

Learning to Forecast the Exchange Rate: Two Competing Approaches

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Abstract

In this paper we compare two competing approaches to model foreign exchange market participants behavior: statistical learning and fitness learning, applied to a set of predictors, which include chartists and fundamentalists. We examine which of these approaches is the best in terms of replicating the exchange rate dynamics within the framework of a standard asset pricing model. First, we find that both learning methods reveal the fundamental value of the exchange rate in the equilibrium. Second, we find that only fitness learning creates the disconnection phenomenon. None of the mechanisms is able to produce unit root process but both of them generate persistence in the volatility of exchange rate returns. These results suggest that fitness learning comes closer to replicate the foreign exchange market participants' behavior.

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1 Introduction

Exchange rate economics has been dominated by the rational expectations efficient market theory (REEM). This theory however has not been empirically validated. First, survey evidence indicates that traders' expectations strongly deviate from rational expectations (Frankel and Froot 1987a and 1987b; Ito 1990; Sarno and Taylor 2003 and Jongen et al. 2008). Second, technical trading rules appear to make risk-adjusted excess returns, violating the efficient markets hypothesis (Sweeney 1986; Levich and Lee 1993; Pilbeam 1995; Neely et al. 1997; LeBaron 1999). As this empirical evidence against REEM theory has tended to accumulate, researchers have increasingly looked for alternative modelling approaches. One of these approaches challenges the assumptions about the way the agents form their expectations.

First, in line with strong survey evidence, a number of researchers have modeled the agents in the foreign exchange market as chartists and fundamentalists. Frankel and Froot (1987a, 1987b, 1990a, and 1990b) were the first to emphasize the impact of the trading strategies on the dynamics of the exchange rate. They argued that swings in the US dollar are due to the shifts in weights that markets give to different trading techniques. Subsequently, many studies demonstrated that the introduction of heterogenous investors into the exchange rate models can generate features observed in the data (See Goodhart 1988; Frenkel 1997; De Grauwe and Grimaldi 2006a and 2006b).

Second, several researchers, instead of assuming full rationality introduced some sort of adapting mechanisms into agents' behavior. Arifovic (1996) develops a two countries' overlapping generations (OLG) model where agents update their decisions using a selection mechanism based on a genetic algorithm. She finds that, in this model, the stationary rational expectations equilibria are unstable and result in persistent fluctuations of the exchange rate. Elaborating on the paper by Arifovic (1996), Lux and Schornstein (2005) find that the OLG model under genetic learning can generate fat tails and volatility clustering in the exchange rate series. Mark (2005) finds evidence that adaptive learning about Taylor-rule fundamentals sheds some light on the real US dollar-DM exchange rate dynamics. Chakraborty and Evans (2008) propose a resolution of the Forward Premium Puzzle assuming that agents use perpetual learning. Kim

(2009) and Lewis and Markiewicz (2009) demonstrate that learning about the monetary model can generate excess volatility of the exchange rate.

All these studies demonstrate that a departure from the Rational Expectations (RE) assumption can help in replicating the data features. It is not clear however what type of departure from RE is the best in explaining the dynamics of the foreign exchange market. Thus, the modeling choice of expectations formation in the foreign exchange markets remains an open question.

In this paper, we analyze this question. As in the related literature, we depart from RE and assume that heterogeneity prevails in the foreign exchange market. Deviating from RE creates the risk of introducing ad hoc assumptions, the number of which can be multiplied ad infinitum. We avoid this risk by imposing selection mechanisms that ensure that only the best performing forecasting rules survive. Thus, as in the spirit of RE-models, we impose a modeling discipline on agents, in that these continuously test and revise their expectation formation. Furthermore, we compare the capacities of two different selection (learning) mechanisms in replicating features of the exchange rate series.

We assume that agents can use two different forecasting rules and combine them to form their expectations about the future exchange rate. The first one will be called a fundamentalist forecasting rule, the second one a chartist rule (technical analysis). Further, we specify two alternative selection procedures (learning mechanisms). The first one is the dynamic predictor selection in the spirit of Brock and Hommes (1997 and 1998) that will be called fitness learning. This mechanism assumes that agents evaluate forecasts by computing their past profitability. Accordingly, they increase (reduce) the weight of one rule if it is more (less) profitable than the alternative rule. In the second mechanism, agents learn to improve their forecasting rules using statistical methods as in the literature of adaptive learning in macroeconomics (see Evans and Honkapohja 2001 for an overview). We investigate the behavior of the exchange rate within the framework of a standard asset pricing model.

The remainder of the chapter is organized as follows. In section two, we develop the baseline model of the exchange rate and we specify the way agents form their expectations about the future exchange rate. Section three introduces the learning mechanisms of the agents. In section four, we study the

equilibrium properties of the models. Section five presents a numerical analysis of the dynamics of the exchange rate and confronts the statistical properties of the exchange rate under the two learning rules with the data and section six provides some concluding remarks.

2 Exchange rate model and expectations formation

2.1 Asset pricing model of the exchange rate

We model the market exchange rate using an asset pricing view of the exchange rate. This allows us to write the exchange rate as:

$$s_t = s_t^* + b \left(\hat{E}_t s_{t+1} - s_t \right) \quad (1)$$

where s_t is the log level of the exchange rate in period t , defined as the domestic price of a unit of foreign currency, s_t^* defines the set of fundamentals and \hat{E}_t denotes expectations (not necessarily rational) formed at time t . Equation (1) expresses the market exchange rate as the sum of the current fundamentals and the expected change of the market rate. Model (1) can be viewed as the reduced form of the monetary model linking the exchange rate to money supplies and incomes. It can also correspond to the models of stock valuation, where s_t^* plays the role of dividends and b is a weight applied to expected future capital gains. We reformulate this equation and assume that there are unexpected disturbances in the market process captured by $\eta_t \sim iid(0, \sigma_\eta^2)$:

$$s_t = (1 - \alpha) s_t^* + \alpha \hat{E}_t s_{t+1} + \eta_t \quad (2)$$

where $\alpha = \frac{b}{1+b}$, and $1 - \alpha = \frac{1}{1+b}$. Thus, the market exchange rate is a convex combination of the fundamental rate and the expectations of the future market rate with $0 < \alpha < 1$ being a discount factor. We also assume that the log fundamental s_t^* is driven by a random walk, i.e.

$$s_t^* = s_{t-1}^* + \epsilon_t \quad (3)$$

where $\epsilon_t \sim iid(0, \sigma_\epsilon^2)$.

Solving model (2) assuming rational expectations of agents $\hat{E}_t = E_t$ yields:

$$s_t = (1 - \alpha) \sum_{n=0}^{\infty} \alpha^n E_t s_{t+n}^*$$

Note that for stationarity of the above solution, we need $\alpha < 1$. Using the definition of the fundamental process in equation (3), and assuming the absence of rational bubbles when $n \rightarrow \infty$, namely that $\lim_{n \rightarrow \infty} \alpha^n E_t s_{t+n} = 0$, we find:

$$\begin{aligned} s_t &= (1 - \alpha) s_t^* + \alpha \left((1 - \alpha) \sum_{n=1}^{\infty} \alpha^n s_t^* \right) + \eta_t \\ &= (1 - \alpha) s_t^* + \alpha s_t^* + \eta_t = s_t^* + \eta_t \end{aligned} \quad (4)$$

We find that under rational expectations, the market exchange rate is driven by the current fundamental rate and some unexpected noise.

2.2 The expectations formation

In this section, we specify the mechanism determining expectations of agents and we depart from the assumption of rational expectations. We take the view that the rational expectations assumption puts too great an informational burden on individual agents. Agents experience cognitive problems in processing information. As a result, they use simple forecasting rules (heuristics; see Kahneman 2002).

We start by assuming that agents can use two different forecasting rules: fundamentalist and chartist (technical analysis).

When using a fundamentalists rule, agents compare the market exchange rate with the fundamental rate and they forecast the future market rate to return to the fundamental rate:

$$\hat{E}_t^f \left(\tilde{\Delta} s_{t+1} \right) = -\psi (s_{t-1} - s_{t-1}^*) \quad (5)$$

where $\tilde{\Delta} s_{t+1}$ is defined as $s_{t+1} - s_{t-1}$.

We assume here that boundedly rational agents' information set on the exchange rate at t is $I_t = \{s_0, s_1, \dots, s_{t-1}, s_0^*, s_1^*, \dots, s_{t-1}^*\}$. We assume that the agents do not know the contemporaneous exchange rates s_t and s_t^* . This is a natural way to proceed under bounded rationality, where agents use past ob-

servations of the relevant variables to make forecast¹. The contemporaneous exchange rate s_t can only be known in a RE-environment. The latter allows agents to compute the effect of their forecasts on the equilibrium exchange rate (since they know the underlying model). In a boundedly rational environment, agents cannot do this.

We can rewrite the expectations for s_{t+1} :

$$\hat{E}_t^f(s_{t+1}) = s_{t-1} + \psi(s_{t-1}^* - s_{t-1}) \quad (6)$$

In this sense, agents follow a negative feedback rule, where $\psi > 0$ is a parameter describing the speed at which the agents expect the exchange rate to return to its fundamental value.

The second forecasting rule agents can use is a chartist rule. We assume that this takes the form of extrapolating the last change of the exchange rate into the future:

$$\hat{E}_t^c(\tilde{\Delta}s_{t+1}) = \beta\Delta s_{t-1} \quad (7)$$

where $\Delta s_{t-1} = s_{t-1} - s_{t-2}$. Alternatively we can write:

$$\hat{E}_t^c(s_{t+1}) = s_{t-1} + \beta\Delta s_{t-1} \quad (8)$$

The degree of extrapolation is given by the parameter $\beta > 0$. Clearly, more sophisticated rules could be specified. Here we focus on the simplest possible chartist rule.

The agents combine these two rules with their respective weights. As a result, the market forecast, $\hat{E}_t s_{t+1}$, is assumed to be a weighted average of the mean-reverting and the extrapolative components, i.e.,

$$\hat{E}_t s_{t+1} = \omega^f \hat{E}_t^f s_{t+1} + \omega^c \hat{E}_t^c s_{t+1} \quad (9)$$

where $\hat{E}_t^f s_{t+1}$ and $\hat{E}_t^c s_{t+1}$ are the mean-reverting and the extrapolative components, respectively, ω^f is the weight given to the fundamentalist rule, ω^c is the weight given to the chartist rule and $\omega^f + \omega^c = 1$.

We now substitute equation (6) and (8) into equation (9) and the latter into equation (2). This yields the actual exchange rate process:

¹See for instance the information structure of the forecasting models proposed by Hommes et al. (2005 and 2009).

$$s_t = (1 - \alpha)s_t^* + \alpha\omega^f [s_{t-1} + \psi (s_{t-1}^* - s_{t-1})] + \alpha\omega^c [s_{t-1} + \beta\Delta s_{t-1}] + \eta_t \quad (10)$$

3 Learning mechanisms of agents

Boundedly rational agents use simple rules described in the previous section. However, they continuously test these rules. This procedure is the mechanism by which discipline is imposed on the behavior of individual agents. We specify two alternative selection mechanisms (learning mechanisms). In the first one, agents select the rules based on a fitness method. In the second mechanism, agents learn to improve these rules using statistical methods.

The main difference between both learning mechanisms lies in the assumption about which parameters are time-varying. In the fitness learning, the market expectations change because of the shifts of the weights on two rules, while the parameters ψ and β are fixed. In the statistical learning, the opposite takes place. The weights on two rules are equal and constant and the agents estimate the parameters ψ and β of the two rules. Importantly, agents can eliminate one of the forecasting rules in their testing procedure. When using fitness learning, this corresponds to the weight on one of the rules being zero. When applying statistical learning, this occurs when one of the estimated parameters is zero.

Although this is not the objective of this paper, one could also combine both learning approaches, as is done by Branch and Evans (2006a and 2006b and 2007) and Lewis and Markiewicz (2009) in the foreign exchange context but different model.

3.1 Fitness mechanism

The first learning mechanism is based on a dynamic predictor selection, which we called fitness learning. It is based on discrete choice theory². This mechanism assumes that agents evaluate the two forecasting rules by computing the past risk-adjusted rates of return of these rules and to increase (reduce) the weight

²This specification is often applied in discrete choice models. For an application in the markets for differentiated goods, see Anderson, et al., (1992). There are other ways to specify a rule that governs the selection of forecasting strategies. One was proposed by Kirman (1993). Another one was formulated by Lux and Marchesi (1999).

of one rule if it is more (less) profitable than the alternative rule. We specify this procedure as follows:

$$\omega_t^f = \frac{\exp \delta \pi_{t-1}^f}{\exp \delta \pi_{t-1}^f + \exp \delta \pi_{t-1}^c} \quad (11)$$

$$\omega_t^c = \frac{\exp \delta \pi_{t-1}^c}{\exp \delta \pi_{t-1}^c + \exp \delta \pi_{t-1}^f} \quad (12)$$

where ω_t^f and ω_t^c are the weights given to the fundamentalist and chartist rules, respectively and π_{t-1}^f and π_{t-1}^c are the realized profits. We assume here that the agents calculate the weights based on the last period profit but it is possible to construct a weighted average of past values, as for instance in De Grauwe and Grimaldi (2006), Branch and Evans (2006b and 2007).

We define the profits as the one-period returns of investing in the foreign asset.

$$\pi_{t-1}^i = (s_{t-1} - s_{t-2}) \operatorname{sgn} \left(\hat{E}_{t-2}^i s_{t-1} - s_{t-2} \right) \quad (13)$$

$$\text{where } \operatorname{sgn}[x] = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -1 & \text{for } x < 0 \end{cases} \quad \text{and } i = c, f$$

The profit functions are constructed in a way to generate buy and sell signals. The forecasting rules are used to generate an expected sign of the exchange rate return (direction of change) in the next period³. The traders receive buy and sell signals and act accordingly. This type of strategies (direction of change) can be profitable as pointed out by Leitch and Tanner (1991) and Levich (2001), among others.

When agents forecast an increase in the exchange rate and this increase is realized, their per-unit profit is equal to the observed increase in the exchange rate. If instead the exchange rate declines, they make a per-unit loss which equals this decline (because in this case they have bought foreign assets which have declined in price).

Equations (11) and (12) can now be interpreted as follows. When the rate of return of the extrapolative (chartist) rule increases relative to the rate of return of the mean-reverting (fundamentalist) rule, then the weight the agents give to the extrapolative rule in period t increases, and vice versa.

³This is a standard way the traders proceed i.e. they go long in a given currency if the signal from the forecasting rule is buy and they go short if the signal is sell (see de Zwart et al. 2009).

The parameter δ measures the intensity with which the agents switch the weights from one rule to the other. With an increasing δ agents react stronger to the relative profitability of the two forecasting rules. In the limit, when δ goes to infinity, all agents choose the forecasting rule which proves to be more profitable so that the weights are either 1 or 0. When δ is equal to zero, agents are insensitive to the relative profitability of these rules. In the latter case, the weights of mean-reverting and extrapolative rules are constant and equal to 0.5. Thus, δ is a measure of inertia in the decision to give more weight to the more profitable rule⁴. Note that $\delta \rightarrow \infty$ represents the case closest to the RE assumptions. When $\delta \rightarrow \infty$, agents' choices are optimal i.e. they always shift to the best (among available) forecasting rule. In the simulations, we will assume that agents' behavior is close to optimal (high δ).

The weights obtained from equations (11) and (12) are then substituted into the exchange rate equation (10):

$$s_t = (1 - \alpha)s_t^* + \alpha\omega_t^f [s_{t-1} + \psi (s_{t-1}^* - s_{t-1})] + a\omega_t^e [s_{t-1} + \beta\Delta s_{t-1}] + \eta_t \quad (14)$$

Note that in this learning mechanism agents are assumed to use the same values of parameters β and ψ in every period t .

3.2 Statistical learning

The second learning mechanism that we consider here is statistical learning (See Evans and Honkapoja 2001). As before, agents' expectations are composed of two components, i.e. a mean-reverting and an extrapolative one. Agents are assumed to have some basic knowledge of econometrics and they estimate the importance of these two components based on data up to period $t - 1$. Their expectations are formed in the following way:

$$\hat{E}_t (\tilde{\Delta}s_{t+1}) = \psi_{t-1} (s_{t-1}^* - s_{t-1}) + \beta_{t-1}\Delta s_{t-1} \quad (15)$$

and the resulting PLM is as follows:

⁴The logic of the switching weight is in the spirit of the adaptive rules that are used in game theoretic models (See, for examples, Cheung and Friedman (1997); Fudenberg and Levine, (1998)). In these models, actions that did better in the observed past tend to increase in frequency while actions that did worse tend to decrease in frequency.

$$s_{t+1} = s_{t-1} + \psi_{t-1} (s_{t-1}^* - s_{t-1}) + \beta_{t-1} \Delta s_{t-1} + \epsilon_t \quad (16)$$

The agents regress Δs_{t+1} on $s_{t-1}^* - s_{t-1}$ and Δs_{t-1} so that the updating algorithm can be written as follows⁵:

$$\begin{aligned} \phi_t &= \phi_{t-1} + \frac{1}{\gamma_t} \mathbf{R}_t^{-1} \mathbf{z}_{t-1} (\Delta s_{t+1} - \phi_{t-1}' \mathbf{z}_{t-1}) \\ \mathbf{R}_t &= \mathbf{R}_{t-1} + \frac{1}{\gamma_t} (\mathbf{z}_{t-1} \mathbf{z}_{t-1}' - \mathbf{R}_{t-1}) \end{aligned} \quad (17)$$

where $\phi_t = (\psi_t, \beta_t)'$ is the vector of parameter estimates, $\mathbf{z}_{t-1} = (s_{t-1}^* - s_{t-1}, \Delta s_{t-1})$ is a vector of explanatory variables, $\mathbf{R}_t = \frac{1}{\gamma_t} \sum_{i=1}^t \mathbf{z}_{i-1} \mathbf{z}_{i-1}'$ is a second moment matrix and $\frac{1}{\gamma_t}$ is the gain sequence. The gain captures the speed of updating in the sense of how much weight the agents put on the new incoming information. Introducing agents' forecast as given by (15) into equation (2), we obtain the resulting actual law of motion (ALM) of the market exchange rate:

$$s_t = (1 - \alpha) s_t^* + \alpha(1 + \beta_{t-1} - \psi_{t-1}) s_{t-1} - \alpha \beta_{t-1} s_{t-2} + \alpha \psi_{t-1} s_{t-1}^* + \eta_t \quad (18)$$

The fitness learning assumes that the agents constantly evaluate their forecasting rules. We employ a similar criterion for agents using statistical learning. Following Marcet and Nicolini (2003), we assume that the agents alter the gain sequence $\frac{1}{\gamma_t}$. This sequence is usually constructed in two different ways.

First, as in standard OLS, it can give the same weight to every observation when $\gamma_t = \gamma_{t-1} + 1$. Since the weight given to every observation decreases with the amount of available data, it is known as a decreasing gain learning.

Second, the sequence $\frac{1}{\gamma_t}$ can be such that the most recent observations receive more weight than the older observations. Then $\gamma_t = \bar{\gamma}$ and it is known as a constant gain learning or "perpetual learning". The "perpetual learning" is used when there is a structural change in the economy i.e. economy follows a stochastic process with parameter values that evolve over time. Then, the fixed parameters are not optimal at all times and a constant gain learning rule or "perpetual learning" will better track the evolution of the parameters than a

⁵ Note that ψ_{t-1} should now be interpreted as the time varying expression $\omega^f \psi$ used in the previous section and β_{t-1} as $\omega^c \beta$.

decreasing gain rule. When the exchange rate is in the calm regime, it would be optimal to use a decreasing gain. However, when the exchange rate is subject to sudden changes (bubbles and crashes), it would be more sensible to use a constant gain, which responds faster to the larger changes in the stochastic process.

We assume that agents' behavior is optimal in the way that they verify at regular intervals ($T = 100$) whether their rule is correctly specified⁶. If their forecast error is higher than some number ς , they shift to the constant gain rule. Otherwise they keep on using the decreasing gain learning rule:

$$\begin{cases} \gamma_t = \gamma_{t-1} + 1 & \text{if } \frac{1}{T} \sum_{t=1}^T |\hat{E}_{t-1}s_t - s_t| < \varsigma \\ = \bar{\gamma} & \text{otherwise} \end{cases} \quad (19)$$

Since it is not clear what value of the constant gain $\frac{1}{\bar{\gamma}}$ should be applied, we will calibrate it.

4 Equilibrium properties

In this section, we analyze the equilibrium properties of the market exchange rate under two learning mechanisms. This will allow us to analyze the question of whether these two learning mechanisms are capable of revealing the fundamental value of the exchange rate in the steady state. In particular, it has been demonstrated that in the long run the exchange rate tends to move towards its fundamental value.

4.1 The equilibrium under fitness learning

We set the fundamental rate, $s_t^* = s^* = 0$, so that the exchange rate movements can be interpreted as deviations from the fundamental. We rewrite (14)

$$s_t = \alpha \left[\left(1 - \omega_{t-1}^f \psi + \omega_{t-1}^c \beta \right) s_{t-1} - \omega_{t-1}^c s_{t-2} \right] \quad (20)$$

and define $m_t \equiv \omega_{t-1}^f - \omega_{t-1}^c$ where ω_{t-1}^f is defined in equation (11). We can write:

⁶We assume that $T = 100$ roughly corresponds to the sum of the lengths of the potential bubble and crash (8 years). It is difficult to define exactly the length either of a bubble or of a crash because these depend on the definition of the fundamental exchange rate. In the empirical literature the length of a bubble varies from 5 years to 20 years (See Van Norden 1996).

$$m_t = \tanh \left[\frac{\delta}{2} \left(\pi_{t-1}^f - \pi_{t-1}^c \right) \right] \quad (21)$$

where profits are defined in following way:

$$\pi_{t-1}^f = (s_{t-1} - s_{t-2}) \operatorname{sgn} [-\psi s_{t-2}] \quad (22)$$

$$\pi_{t-1}^c = (s_{t-1} - s_{t-2}) \operatorname{sgn} [\beta(s_{t-2} - s_{t-3})] \quad (23)$$

Because for all the 3 cases, $\operatorname{sgn}[x] = 1, 0, -1$, $m_t = 0$ we find $\omega_{t-1}^f = \omega_{t-1}^c = 0.5$. We substitute them into (38) and find:

$$\bar{s} = \frac{1}{2} \alpha (1 - \psi + \beta) \bar{s} \quad (24)$$

There are two solution to (24). The first one is when $\bar{s} = 0$ and represents the fundamental exchange rate equilibrium which is also characterized by zero profits and equal weights of both rules:

$$\bar{s} = s^*, \bar{\omega}^c = \bar{\omega}^f = \frac{1}{2}, \bar{\pi}^f = \bar{\pi}^c = 0 \quad (25)$$

The unique fundamental solution is stable if $\frac{1}{2} \alpha (1 - \psi + \beta) \Rightarrow \alpha < \frac{2}{1 - \psi + \beta}$ and holds because $0 < \psi < 1$ and $0 < \beta < 1$ $\beta > \psi$ so that $1 > \beta - \psi > 0 \Rightarrow 2 > \frac{2}{1 - \psi + \beta} > 1$ and $\alpha < 1$ by definition since this is a discount factor so that $\alpha < \frac{2}{1 - \psi + \beta}$ holds and the fundamental equilibrium is stable. More details on computations of the equilibria and their stability can be found in the appendix.

4.2 The equilibrium under statistical learning

In this subsection, we analyze the properties of the equilibrium of the model under statistical learning. For the sake of simplicity, we assume that the agents learn using decreasing gain i.e., simple Least Squares updating. The agents' PLM is of the following form:

$$\tilde{\Delta} s_{t+1} = \psi_{t-1} (s_{t-1}^* - s_{t-1}) + \beta_{t-1} \Delta s_{t-1} + \nu_{t+1} \quad (26)$$

Accordingly, the agents form their expectations:

$$\hat{E}_t \left(\tilde{\Delta} s_{t+1} \right) = \psi_{t-1} (s_{t-1}^* - s_{t-1}) + \beta_{t-1} \Delta s_{t-1} \quad (27)$$

Substituting the PLM into equation (2) yields the resulting ALM of the market exchange rate:

$$s_t = (1 - \alpha)s_t^* + \alpha(1 + \beta - \psi)s_{t-1} - \alpha\beta s_{t-2} + \alpha\psi s_{t-1}^* + \eta_t \quad (28)$$

Using the definition of the fundamental rate s_t^* in equation (3), we define T-map⁷:

$$T \begin{pmatrix} \psi \\ \beta \end{pmatrix} = \begin{pmatrix} (\alpha\psi - \alpha + 1)(\alpha + \alpha\beta - \alpha\psi + 1) \\ \alpha^2(1 + \beta - \psi)\beta \end{pmatrix} \quad (29)$$

From the system (29), we compute the stationary points of $T(\psi, \beta)'$. We have two solutions: the fundamental and the bubble solution. These two solutions are standard in the asset pricing literature where we have a rational expectations equilibrium and a rational bubble. The fundamental one is given by the combination $\phi_1 = (1, 0)'$ and means that the agents learn that the extrapolating component does not play a role in determination of the exchange rate ($\beta_1 = 0$). They find that the exchange rate will return to the fundamental rate in the next period ($\psi_1 = 1$). Substituting these values in the ALM (28), we obtain:

$$s_t = s_t^* + \eta_t - \alpha\epsilon_t \quad (30)$$

Thus, we find that this set of fixed points leads to the rational expectations solution of the model. Equation (30) includes an additional element with respect to the REE described by equation (4), namely $\alpha\epsilon_t$. The existence of this noise term in the equilibrium exchange rate process is due to the different assumptions about the information set the agents use. Rational agents are assumed to know the fundamental value of the exchange rate s_t^* at t , while the boundedly rational agents do not have s_t^* in their information set at t . As a result, these agents face an additional uncertainty about the exchange rate, reflected by $\alpha\epsilon_t$.

In the steady state, assuming for the sake of simplicity, $s_t^* = s_{t-1}^* = \bar{s}^* = 0$ and $\eta_t = \epsilon_t = 0$, we find $s_t = 0$. This means that in the steady state the adaptive learning model leads the exchange rate to its fundamental value.

The bubble solution is given by $\phi_2 = \left(1 - \frac{1}{\alpha^2}, 0\right)'$ and indicates that the agents again learn that extrapolating parameter to be zero ($\beta_2 = 0$) and a negative value of ψ_2 . This indicates that the fundamentalists learn to extrapolate

⁷T-map represents ALM parameters of regressors and we seek to find fixed points of this map. These solutions are the points that the parameters estimated by agents converge to.

the difference between market and fundamental exchange rates. We substitute the values of the second solution i.e., $\psi_2 = 1 - \frac{1}{\alpha^2}$ and $\beta_2 = 0$ into the ALM (28). We find that the resulting current market exchange rate is a sum of the fundamental rate and the extrapolated difference between past market and fundamental rates:

$$s_t = s_t^* + \frac{1}{\alpha} (s_{t-1} - s_{t-1}^*) - \alpha v_t + \eta_t \quad (31)$$

We check the expectational stability (E-stability) of the two possible solutions by calculating the eigenvalues of $\mathbf{DT} - \mathbf{I}$. We compute the Jacobian matrix $\mathbf{DT} \begin{pmatrix} \psi \\ \beta \end{pmatrix}$ and the eigenvalues of the matrix $\mathbf{DT} - \mathbf{I}$. For the first set of solutions ϕ_1 , we find that both eigenvalues are equal to zero and thus the solution ϕ_1 , leading to the fundamental equilibrium is E-stable. The second set of solutions, ϕ_2 , yields the eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 1$ and thus is E-unstable. These are standard results for the asset pricing models and a similar example can be found in Evans and Honkapohja 2001. Ch.9. The details of the calculations are reported in the appendix.

5 Dynamic analysis

From the previous analysis it follows that both learning mechanisms can generate rational expectations equilibria where the market rate equals the fundamental rate. Here we analyze the dynamics produced by these two learning mechanisms and we ask the question of whether they are capable of mimicking the empirical puzzles observed in the foreign exchange market.

5.1 Stylized facts of foreign exchange markets

First, there is the *disconnect puzzle*, which is that the exchange rate is disconnected from fundamental variables. Related to that, we observe *excess volatility* of the exchange rate returns i.e. exchange rate returns tend to be higher than can be accounted for by the changes in the fundamentals. In addition, the volatility of the returns appears to be time dependent (*volatility clustering*).

Second and related to the first one is *unit root property*. Most prominently, (log) exchange rates seem to be nonstationary and thus unit-root tests are typically unable to reject the null hypothesis of a first-order autoregressive process with a coefficient equal to unity. As a result, future exchange rate changes

are hard to predict by past exchange rate changes. We analyze these time series properties of the exchange rate produced by the model under two different learning mechanisms and compare them with the ones observed in the data.

5.2 Fundamental

The monthly fundamental rates are constructed on the basis of the Taylor-rule model of the exchange rate. Several recent studies showed that this model can generate better exchange rate forecasts than other fundamentals-based models⁸. There is a large number of Taylor rule specifications that have been used in the literature and tested empirically (see for instance Taylor, 1999). As a consequence, the exchange rate models derived from the two Taylor rules can take different forms. In this exercise the Taylor rule model is used only as a benchmark to evaluate the performance of the respective learning mechanisms and therefore we can use its simplest version.

The model is as follows. Assume that the home (US) monetary authority sets interest rates according to a simple interest rule proposed by Taylor (1993):

$$i_t = \gamma_0 + \gamma_1 \pi_t + \gamma_2 y_t + \nu_t \quad (32)$$

where the monetary policy instrument i_t is a short term interest rate, π_t is inflation, y_t is the output gap and ν_t is a shock to the monetary policy rule. Following Clarida, Gali and Gertler (1999), we assume the interest rate smoothing so that:

$$i_t = \rho i_{t-1} + (1 - \rho)(\gamma_0 + \gamma_1 \pi_t + \gamma_2 y_t + \nu_t) \quad (33)$$

The foreign (UK) central bank follows a similar reaction function:

$$i_t^* = \rho^* i_{t-1}^* + (1 - \rho^*)(\gamma_0^* + \gamma_1 \pi_t^* + \gamma_2 y_t^* + \nu_t^*) \quad (34)$$

where stars denote the foreign variables and parameters and the UIP condition is:

$$i_t = i_t^* + \hat{E}_t s_{t+1} - s_t + u_t \quad (35)$$

where u_t is an exogenous risk premium shock. Combine the two Taylor rules in (33) and (34) with (35) to obtain:

⁸See Molodsova and Papell (2008 and 2009)

$$\begin{aligned}\hat{E}_t s_{t+1} - s_t &= \rho i_{t-1} + (1 - \rho)(\gamma_0 + \gamma_1 \pi_t + \gamma_2 y_t + \nu_t) - \\ &\quad - \rho^* i_{t-1}^* - (1 - \rho^*)(\gamma_0 + \gamma_1 \pi_t^* + \gamma_2 y_t^* + \nu_t^*) + u_t\end{aligned}\quad (36)$$

Assuming rational expectations and solving the model forward gives:

$$s_t = B_1 \sum_{j=0}^T f_{t+j} + B_2 \sum_{j=0}^T f_{t+j}^* + E_t s_{t+T}$$

Letting $T \rightarrow \infty$ and imposing the no-bubbles condition so that $\lim_{T \rightarrow \infty} E_t s_{t+T} = 0$, we find:

$$s_t = (I - B_1)^{-1} f_t + (I - B_2)^{-1} f_t^* + \epsilon_t \quad (37)$$

where $f_t = (1, i_t, \pi_t, y_t)'$, $f_t^* = (1, i_t^*, \pi_t^*, y_t^*)'$,

$$B_1 = - \left(\begin{array}{cccc} \frac{\gamma_0}{(1-\rho)} & \rho & \frac{\gamma_1}{(1-\rho)} & \frac{\gamma_2}{(1-\rho)} \end{array} \right) \text{ and}$$

$$B_2 = \left(\begin{array}{cccc} \frac{\gamma_0^*}{(1-\rho^*)} & \rho & \frac{\gamma_1^*}{(1-\rho^*)} & \frac{\gamma_2^*}{(1-\rho^*)} \end{array} \right)$$

The current exchange rate is a function of home and foreign Taylor-rule fundamental variables: the interest rate, the output gap, the inflation rate and shocks in ϵ_t . ϵ_t is a linear combination of shocks to monetary policy rules and to UIP: $\epsilon_t = (\nu_t^* - \nu_t) - u_t$.

We estimate the Taylor rules for 3 countries: the United Kingdom, Germany, and the United States, the last being the numeraire⁹. Next, we combine these rules so that we can derive the specification for the British Pound, and German Mark against the US Dollar. We use monthly data. Inflation is year on year CPI inflation and interest rate is a monthly market rate. Potential output is proxied by the HP trend of industrial production and the output gap is calculated as a percentage difference between the actual output and the HP trend. The data are mainly coming from the databases of the respective central banks and they are reported in detail in the appendix.

In order to facilitate the comparison of the two mechanisms we calibrate the parameters driving the dynamics in each of the learning mechanisms. It is well known that the appropriate values of the intensity of choice and the gain

⁹Note that to avoid the endogeneity problems, we estimate Taylor rules with the past values of fundamentals.

parameters can generate high volatility. We can therefore choose the values that can reproduce the well-known excess volatility puzzle.

We assume monthly frequency for the calibration. This implies the following parameter values: a discount factor α is calculated based on the assumption that the nominal interest rate $r = 5\%$ per year, $\alpha = 0.996 \left(\alpha = \frac{1}{(1 + 0.05)^{\frac{1}{12}}} = 0.996 \right)$. In the fitness learning mechanism, we set the value of the parameter β at 0.95 and ψ at 0.08. The choice of parameter values is consistent with empirical and survey evidence suggesting that, in the short run, agents expect the past change to be almost entirely extrapolated into the future, while they believe that, in the longer run, the market rate will return to the fundamental value¹⁰. We will perform a sensitivity analysis to check how sensitive the results are with respect to β . The value of $\psi = 0.08$ for monthly observations suggests that the agents expect the market rate to return to the fundamental value in one year¹¹. In the statistical learning model, we assume that the agents alter the gain sequences according to condition (19), where $\varsigma = 0.01$. We will check the sensitivity of the results to this parameter value as well.

First, we calculate the average excess volatility of the market rate against the Taylor-rule based fundamental in (37). Next, we set the intensity of choice parameter δ and constant gain sequence $\frac{1}{\bar{\gamma}}$ to match this excess volatility. We then analyze the remaining foreign exchange market regularities.

We searched the value of δ that minimizes the squared distance between the exchange rate returns volatility in the data and the one implied by fitness learning. We carried the search within the interval $[0, 200]$. We carried a similar search for the "optimal" gain sequence where the searched interval was $(0, 1)$.

The average excess volatility for both currencies equals 0.010, as reported in Table 1. The last two columns of the table display the values of the learning parameters needed to match the excess volatility. We take averages of those

¹⁰For the detailed description of the traders' forecasts see Cheung and Chinn (2001) and Cheung et al. (2004).

¹¹The empirical evidence suggests that the convergence of market rate to the fundamental rate may take longer than one year. Frankel and Froot (1987) find an expected half-life for deviations from the fundamental proxied by PPP of around 3 years. Similarly, Mark (1995) and Chinn and Meese (1995) demonstrate that the models incorporating a set of fundamentals have some statistically significant power over the horizon of 3 years. These results are not inconsistent with traders' expectations. In fact, traders use both rules, i.e. fundamentalists and chartists forecasting rules, simultaneously. The extrapolative behaviour of agents prevents the market rate from reaching its fundamental value over the horizon implied by the fundamentalists rule (Cheung et al. (2001)).

Table 1: **Data and model implied exchange rate volatility**

Currency	Market volatility	Fundamental volatility	Excess Volatility	Matching gain $\frac{1}{\bar{\gamma}}$	Matching IOC δ
DM	0.027	0.015	0.012	0.06	34.5
UKP	0.025	0.017	0.008	0.05	49.5

Return volatility is measured by a sample standard deviation of returns. DM denotes the German Mark and UKP the UK Pound. The sample period for DM is 1975M1-1999M1 and for UK pound 1975M1-2009M1.

values and we set the constant gain sequence to 0.055, and the intensity of choice parameter to 42. We carry out additional calibrations with these values of learning dynamics parameters to compare the properties of the exchange rate series that the model generates under both learning mechanisms.

5.3 Disconnect puzzle

We first quantify the disconnect puzzle¹². We calculate the percent of time the exchange rate deviates from the fundamental variable by more than 3 average standard deviations (of the fundamental, $\sigma_f = 0.016$, see Table 1), during at least 15 consecutive periods.

Table 2: **Data and model implied disconnect puzzle**

Currency	Data	Fitness learning	Statistical learning
DM	80%	25%	0
UKP	30%	15%	0

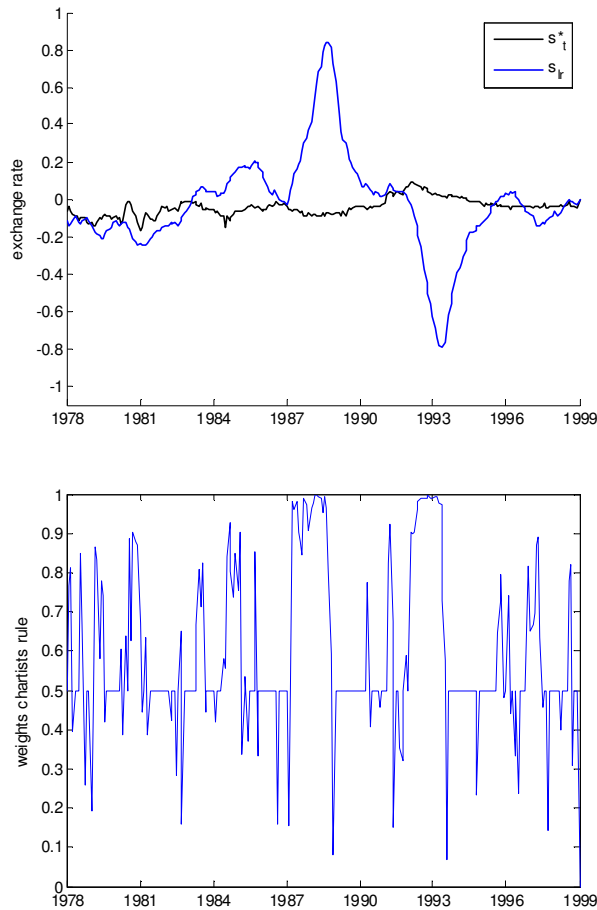
Return volatility is measured by a sample standard deviation of the returns. DM denotes the German Mark and UKP the UK Pound. The sample period for DM is 1975M1-1999M1 and for UK pound 1975M1-2009M1. The excess volatility is calculated as a difference between the volatility of market and fundamental returns.

We found that the dollar exchange rate against the DM and the UKP are disconnected from the fundamental variable on average 55% of the time. As summarized in the first column of Table 2, the Taylor rule fundamentals approximate quite well the UKP behavior (30% disconnect) while they do not fit at all the DM dynamics (80% disconnect).

¹²Note that the excess volatility reflects one dimension of the disconnect puzzle where the market returns' volatility largely exceeds the fundamental returns' volatility. Since we set the learning parameters to match the second moments, we analyze now the disconnection of the market rate from its fundamental in (log)level.

The disconnection under fitness learning is somewhat lower than the one we observe in the data, at least on average. This is mainly the case for the DM. For the UKP, the results are closer to the data since the exchange rate under fitness learning is disconnected 15% of time while in the data this occurs 30 % of time. Figure 1 helps in understanding what drives these dynamics.

Figure 1: **Exchange rate under fitness learning, DM fundamental rate and the weight on chartists' rule**



The upper panel plots the market and the fundamental exchange rates, the latter being the Taylor rule-based German Mark. We find that the market

exchange rate is often disconnected from the fundamental one. As can be seen in Figure 1, the market exchange rate moves around the fundamental in a cyclical way and it never remains at the fundamental value. This is in line with the equilibrium analysis that showed that the fundamental solution is not stable. These cyclical movements have the appearance of bubbles and crashes.

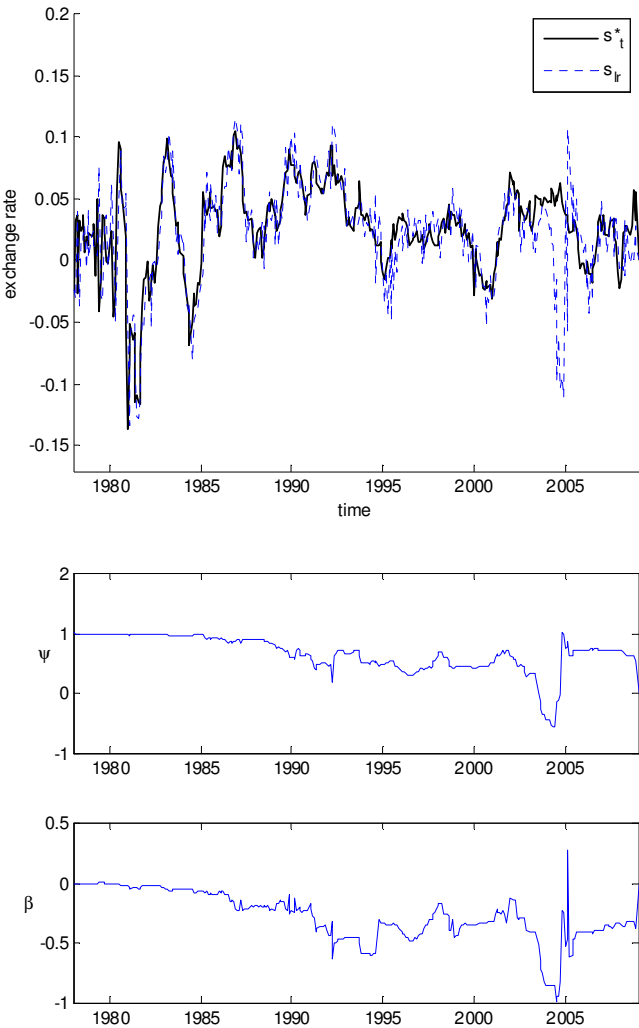
A comparison of the upper and lower parts of Figure 1 allows us to understand the nature of these cyclical movements. The lower part shows the weights on the chartists' rule. We find that periods of sustained deviations of the exchange rate from the fundamental coincide with periods during which the chartists' rule dominates the market expectations.

De Grauwe and Grimaldi (2006a and 2006b) have analyzed this feature in the framework of a similar model. Our interpretation of this result is that a series of stochastic shocks in one direction can lead to an increased profitability of the extrapolative (chartist) forecasting rule thereby leading to an increased popularity of this rule at the expense of the fundamentalist rule which becomes increasingly loss-making during the bubble phase of the cycle. This creates a self-fulfilling dynamics. As the chartist rule becomes more profitable it gets more weight in the market forecast, thereby intensifying the upward (downward) movement. At some point, however, movements in the fundamental have the effect of pulling the exchange rate back to its fundamental. We will return to this feature later and apply a sensitivity analysis to check under what conditions these dynamics occur.

Table 2 clearly displays that statistical learning does not generate any disconnection from the fundamental exchange rate.

The equilibrium analysis already suggested that the exchange rate would converge to the fundamental solution since it was E-stable. We plot the simulated market and fundamental exchange rates in the upper panel of Figure 2 and corresponding parameter values β and ψ in the lower panel. It is clear that the market exchange rate is almost permanently connected to the fundamental. This is the case because, as lower panel of Figure 2 indicates, the agents learn the fundamental exchange rate. Put differently, the parameters β and ψ take equilibrium values 0 and 1 that in turn imply the fundamental solution of the exchange rate. The only episode when the market and fundamental exchange

Figure 2: Exchange rate under statistical learning, UKP fundamental rate and the updated parameters



rates seem to diverge occurs around 2005 when both parameters are strongly biased downwards and generate the disconnection. This disconnection is however transitory as the parameters move back towards their equilibrium values.

5.4 Unit root properties and unpredictability

Exchange rates are nonstationary while their first differences are stationary. More precisely, unit-root tests are typically unable to reject the null hypothesis of a first-order autoregressive process with a coefficient equal to unity. Because the exchange rate behaves similarly to a random walk, the autocorrelation of changes in exchange rates is very low¹³. As a result, future exchange rate changes are hard to predict either by the forward discount or by past exchange rate changes.

The unit root property seems to be at odds with the fact that some trading strategies can generate profits in the foreign exchange markets. However, these profits are made only using certain currencies, during certain periods. For instance Cialenco and Protopapadakis (2006) find that, on average, technical strategy profits are economically small and statistically insignificant when they are applied strictly out-of-sample. These authors analyze the profitability of the trading strategies for 8 currencies, over the period from 1986 to 2004. They show that strategies that make profits out-of-sample in the first half of the sample generally make losses in the second half of the sample and thus do not generate profits on average.

The natural way to test for unpredictability of exchange rates is to test for the unit root. We carry out Augmented Dickey Fuller unit root test for two series generated by the calibrations under fitness and statistical learning. We use the specifications with a constant, a constant and a trend, and without any of them. All of the specifications provide similar results which are summarized in Table 3.

As displayed in Table 3, the unit root hypothesis cannot be rejected in any of the cases, at least at 10 % significance level. Loosely speaking, there is some predictability in the exchange rates series generated by the proposed model under learning mechanisms and, as we can read from the last 2 columns of

¹³Bacchetta P. and van Wincoop E. (2007) note for instance that the autocorrelation of quarterly exchange rate changes is less than 0.03.

Table 3: **Unit root test on calibrated series**

	t-statistics	p-values	AR(1)	t-statistics
FL-DM	-3.2195*	0.0825	-0.0116***	-3.2050
SL-UKP	-4.7354***	0.0007	-0.1151***	-4.3267

Table reports the results of ADF test with the constant and the trend. FL-DM denotes fitness learning where in the calibrations the fundamental was calculated based on the German data. SL-UKP stands for statistical learning where in the calibrations the fundamental was calculated based on the UK data. The sample period for DM is 1975M1-1999M1 and for UK pound 1975M1-2009M1. Asterisks refer to level of significance: ***: 1%, **:5%, *:10%. One-sided p-values are from MacKinnon (1996).

Table 3, both mechanisms produce the series with a low mean-reverting AR(1) coefficient.

5.5 Volatility clustering

Two important features of the second moments of the exchange rate series have been observed in the data. The first one is the excess volatility. The second one is volatility clustering. Since we set the learning parameters to match the excess volatility in the data, we will focus our analysis on the second phenomenon. We fit the GARCH (1,1) model to the monthly exchange rate returns for 2 currencies: the DM and UKP. Then, we consider the same GARCH (1,1) model for the market returns generated by the proposed model under two learning rules. We use the same monthly parameter values as previously¹⁴.

First, as reported in Table 4, we note that all the coefficients of the variance equation specification for both currencies are significant. Second, in line with existing empirical results, we find that the sum of the ARCH (1) and GARCH (1) ($a+b$) components (degree of inertia) is very close to one for both currencies. This clearly indicates the presence of volatility clustering in the data.

We now fit the same GARCH (1,1) model to the exchange rate returns generated by the calibrated model. The bottom panel of Table 4 shows maximum likelihood estimates of the parameters in GARCH (1,1) model and z-statistics. The estimates are obtained by the Bernt, Hall, Hall and Hausman (1974) algorithm. As in the data, the estimated values for $a+b$ are close to one. Moreover, in both cases, the coefficients a and b are significant at the 1% level. Thus, we

¹⁴See the details in Section 5.3.

Table 4: **Volatility clustering in the data and calibrated series**

	Variable	Coefficient	z-statistic
DM	C	0.0001**	2.2545
	ARCH(1)	0.0489*	1.6982
	GARCH(1)	0.7902***	7.9547
UKP	C	0.0001***	6.0148
	ARCH(1)	0.2626***	4.1227
	GARCH(1)	0.5538***	5.4819
FL-DM	Variable	Coefficient	z-statistic
	C	0.0003***	2.3966
	ARCH(1)	0.2862***	3.0880
	GARCH(1)	0.6517***	8.6406
SL-UKP	C	0.0009***	3.9728
	ARCH(1)	0.1750***	4.2471
	GARCH(1)	0.7143***	14.480

We report here only the results of estimated conditional variance equation: $\sigma_t^2 = v + a\epsilon_{t-1}^2 + b\sigma_{t-1}^2$, where σ_t^2 is the one-period ahead forecast variance, v is the mean, ϵ_{t-1}^2 and σ_{t-1}^2 denote the lag of squared residual from the mean equation and the last period's forecast variance, respectively. The ARCH(1) parameter in table corresponds to a and the GARCH(1) parameter to b , in the variance equation. DM denotes the German Mark, and UKP the UK Pound. SL-UKP stands for statistical learning where in the calibrations the fundamental was calculated based on the UK data. The sample period for DM is 1975M1-1999M1 and for the UK pound 1975M1-2009M1. Asterisks refer to level of significance: ***: 1%, **:5%, *:10%.

find that under both learning rules the exchange rate returns display persistence in volatility.

6 Conclusion

In this paper we compared two competing approaches to model the behavior of foreign exchange market participants: statistical learning and fitness learning, applied to a set of predictors, which include chartists and fundamentalists. We examined which of these approaches is the best in terms of replicating the exchange rate dynamics within the framework of a standard asset pricing model.

Although many theoretical studies of exchange rate behavior assumed bounded rationality, this paper is the first to our knowledge, which compares two of the most commonly used types of learning.

In order to facilitate the comparison of the two mechanisms we calibrated the parameters driving the dynamics in each of the learning mechanisms. More precisely, we chose the values of those parameters that could reproduce the well-known excess volatility puzzle.

Next, we analyzed a set of stylized facts in the foreign exchange markets. Those are: the long term convergence to the fundamental value, disconnect puzzle, excess and persistent volatility and unit root features.

Our results can be summarized as follows. First, we found that both learning mechanisms are efficient in revealing the fundamental value of the exchange rate in the steady state.

Second, only fitness learning creates the disconnection phenomenon. Under statistical learning the agents learn the fundamental equilibrium and therefore the market exchange rate moves close to it. Under fitness learning, the chartist' rule often dominates the market forecast and, as a result, we observe the disconnection.

It is well documented by the empirical literature that the exchange rate behaves as a random walk. As a result, it is impossible to reject the unit root hypothesis for the exchange rate series in level. A unit root test was performed on the series generated by the proposed model under two learning mechanisms. Both of them failed this test as we could not reject the unit root.

We also analyzed in detail the variability characteristics of the market exchange rate produced by the model under the two different learning mechanisms. We found that under both learning rules the exchange rate returns display persistence in volatility. Thus both learning mechanisms mimic the volatility clustering that is also observed in the data.

Overall, the fitness learning seems to be superior to statistical learning as it generates the exchange rate series that mimic better the data. In particular it produces the disconnection of the market rate from its fundamental, an important stylized fact. However, it still produces a predictable series, the fact which is in contradiction with the empirical evidence. One of the natural extensions would be to combine both mechanisms as in Branch and Evans (2007) and (2008).

Appendix

Equilibrium analysis

Equilibrium under fitness learning

We set the fundamental rate, $s_t^* = s^* = 0$. The exchange rate follows:

$$s_t = \alpha \left[\left(1 - \omega_{t-1}^f \psi + \omega_{t-1}^c \beta \right) s_{t-1} - \omega_{t-1}^c s_{t-2} \right] \quad (38)$$

Define $m_t \equiv \omega_{t-1}^f - \omega_{t-1}^c$ where ω_{t-1}^f is defined in equation (??). We can write:

$$m_t = \tanh \left[\frac{\delta}{2} \left(\pi_{t-1}^f - \pi_{t-1}^c \right) \right] \quad (39)$$

where profits are defined in following way:

$$\pi_{t-1}^f = (s_{t-1} - s_{t-2}) \operatorname{sgn} [-\psi s_{t-2}] \quad (40)$$

$$\pi_{t-1}^c = (s_{t-1} - s_{t-2}) \operatorname{sgn} [\beta(s_{t-2} - s_{t-3})] \quad (41)$$

Note that $\forall \operatorname{sgn}[x] = 1, 0, -1$, $m_t = 0$, so that $\omega_{t-1}^f = \omega_{t-1}^c = 0.5$. Substituting the weights into (38) and setting $s_t = s_{t-1} = s_{t-2} = \bar{s}$, we obtain:

$$\bar{s} = \frac{1}{2} \alpha (1 - \psi + \beta) \bar{s} \quad (42)$$

The first solution is obviously $\bar{s} = 0$ so that the equilibrium exchange rate equals its fundamental value. Note also that at the steady state:

$$\bar{s} = s^*, \bar{\omega}^c = \bar{\omega}^f = \frac{1}{2}, \bar{\pi}^f = \bar{\pi}^c = 0 \quad (43)$$

The second class of solutions from (42) exists for any $\bar{s} \neq 0$ if $\frac{1}{2} \alpha (1 - \psi + \beta) = 1$ but it implies instability. The unique fundamental solution is stable if $\frac{1}{2} \alpha (1 - \psi + \beta) \Rightarrow \alpha < \frac{2}{1 - \psi + \beta}$ and holds because $0 < \psi < 1$ and $0 < \beta < 1$ $\beta > \psi$ so that $1 > \beta - \psi > 0 \Rightarrow 2 > \frac{2}{1 - \psi + \beta} > 1$ and $\alpha < 1$ by definition since this is a discount factor so that $\alpha < \frac{2}{1 - \psi + \beta}$ holds and the fundamental equilibrium is stable.

Equilibrium under statistical learning

The agents' PLM is of the following form:

$$\tilde{\Delta}s_{t+1} = \psi_{t-1} (s_{t-1}^* - s_{t-1}) + \beta_{t-1} \Delta s_{t-1} + \varsigma_{t+1} \quad (44)$$

Accordingly, the agents form their expectations:

$$\hat{E}_t (\tilde{\Delta}s_{t+1}) = \psi_{t-1} (s_{t-1}^* - s_{t-1}) + \beta_{t-1} \Delta s_{t-1} \quad (45)$$

Given PLM and exchange rate equation (2), we have the following ALM:

$$s_t = (1 - \alpha)s_t^* + \alpha(1 + \beta - \psi)s_{t-1} - \alpha\beta s_{t-2} + \alpha\psi s_{t-1}^* + \eta_t \quad (46)$$

Using the ALM for s_{t-1} and the definition of the fundamental rate s_t^* in equation (3), we obtain the following specification of the market exchange rate:

$$\begin{aligned} s_t = & [(1 - \alpha) + \alpha\psi + \alpha(1 + \beta - \psi)(1 - \alpha)] s_{t-2}^* + \\ & [\alpha^2(1 + \beta - \psi) - \alpha\beta] s_{t-2} + \alpha^2\beta(1 + \beta - \psi)(s_{t-2} - s_{t-3}) + \\ & \alpha^2\psi(1 + \beta - \psi)(s_{t-2}^* - s_{t-2}) + [(1 - \alpha) + \alpha\psi + \alpha(1 + \beta - \psi)(1 - \alpha)] \epsilon_{t-1} + \\ & (1 - \alpha) \epsilon_t + [\alpha(1 + \beta - \psi)] \eta_{t-1} + \eta_t \end{aligned} \quad (47)$$

We subtract s_{t-2} from both sides and carrying out some manipulations and define T-map:

$$\mathbf{T} \begin{pmatrix} \psi \\ \beta \end{pmatrix} = \begin{pmatrix} (\alpha\psi - \alpha + 1)(\alpha + \alpha\beta - \alpha\psi + 1) \\ \alpha^2(1 + \beta - \psi)\beta \end{pmatrix} \quad (48)$$

From the system (29), we compute the stationary points of $\mathbf{T}(\psi, \beta)$. From the second equation of this system, we obtain two solutions for β i.e., $\beta_1 = 0$ or $\beta_2 = -1 + \psi + \frac{1}{\alpha^2}$. we calculate the resulting solutions for ψ , for each of the fixed points of β . When $\beta_1 = 0$, we have two possible solutions for $\psi_1 = 1$ or $\psi_2 = 1 - \frac{1}{\alpha^2}$. For $\beta_2 = -1 + \psi + \frac{1}{\alpha^2}$, we find $\psi_3 = 1 - \frac{1}{\alpha^2}$.

Substituting this result in β_2 , this yields $\beta_2 = 0$. As a result, we have two possible solutions. The first one is given by the combination $\phi_1 = \begin{pmatrix} 1 & 0 \end{pmatrix}$ and implies the RE equilibrium exchange rate process:

$$s_t = s_t^* + \eta_t - \alpha\epsilon_t \quad (49)$$

The second, bubble solution is given by $\phi_2 = \begin{pmatrix} 1 - \frac{1}{\alpha^2} & 0 \end{pmatrix}'$ and indicates that the agents again learn that extrapolating parameter to be zero ($\beta_2 = 0$) and a negative value of ψ_2 . In equilibrium, the current market exchange rate

is a sum of the fundamental rate and the extrapolated difference between past market and fundamental rates:

$$s_t = s_t^* + \frac{1}{\alpha} (s_{t-1} - s_{t-1}^*) - \alpha v_t + \eta_t \quad (50)$$

If we again assume that in the steady state $s_t^* = s_{t-1}^* = \bar{s}^* = 0$ and $\epsilon_t, \eta_t = 0$, we find that $s_t = 0$. We conclude that the only existing equilibrium is the one when the market exchange rate equals the fundamental rate.

We check the expectational stability (E-stability) of the two possible solutions by calculating the eigenvalues of $\mathbf{DT} - \mathbf{I}$. First, we compute the Jacobian matrix

$$\mathbf{DT} \begin{pmatrix} \psi \\ \beta \end{pmatrix} : \mathbf{DT} \begin{pmatrix} \psi \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha^2 (2(1-\psi) + \beta) & \alpha(\alpha\psi - \alpha + 1) \\ -\alpha^2\beta & \alpha^2(1 + 2\beta - \psi) \end{pmatrix}$$

We calculate the eigenvalues of the matrix $\mathbf{DT} - \mathbf{I}\lambda$ at the fixed points ϕ_1 and ϕ_2 . Evaluated at the the first solution ϕ_1 the eigenvalues are: $\lambda_1 = \lambda_2 = 0$ and thus the fundamental solution is E-stable. The bubble solution ϕ_2 yields the eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 1$ and thus is E-unstable.

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