \forall h$, so that for ϵ close to 0 we have the equilibrium prices for u_ϵ, $p_\epsilon^h \neq \lambda\, \mathbf{1}, \forall h, \forall \lambda \in \Re$ ■

Pareto Optimality

We first confirm that a Monetary Equilibrium is conditionally Pareto Optimal in the two-period version of the more general model. For this model an allocation, characterized by the transfers/prices $p_1, \ldots, p_K$ is Pareto Optimal if there are $W_1, \ldots, W_K > 0$ s.t.

$$W_k \sum_s \sum_i -\frac{\partial u}{\partial C_1}(e - p_k, e_s + p_i) q_i \pi_s^k + \sum_j W_j \sum_s \frac{\partial u}{\partial C_2}(e - p_j, e_s + p_k) q_k \pi_s^j = 0, k = 1, 2, \ldots, K \quad (25)$$

A proof of this claim is not provided, but would follow the logic of the proof of Theorem 3 in Peled(1982). Let

$$d_k(p) = \sum_s \sum_i \frac{\partial u}{\partial C_1}(e - p_k, e_s + p_i) q_i \pi_s^k$$

and

$$c_{kj}(p) = \sum_s \frac{\partial u}{\partial C_2}(e - p_k, e_s + p_j) q_j \pi_s^k$$

Then the equilibrium condition can be written $H(u, p) \cdot p = 0$ where

$$H(u,p) = \begin{pmatrix} -d_1 + c_{11} & c_{12} & c_{13} & \cdot & \cdot & \cdot & c_{1K} \\ c_{21} & -d_2 + c_{22} & c_{23} & \cdot & \cdot & \cdot & c_{2K} \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & \cdot \\ c_{K1} & \cdot & \cdot & \cdot & \cdot & \cdot & -d_K + c_{KK} \end{pmatrix}$$

while (25) can be written $(W_1, \ldots, W_K) \cdot H(p) = 0$. The argument is now the same as in Peled(1984). His Theorem 1 states that if H is an K by K matrix with positive off-diagonal elements and if there is $p \gg 0$ s.t. $H \cdot p = 0$ then there is $W \gg 0$ s.t. $W \cdot H(p) = 0$ ■

We next show that generically the Monetary Equilibrium in the model studied (the 3-period version) is not Pareto Optimal. More precisely we show the following. Let $Q = (q_1, q_2, \ldots, q_K)$

PROPOSITION 9 *For generic $u \in \mathcal{U}^{**}$, whenever $H(u,p) \cdot p = 0$, $Q \cdot H(u,p) \neq 0$*

Proof:

Let $A = \{u \in \mathcal{U}^{**} : F(u,p) = 0 \Rightarrow Q \cdot H(u,p) \neq 0\}$. Suppose A were not open, i.e. there were a u in A and a sequence $u^n \to u$ and p^n s.t. $F(u^n, p^n) = 0 = QH(u^n, p^n)$. As before (using Proposition 6) p^n has a cluster point $\bar{p} \gg 0$ giving the contradicting consequence, $f(u, \bar{p}) = 0 = QH(u, \bar{p})$ Before proving denseness we prove the following lemma:

LEMMA. 3 *Suppose that p is a particular Monetary Equilibrium for the regular economy u s.t.*

(i) $p_i \neq p_j$ for some i, j

(ii) $Q \cdot H(u,p) = 0$

There is then for every open ball, B around u an $u' \in B$ which is regular s.t. p is still an equilibrium price for u' but $Q \cdot H(u', p) \neq 0$.

Proof: Let $\underline{p} = min\{p_1, \ldots, p_K\} < \bar{p} = max\{p_1, \ldots, p_K\}$. Pick a C^2 function g defined on $\Re^2_{++}$ and with all first and second derivatives bounded in absolute value s.t.

(a) $\frac{\partial g}{\partial C_1}(e - \underline{p}, e_S + \bar{p})\underline{p} = -\frac{\partial g}{\partial C_2}(e - \underline{p}, e_S + \bar{p})\bar{p} \neq 0$

(b) $\frac{\partial g}{\partial C_1}(e - p_k, e_S + p_j) = \frac{\partial g}{\partial C_2}(e - p_k, e_S + p_j) = 0$ for all k, j, s s.t. $p_k \neq \underline{p}$ or $p_j \neq \bar{p}\}$ or $sneqS$. Note that from (a) it follows that

$$\sum_{\{j:p_j=\bar{p}\}} \frac{\partial g}{\partial C_1}(e - \underline{p}, e_S + \bar{p})\underline{p} q_j \pi^k_S = - \sum_{\{j:p_j=\bar{p}\}} \frac{\partial g}{\partial C_2}(e - \underline{p}, e_S + \bar{p})\bar{p} q_j \pi^k_S, \forall k \text{ s.t. } p_k = \underline{p}$$

Let $u_\epsilon = u + \epsilon g$. There is $\delta > 0$ s.t. for $\epsilon \in [-\delta, \delta]$, u_ϵ fulfils the maintained assumptions about preferences. Also, for all ϵ, $H(u_\epsilon, p) \cdot p = 0$ i.e. p is still an equilibrium for the perturbed economy: $H(u_\epsilon, p) = H(u, p) + \tilde{H}$ where every row $\tilde{H} - k, \cdot = 0$ if $p_k \neq \underline{p}$. If $p_k = \underline{p}$ then $\tilde{H}_{kk} = \epsilon \sum_{\{j:p_j=\bar{p}\}} \frac{\partial g}{\partial C_1}(e - p_k, e_S + \bar{p}) q_j \pi^k_S$ and $\tilde{H}_{kj} = \frac{\partial g}{\partial C_2}(e - p_k, e_S + \bar{p}) q_j \pi^k_S$ for all j s.t. $p_j = \bar{p}$, else $\tilde{H}_{kj} = 0$. It follows that $\tilde{H}_{k,\cdot} \cdot p = 0, \forall k$ i.e. that $H(u_\epsilon, p) \cdot p = 0$. Let j be such that $p_j = \bar{p}$. The column $H(u_\epsilon, p)_{\cdot,j}$ has the form $H(u, p)_{\cdot,j} + \tilde{H}_{\cdot,j}$, where $\tilde{H}_{kj} = \frac{\partial g}{\partial C_2}(e - \underline{p}, e_S + \bar{p}) q_j \pi^k_S$ whenever $p_k = \underline{p}$, 0 else. It follows that $Q \cdot \tilde{H}_{\cdot,j} \neq 0$, while $Q \cdot H(u, p)_{\cdot,j} = 0$ so that $Q \cdot H(u_\epsilon, p)_{\cdot,j} \neq 0$ ■

To finish the proof of the Proposition, we show that for all u and all open balls, B containing u, $B \cap A \neq \emptyset$. Since the set of u which are regular and have the feature that for all equilibrium prices p, $p_j \neq p_i$ for some i, j is open and dense, we can assume that u has these features. By regularity there are then only finitely many Monetary Equilibria for u, say $p^1, \ldots, p^N$. Let B be an open ball around u so small that the equilibrium prices can be parametrized by N continuous functions, $g_n : B \to (0, \bar{e}]^K, n = 1, \ldots, N$.

Start with p^1. If $Q \cdot H(u, p^1) = 0$ pick according to the lemma u^1 in B s.t $H(u^1, p^1) \cdot p^1 = 0$ but $Q \cdot H(u^1, p^1) \neq 0$. Else let $u^1 = u$. In the j'th step we have u^{j-1} with equilibrium prices (ea ch in $\Re^K_{++}$) $(p^{j-1,1}, \ldots, p^{j-1,N})$ s.t. $Q \cdot H(u^{j-1}, p^{j-1,i}) \neq 0$ for $i \leq j - 1$. These inequalities continue to hold in an open neighborhood, $B^{j-1} \subset B$ of u^{j-1}. Consider $p^{j-1,j}$. If $Q \cdot H(u^{j-1}, p^{j-1,j}) = 0$ pick according to the lemma a u^j in B^{j-1} s.t. $H(u^j, p^{j-1,j}) \cdot p^{j-1,j} = 0$ but $Q \cdot H(u^j, p^{j-1,j}) \neq 0$. Else let $u^j = u^{j-1}$. u^N then has the desired properties ■

4.2 Brief Introduction to Rational Beliefs

The generic set of variables is $\Re^L$. We consider $\times_{t=1}^\infty \Re^L$ and it Borel sets, $\mathcal{B}^\infty$. and on the shift transformation $T : \times_{t=1}^\infty \Re^L \to \times_{t=1}^\infty \Re^L$, i.e. $T(H_1, H_2, ...) = T(H_2, H_3 ...)$. Let μ be a probability measure on $(\times_{t=1}^\infty \Re^L, \mathcal{B}^\infty)$ so that $(\times_{t=1}^\infty \Re^L, \mathcal{B}^\infty, T, \mu)$ is a dynamical system. Finally, let $\mathcal{C}(\times_{t=1}^\infty \Re^L)$ be the cylinders. The following definitions are taken from Kurz[1994] :

Definition 2 *Stability: The dynamical system* $(\mathcal{H}^\infty, \mathcal{B}(\mathcal{H}^\infty), \mu, T)$ *as well as the measure* μ *are said to be stable if for all cylinders* $C \in \mathcal{C}(\times_{t=1}^{\infty}\Re^L)$*:*

$$lim_{J\to\infty}\frac{1}{J}\sum_{j=0}^{J-1} 1_C(T^j(x))$$

exists for μ*-a.a.* $x \in \times_{t=1}^{\infty}\Re^L$

For the case we are studying when the system is stable there is an associated stationary measure, $\bar{\mu}$ s.t. $\bar{\mu}(C)$ is the limit of the sequence in the above definition. This $\bar{\mu}$ is the empirical distribution of the stochastic process and is assumed to be known by all agents.

To know that the true but unknown dynamical system $(\times_{t=1}^{\infty}\Re^L, \mathcal{B}^\infty, T, \mu)$ generated $\bar{\mu}$ is not the same as knowing μ . There are many possible stable dynamical systems which will generate the same stationary measure.

Definition 3 *A probability measure* ρ *on* $(\times_{t=1}^{\infty}\Re^L, \mathcal{B}^\infty)$ *is said to be a Weakly Rational Belief for the stable dynamical system* $(\times_{t=1}^{\infty}\Re^L, \mathcal{B}^\infty, T, \mu)$ *if* $\bar{\rho} = \bar{\mu}$ *i.e. if for all cylinders* $C \in \mathcal{C}(\times_{t=1}^{\infty}\Re^L)$*:*

$$lim_{J\to\infty}\frac{1}{J}\sum_{j=0}^{J-1} 1_C(T^j(x)) = \bar{\mu}(C)$$

for ρ*-a.a.* $x \in \times_{t=1}^{\infty}\Re^L$

Thus a belief ρ is rational, if it generates the same empirical distribution as the one being observed. In this paper we use rational beliefs which are generated by a random signal/sunspot, z and two one-period beliefs, $B_i, i = 1, 2$. We assume that the empirical distribution is i.i.d. with one-period distribution $\bar{B}$. Thus we can phrase the rationality conditions in terms of one-period beliefs only as in (3) (see Nielsen(1996) for details).

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