

General Physics I

“Back of the Envelope” Problem – A Magnet Falling through A Metal Tube

An experiment is constructed to investigate the physics of a magnet falling through a pipe. The pipe has radius a , wall-thickness δ , length, L , electrical conductivity, σ and magnetic permeability, μ . The magnet has a dipole moment of M , and a mass m . The experimental configuration is shown in Figure 1. The magnet is placed at the top of the pipe and released.

In this problem, you will predict some of the findings of the experiment. Sufficient information is provided for you to solve this problem using “Back of the Envelope” physics and dimensional analysis techniques without having to rigorously solving all of the electrodynamics involved. You should neglect air resistance. Since this is a general physics question, not a second electromagnetism question, the following expressions are provided, although, as stated, you should not have to solve electromagnetic equations.

- The energy stored in a magnetic field in free space has a density $\epsilon = \frac{B^2}{2\mu_0}$
- The force \vec{F} on a magnetic dipole \vec{M} due to an external magnetic field \vec{B} is $\vec{F} = \nabla(\vec{M} \cdot \vec{B})$
- The current density \vec{J} and the electric field \vec{E} are related via the electrical conductivity σ by $\vec{J} = \sigma\vec{E}$
- The Lorentz force on a point charge q moving through an electric and magnetic field is $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$

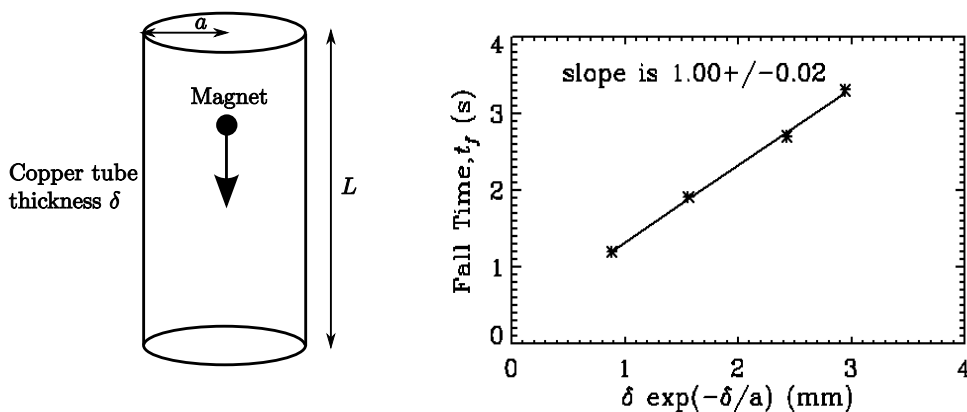


Figure 1: Schematic of the experimental setup. The data on the right were taken with 8 mm diameter tubes of varying wall-thickness, δ , with all other parameters held constant. The tubes were 25 cm long, and the magnet had a mass of 10 g.

- a) First describe qualitatively, the physical processes that are taking place as the magnet falls through the tube.
- b) The electromagnetic forces in the system cause an effective drag force $F = kv$ to be exerted on the magnet where v is the magnet velocity, and k is a force constant. Provided the tube is thin-walled:

$$k = C\delta \exp(-\delta/a)$$

where δ and a are the tube wall thickness and tube radius respectively, and C is a constant that is *not dimensionless*. Determine the dependence of C on the parameters of the system, up to a dimensionless constant.

- c) The data in Figure 1 were obtained with tube length $L = 25 \pm 0.1$ cm and a magnet mass of 10 ± 0.1 g. Determine C , with an appropriate uncertainty, from the data provided in Figure 1. You can assume that the magnet very quickly reaches its terminal velocity after it is released.
- d) In part *c*, the assumption is made that the magnet very quickly reach its terminal velocity once it is released from rest. Using the information provided, show that this is a good approximation provided that the length of the tube, L , satisfies:

$$L \gg \frac{m^2 g}{k^2}$$

General Physics I Solutions

“Back of the Envelope” Problem – A Magnet Falling through A Metal Tube

- a) As the magnet falls, the changing magnetic field induces an emf in the pipe wall, generating a current around the circumference of the pipe that depend on the rate of change of the field (the velocity of the magnet), the permeability, μ , of the pipe wall and the electrical conductivity, σ , of the pipe wall. The induced current in turn sets up a magnetic field within the pipe that exerts a force on the dipole moment of the magnet in the direction opposite to its direction of motion (i.e. upward).
- b) To use dimensional analysis effectively, we need to apply some knowledge of physics to the situation. For instance, we know that the origin of the drag force is the interaction of the magnetic field produced by the induced current in the wall of the tube with the magnetic dipole moment. Thus it makes sense to list the electrical and magnetic properties of the system and to write those variables in terms of their S.I dimensions (\mathbf{M} , \mathbf{L} , \mathbf{T} and \mathbf{I} where \mathbf{I} refers to current) and then to look for a group with dimensions equal to those of C .

Note since we are using current as our basic unit, we need the following dimensions:

- Magnetic field – $\mathbf{MT}^{-2}\mathbf{I}^{-1}$ (can get this from the provided equation for energy density in a magnetic field)
- Induced emf in Volts – $\mathbf{ML}^2\mathbf{T}^{-3}\mathbf{I}^{-1}$

The dimensionality of the parameters of the experiment are:

The function, C . From the equation of motion and the functional form of k , the dimensions of C must be $\mathbf{ML}^{-1}\mathbf{T}^{-1}$

Magnetic dipole moment, M . The student may remember the units as Joules/Tesla, or they can deduce this from the provided equation $\vec{F} = \nabla(\vec{M} \cdot \vec{B})$. In the SI unit set given above, the units of M are then $\mathbf{L}^2\mathbf{I}$.

Conductivity of the tube, σ . Units of σ are Amps/Volt-m which becomes $\mathbf{M}^{-1}\mathbf{L}^{-3}\mathbf{T}^3\mathbf{I}^2$

Permeability of the tube The units of permeability can be determined from the expression provided for the energy density of a magnetic field and yields the dimensions $\mathbf{MLT}^{-2}\mathbf{I}^{-2}$

We can also reason that the damping force should also depend on the tube radius, a , as both the strength of the induced current and the force exerted by the induced field on the magnet will decrease as a increases. This is in addition to the dimensionless quantity δ/a which appears in the exponential of the given equation for k .

So let's write:

$$C = \sigma^\alpha \mu^\beta M^\gamma a^\eta$$

Inserting the dimensions gives:

$$\mathbf{MT}^{-1}\mathbf{L}^{-1} = (\mathbf{M}^{-1}\mathbf{L}^{-3}\mathbf{T}^3\mathbf{I}^2)^\alpha \times (\mathbf{MLT}^{-2}\mathbf{I}^{-2})^\beta \times (\mathbf{L}^2\mathbf{I})^\gamma \times L^\eta$$

Collect together like terms:

$$1 = -\alpha + \beta \tag{1}$$

$$-1 = 3\alpha - 2\beta \tag{2}$$

$$0 = 2\alpha - 2\beta + \gamma \tag{3}$$

$$0 = -3\alpha + \beta + 2\gamma + 1 + \eta \tag{4}$$

which gives $\alpha = 1$, $\beta = 2$, $\gamma = 2$, $\eta = -4$ and so:

$$C \propto \frac{\mu^2 M^2 \sigma^2}{a^4}$$

where the proportionality constant is dimensionless. If for some reason the student forgets a and uses only the electrical and magnetic properties, they will immediately see a problem because the system of 4 equations is over-constrained and not soluble, hopefully prompting them to look for a length parameter in the system.

- c) Equating the drag force and the gravitational force gives the magnet terminal velocity $v_t = \frac{mg}{k}$. Then:

$$t_f = \frac{L}{v_t}$$

where L is the length of the tube. Therefore:

$$t_f = \frac{kL}{mg} = \frac{CL}{mg} \delta \exp(-\delta/a)$$

So C is related to the slope, A , by $C = mgA/L$. The slope is 1.00 ± 0.02 s/mm or 1000 ± 20 s/m. Using $m = 10.0$ g, $g = 10.0$ m/s², $L = 0.25$ gives:

$$C = \frac{1000.0 \times 0.01 \times 10.0}{0.25} = 400 \text{ kg m}^{-1} \text{ s}^{-1}$$

$$\frac{\Delta C^2}{C^2} = \frac{\Delta A^2}{A^2} + \frac{\Delta m}{m} + \frac{\Delta L}{L}$$

$$\Delta L/L = 0.4\% \quad \Delta C/C = 2.0\% \quad \Delta m/m = 1.0\%$$

Therefore:

$$\frac{\Delta C^2}{C^2} = 5.16 \times 10^{-4}$$

from which: $\Delta C/C = 2.3\%$

$$\text{So } C = 400 \pm 9 \text{ kg m}^{-1} \text{ s}^{-1}$$

- d) Start by writing down the full equation of motion in terms of the magnet velocity:

$$m\dot{v} = mg - kv$$

Rearrange and integrate from $t = 0$ to some arbitrary time:

$$\int_0^v \frac{1}{g(1 - kv/mg)} dv = \int_0^t dt$$

which gives:

$$v = v_t(1 - e^{-\frac{k}{m}t})$$

where $v_t = mg/k$ is the terminal velocity. So the time taken to reach terminal velocity is of order $t_t = m/k$ while the total fall time is $t_f \sim L/v_t = kL/mg$

So requiring $t_t \ll t_f$ gives:

$$L \gg \frac{m^2 g}{k^2}$$

which is the required answer since $k = C\delta \exp(-\delta/a)$.

General Physics II

On something hot and something cold

You should provide suitable estimates for any quantities that are not given in this question. If you don't have a calculator, or if you are short of time, an order of magnitude estimate based on a derived expression is adequate.

The thermal power (in W) emitted by a black body at an absolute temperature T is given by the Stefan-Boltzman law:

$$P = \sigma AT^4 \quad (1)$$

where the constant $\sigma = 5.7 \times 10^{-8} \text{Js}^{-1}\text{m}^{-2}\text{K}^{-4}$ and A is the area of the black body.

- a) Calculate the “solar constant” for the Earth which is the power per square meter striking the Earth's surface when the Sun is directly overhead. The radius of the Sun is $R_{sun} = 7 \times 10^5 \text{ km}$, its surface temperature is $T_{sun} = 5800 \text{ K}$ and the distance from the sun to the Earth is $D_{sun} = 1.5 \times 10^8 \text{ km}$.

Real objects are not perfect black-bodies and so Equation 1 for these objects becomes:

$$P = \sigma A\epsilon T^4 \quad (2)$$

where ϵ is the emissivity of the radiating object. The “absorbance” of the surface, which is the fraction of electromagnetic energy incident on a surface that is absorbed, is also very close to ϵ .

- b) The surface of your skin has $\epsilon \sim 1$. Suppose that you jump out of a space-ship airlock in deep space, far from the sun or other stars, and float naked in space (assuming of course that you are equipped with suitable breathing apparatus, and ignoring the effects of pressure drop that occurs). What is your maximum rate of energy loss? Neglect the effect of the spaceship – that is assume that the spaceship vanishes at the moment you leave the airlock.
- c) Repeat the calculation, assuming that you exit the airlock at a point that is the same distance from the sun as the earth, but in a much different part of the orbit from the earth.

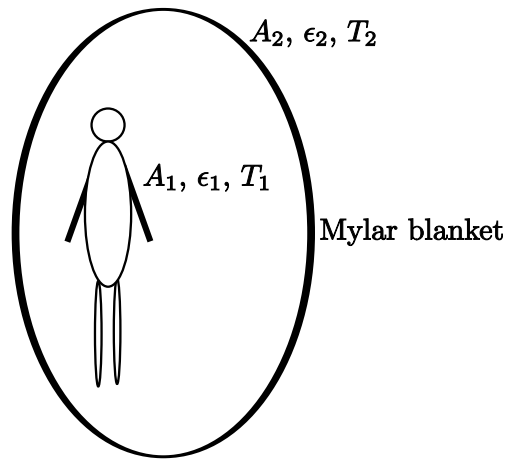
It should be clear from your answer that even if the effects of the vacuum on your skin could be ignored, the effect would still be uncomfortable whether you are in deep space, or being warmed by the sun! The rapid temperature changes that are implied by the previous calculations are prevented by using a very low-emissivity material for spacesuit and spacecraft insulation. This material is made with a Mylar® film coated on both sides with a very thin layer of aluminum (Al) and is similar to emergency blankets used on earth.

- d) Suppose that you jump out of the airlock in deep space but this time with a mylar emergency blanket that you wrap around you like a cocoon as sketched in the figure at the top of the next page. The blanket has a temperature T_2 , surface area, A_2 and emissivity ϵ_2 . Neglecting conduction, derive the expression for your power loss while wrapped in the blanket if your emissivity is ϵ_1 which is close to, but not quite 1 (assume in true physicist style that you are a cylinder).

You may wish to use the expression for an infinite geometric series:

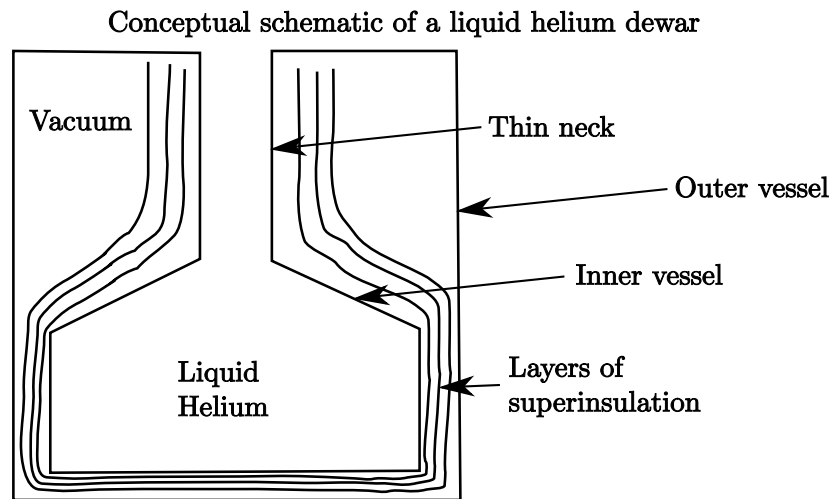
$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$$

where $r < 1$.



- e) To insure your comfort (i.e. to maintain your temperature as closely as possible) which of the two limiting cases is preferred? $A_2 \rightarrow \infty$ or $A_2 = A_1$? Explain your answer.
- f) Find an expression for the equilibrium value of the blanket temperature when $A_2 = A_1$.

Mylar blankets can keep hot things hot, or cold things cold. An example of the latter is a *Dewar* which is a thermally isolated container made by nesting two vessels inside one another and removing the air between them to prevent convection losses. The vessels are connected by a thin neck tube to reduce thermal exchange by conduction. High performance Dewars have many layers of mylar blanket nested in the vacuum space, as sketched in the figure below. Care is taken that successive layers do not make contact with one another, usually by interleaving gauze between them. This multi-layer insulation is often called “superinsulation”.



- h) Assuming that the two vessels and each of the blanket layers each have identical surface area and emissivity, ϵ , show that the effective emissivity of an n -layer super-insulated Dewar is given by:

$$\epsilon' = \frac{\epsilon}{n + 1}$$

You should neglect conduction between the vessels and you may assume that each vessel and each layer is isothermal.

General Physics II Solutions

- a) From the equation given, the power emitted by the Sun is $P_{sun} = 3.8 \times 10^{26}$ W. The Earth intercepts $1/4\pi r_{earth}^2 = 3.6 \times 10^{-24}$ of it, or $P_{earth} = 1400$ W/m² for the sun at the zenith.
- b) We approximate a person as a cylinder of radius 20 cm and height 170 cm. So the surface area of the person is about $A \approx 2$ m². Your temperature is $T_{you} \simeq 310$ K, and so you emit a power σT_{you}^4 per unit area and, at the same time, absorb, also per unit area a power $\sigma T_{Universe}^4$ where $T_{Universe} \simeq 4$ K. So the total power balance is given by

$$P_{total} = \sigma A(T_{you}^4 - T_{Universe}^4) \quad (1)$$

The convention we have used here is that outgoing power is positive. Since $T_{Universe}$ is so much smaller than T_{you} we will neglect the absorption term. Therefore $P_{total} \simeq 2 \times 5.7 \times 10^{-8} \times 310^4 = 1000$ W. Since your metabolism produces about 100 W clearly your temperature will decrease and you will quickly die of hypothermia.

- c) Assume that the sun illuminates half of your body – an area of 1 sq meter. Then the second term of equation 1 is not negligible and, in fact $P_{total} \simeq 1000 - 1400 = -400$ W. The negative implies that you are heating up.
- d) We assume the person to be a convex object (as it is in the case of the cylinder) and the cocoon to be concave as shown in the figure. In this configuration the person emits a power

$$P^{out} = \sigma \epsilon_1 A_1 T^4. \quad (2)$$

Of this, a fraction ϵ_2 is absorbed by the blanket, while the fraction $(1 - \epsilon_2)$ is reflected back. Of the power reflected back the fraction A_1/A_2 intercepts the person and is absorbed with a further efficiency of ϵ_1 . Hence the power that is reabsorbed, after one bounce, by the person is

$$P_1 = P^{out}(1 - \epsilon_2)\epsilon_1 \frac{V_1}{V_2}. \quad (3)$$

where the subscript refers to “bounce 1”. But, of course, the complement of this (i.e. replace ϵ_1 with $(1 - \epsilon_1)$) takes another round trip between you and the blanket so on, so you reabsorb a power from the n th bounce of:

$$P_n = P^{out} \epsilon_1 \gamma^n (1 - \epsilon_2)^n (1 - \epsilon_1)^{n-1} \quad (4)$$

where $\gamma = V_1/V_2$. So the total power balance for the person, that is the outgoing term in (2) (conventionally counted here as positive) minus the returning term in (3) and all the subsequent ones can be written as

$$P^{tot} = \sigma \epsilon_1 A_1 T_1^4 \left[1 - \epsilon_1 \sum_{n=1}^{\infty} \gamma^n (1 - \epsilon_2)^n (1 - \epsilon_1)^{n-1} \right] \quad (5)$$

This can be re-written as

$$P_1^{tot} = \sigma \epsilon_1 A_1 T_1^4 \left\{ 1 - \epsilon_1 (1 - \epsilon_2) \gamma \sum_{n=0}^{\infty} [\gamma (1 - \epsilon_2) (1 - \epsilon_1)]^n \right\} \quad (6)$$

and, summing the series

$$P_1^{tot} = \sigma \epsilon_1 A_1 T_1^4 \left[\frac{1 - \gamma + \gamma \epsilon_2}{1 - \gamma (1 - \epsilon_2) (1 - \epsilon_1)} \right] \quad (7)$$

- e) For $A_2 \rightarrow \infty$, that is, if the shroud grows very large with respect to the person, then $\gamma \rightarrow 0$ and equation (8) reduces to

$$P_1^{tot} = \sigma \epsilon_1 A_1 T_1^4 \quad (8)$$

that is the blanket has no effect on the power dissipation by the person. So the proper way to use the blanket is to wrap it around your body so that its area is as small as possible (still without touching it, to avoid conduction; this can be achieved by -say- a wool undergarment of some sort). The minimum size of the blanket is $A_2 = A_1$. In this case $\gamma = 1$ and the (8) becomes

$$P_1^{tot} = \sigma A_1 T_1^4 \left[\frac{\epsilon_1 \epsilon_2}{1 - (1 - \epsilon_2)(1 - \epsilon_1)} \right] \quad (9)$$

Hence, under these conditions the blanket simply changes the effective emissivity of your body by the factor

$$\frac{\epsilon_2}{1 - (1 - \epsilon_2)(1 - \epsilon_1)} \quad (10)$$

that is always < 1 . In particular, if $\epsilon_1 = 1$, that we have said is not such a bad approximation for your body, then

$$P_1^{tot} = \sigma A_1 T_1^4 \epsilon_2 \quad (11)$$

that is you emit power like you had the emissivity of the blanket. Finally note that if $\epsilon_2 = 0$ then you lose no power at all.

- f) The equilibrium temperature of the blanket, T_2 , can be calculated by imposing the condition that the power that it absorbs from the person is equal to the power it gives out to the outside. This is expressed by

$$\sigma \epsilon_2 A T_2^4 = \frac{\epsilon_1 \epsilon_2}{1 - (1 - \epsilon_1)(1 - \epsilon_2)} \sigma A T_1^4 \quad (12)$$

and

$$\boxed{T_2 = T_1 \left[\frac{\epsilon_1}{1 - (1 - \epsilon_1)(1 - \epsilon_2)} \right]^{1/4}} \quad (13)$$

- g) Let the temperature of the cold vessel be T_c and that of the outer vessel at room temperature be T_h . The temperature of the intermediate layers are T_1, T_2, \dots, T_n . If the system is in equilibrium (see part f above) the power flowing between adjacent layers must be identical. This can be written as:

$$\begin{aligned} P_1 &= \sigma \epsilon A (T_1^4 - T_c^4) \\ P_2 &= \sigma \epsilon A (T_2^4 - T_1^4) \\ P_3 &= \sigma \epsilon A (T_3^4 - T_2^4) \\ &\dots \\ &\dots \\ P_n &= \sigma \epsilon A (T_h^4 - T_n^4) \end{aligned} \quad (14)$$

and $P_1 = P_2 = \dots = P_n = P$. The total power flow is then obtained by adding the $n + 1$ equations together to give

$$(n + 1)P = \sigma \epsilon A (T_h^4 - T_c^4) \quad (15)$$

So:

$$P = \sigma A \frac{\epsilon}{n + 1} (T_h^4 - T_c^4) \quad (16)$$

and the n layers provide an effective emittance:

$$\boxed{\epsilon' = \epsilon / (n + 1)}$$

.

Quantum Physics I

A two state system interacts with a harmonic oscillator. The oscillator is described by ordinary creation and annihilation operators a^+ and a^- . The two state system is described by raising and lowering operators s^+ , s^- . The states of the oscillator are labeled in the usual way by $|n\rangle$. The two state system has states $|u\rangle$, $|d\rangle$. The operators s are defined by:

$$s^+|d\rangle = |u\rangle$$

$$s^-|u\rangle = |d\rangle$$

The Hamiltonian of the system is:

$$H = \omega \{a^+a^- + ms^+s^- + ga^+s^- + ga^-s^+\}$$

where ω , m , g are constants.

- a) Write down two quantities that are conserved in this problem.
- b) Find all the energy eigenvalues and eigenvectors.

Quantum Mechanics I Solutions

- a) Define $a^+a^- = N$ and $s^+s^- = S$. The conserved quantities are the total energy and the quantity $N + S$.
- b) We can first diagonalize $N + S$ and work in the subspace of fixed $N + S$. There are two states with given value of this quantum number, namely $|n, u\rangle$ and $|n + 1, d\rangle$ for each value of n . Within that subspace the Hamiltonian is a two by two matrix with entries $H_{ij} = \omega h_{ij}$

$$\begin{aligned} h_{11} &= n + m \\ h_{22} &= n + 1 \\ h_{12} &= h_{21} = g\sqrt{n + 1} \end{aligned} \tag{1}$$

Standard methods lead to eigenvalues

$$n + \frac{m + 1}{2} \pm \sqrt{\frac{1}{4}(m - 1)^2 + g^2(n + 1)} \tag{2}$$

To solve for the eigenvectors define

$$\frac{h_{12}}{(h_{11} - h_{22})} = \tan \theta$$

The eigenvectors are

$$\left(\cos \frac{\theta}{2}, \sin \frac{\theta}{2}\right)$$

and

$$\left(-\sin \frac{\theta}{2}, \cos \frac{\theta}{2}\right)$$

Quantum Physics II

Consider the three dimensional spherical harmonic oscillator described by the Hamiltonian

$$H_0 = \frac{|\vec{P}|^2}{2M} + \frac{K}{2}|\vec{r}|^2 \quad (1)$$

where \vec{P} and \vec{r} satisfy canonical commutation relations, $[r_a, P_b] = i\hbar\delta_{ab}$.

- a) It has a discrete spectrum of eigenvalues. What are the eigenenergies?
- b) While the ground-state of this Hamiltonian is unique, the excited states are all multiply degenerate. What is the degeneracy of the three lowest excited energy levels?

Some of these degeneracies are a consequence of the rotational symmetry of the Hamiltonian and others are “accidental degeneracies.” If we perturb the problem with with an arbitrary additional term which preserves rotational symmetry, we expect the degeneracies determined by symmetry to remain, but the accidental symmetries to be lifted. Consider the perturbed Hamiltonian

$$H = H_0 + \frac{\lambda}{4}|\vec{r}|^4 \quad (2)$$

where $0 < \lambda \ll \lambda_0 \equiv K^{3/2}M^{1/2}\hbar^{-1}$.

- c) What happens to the pattern of degeneracies of the lowest three excited states analyzed previously – which ones remain degenerate and for the ones that have their degeneracy lifted, what is the pattern of degeneracy that remains?
- d) What role did the condition $\lambda \ll \lambda_0$ play in this analysis, and where did λ_0 come from, anyway?
- e) For that matter, why is it necessary to specify λ positive? It would seem that perturbation theory should be applicable so long as $|\lambda| \ll \lambda_0$; what goes wrong for $0 > \lambda \gg -\lambda_0$?

Quantum Physics II Solutions

- a) $E = \hbar\omega[(3/2) + N]$ where $N = 0, 1, 2, \dots$ is a non-negative integer and $\omega = \sqrt{K/M}$. In fact, N is the sum of three non-negative integers, $N = n_x + n_y + n_z$.
- b) The degeneracies can be seen by inspection. For $N = 0$, the ground-state is, as advertised, non-degenerate. The $N = 1$ state is 3-fold degenerate, the $N = 2$ state is 6-fold degenerate, and the $N = 3$ state is 10-fold degenerate.
- c) Since the state with $N = 1$ has angular momentum $l = 1$, its degeneracy is dictated by symmetry, and remains intact. The $N = 2$ states consist of an $l = 2$ multiplet which is 5-fold degenerate and an $l = 0$ state which is non-degenerate, so this state is split into a 5-fold and a non-degenerate state. The $N = 3$ state splits into a 7-fold degenerate $l = 3$ multiplet and a 3-fold degenerate $l = 1$ multiplet.
- d) We can recast the problem so that it becomes dimensionless by rescaling $\vec{\mathcal{P}} = (\ell/\hbar)\vec{P}$, $\vec{\mathcal{R}} = (1/\ell)\vec{r}$ and $\mathcal{H} = (1/\hbar\omega)H$, so that $[\mathcal{R}_a, \mathcal{P}_b] = i\delta_{ab}$ and

$$\mathcal{H} = \frac{|\vec{\mathcal{P}}|^2}{2} + \frac{|\vec{\mathcal{R}}|^2}{2} + \frac{\lambda}{\lambda_0} \frac{|\vec{\mathcal{R}}|^2}{4}. \quad (1)$$

Here $\ell = \sqrt{\hbar/M\omega}$. We can see from this that λ/λ_0 is the dimensionless measure of the strength of the perturbation. So long as $\lambda \ll \lambda_0$, perturbation theory should be accurate, and hence the ordering of the states should be determined, to zeroth order, by the unperturbed Hamiltonian H_0 . If $\lambda \gg \lambda_0$, states that have much higher unperturbed energies can cross with states with lower unperturbed energies, leading to a complete rearrangement of the spectrum.

- e) If $\lambda < 0$, the potential is unbounded below, and hence the particle will tunnel away from the origin and be lost to infinity.

Special Relativity

A traveler is on a space-ship that moves according to the law $x^2 - c^2t^2 = R^2$, where R is some constant. The traveler feels the spaceship move with a constant acceleration a .

- a) Show that the acceleration $a = 1/R$ at $t = 0$. (For simplicity, you may want use the system of units where $c = 1$):
- b) Explain why the acceleration as measured by the traveller is the same at all times.

Bob is a space traveler. At the beginning of his journey, time $t = 0$, he leaves Earth, located at $x = 0$, flying away with constant acceleration $a = 1/R$ as defined above.

- c) Find a relation between x and t for Bob that is similar to the one discussed above. Draw the space-time diagram for his motion with constant acceleration.
- d) How much older will Bob will be at time t ? Depending on how you solve the problem, you may or may not need the following integral:

$$\int \frac{dy}{\sqrt{1+y^2}} = \sinh^{-1} y.$$

- e) Bob has heard a rumor that in the limit $a \rightarrow \infty$ one can reach other planets in no time at all. But this seems to contradict the fact than one cannot travel faster than light. Explain the true meaning of this statement.

Bob's final destination is a planet Boo, at a distance L from the Earth. If he constantly accelerates, he will crash into the planet. Therefore he accelerates only during the first half of his journey from Earth towards the planet and he decelerates during the second half of the journey. Then he immediately flies back to Earth, again accelerating during the first half of the journey and decelerating during the second half. The accelerations and decelerations have the same magnitude, a , and are constant over the relevant period.

- f) Sketch the space-time diagram of his round trip to the planet, including the stages of deceleration, and the return to Earth.
- g) In the brief moment that Bob is at the planet Boo he receives a phone call from his brother who has stayed behind on Earth. Using the space-time diagram, and assuming that $La \gg 1$, determine when his brother made the phone call.
- h) If Bob were to accelerate very quickly and fly back and forth to Boo with speed equal to the speed of light, he will return within the time $2L/c$, from the point of view of an observer on Earth. Using the space-time diagram, find how much longer the travel actually takes from the point of view of an observer on Earth if Bob travels with acceleration $\pm a$, as explained above, assuming that $La \gg 1$.

Special Relativity Solutions

a) Differentiate x twice with respect to t .

$$\frac{d^2x}{dt^2} = \frac{R^2}{(R^2 + c^2t^3)^{3/2}}$$

b) Set $t = 0$:

$$a = \frac{1}{R}$$

c) Because of the Lorentz invariance, one can always change the coordinate system to make $t = 0$.

d) $(x + a^{-1})^2 - t^2 = a^{-2}$

e) $\tau = a^{-1} \sinh^{-1}(at)$

f) It is the proper time τ that goes to zero as $a \rightarrow \infty$.

h) $t = \frac{1}{2a}$

i) $t = 2L + 4a^{-1}$

Statistical Mechanics

Consider a classical statistical mechanics system consisting of N subsystems labelled by $i = 1 \dots N$, each of which can exist in two states $s_i = \pm 1$. Let the total energy of the system be given by

$$E = -J \sum_{i=1}^N s_i \quad (1)$$

- a) Let the energy of the system be fixed at $E_0 = N\epsilon$ where $-J < \epsilon < 0$ (microcanonical ensemble). Compute the entropy of the system assuming N is large.
- b) Focus on a particular subsystem, $i = 1$. Compute the probability that $s_1 = +1$ divided by the probability that $s_1 = -1$, directly in the microcanonical ensemble, again assuming that N is large.
- c) Now suppose that the entire system is used as a heat bath for the subsystem considered in part (b). What temperature does the entire system have?
- d) Repeat part (c) above, treating subsystems $s_2 \dots s_N$ as a heat bath for system s_1 and then working in the canonical ensemble. Are your answers consistent?
- e) Give a qualitative sketch of the temperature as a function of energy. Note a peculiarity in the system for $E_0 > 0$.

You may find Sterling's approximation useful:

$$\log n! \sim n \log n - n$$

Statistical Mechanics Solutions

- a) Let the number of + values be n_+ , the number of - values be n_- . $N = n_+ + n_-$, $E_0 = -J(n_+ - n_-)$. The number of configurations is

$$\Omega = \frac{N!}{(n_+)!(N - n_+)!}$$

Use Stirling's formula when N, n_+, n_- are large.

$$\log \Omega = N \log N - n_+ \log n_+ - (N - n_+) \log(N - n_+)$$

Rewrite using $x = n_+/N$ as

$$\log \Omega = -N(x \log x + (1 - x) \log(1 - x)) = -Nf(x)$$

The entropy is given by $S = k_B \log \Omega$. The variable x is determined by

$$x = \frac{1}{2}(1 + E_0/(NJ))$$

- b)

$$r = P(+1)/P(-1) = \Omega(N - 1, n_+ - 1)/\Omega(N - 1, n_+)$$

where n_+ refers to the number of up values in the original system. Defining $y = n_+/(N - 1)$ which approaches x at large N we have

$$\log r = -(N - 1)(f(y - \frac{1}{N - 1}) - f(y))$$

Expanding for large N (dropping terms $\sim 1/N$) yields

$$\log r = df(x)/dx = \log x - \log(1 - x)$$

or

$$r = x/(1 - x)$$

- c) $\beta = 1/k_B T$ is defined to be $\beta = -\partial S/\partial E$. Using the above this translates into

$$\beta = (1/(2NJ))\partial(\log \Omega)/\partial x$$

or

$$\beta = (-1/2J)(\log x - \log(1 - x))$$

- d) In the canonical ensemble

$$r = P(+1)/P(-1) = \frac{\exp(-\beta J(+1))}{\exp(-\beta J(-1))} = \exp(-2\beta J)$$

or

$$r = x/(1 - x)$$

agreeing with the result of part 3.

- e) The temperature goes to infinity as E_0 passes through zero, then goes negative.

Electromagnetism

You may find the following values useful:

$$\epsilon_0 = 8.85 \times 10^{-12} \text{ C}^2/\text{Nm}^2$$

$$\mu_0 = 4\pi \times 10^{-7} \text{ N/A}^2$$

$$c = 3.00 \times 10^8 \text{ m/s}$$

This problem studies the behavior of electromagnetic waves with angular frequency ω in a conducting medium, where the free-current density is $\mathbf{j}_f = \sigma \mathbf{E}$, with conductivity σ . All equations should be given in S.I. units. For simplicity, assume that the dielectric permittivity is ϵ_0 and the magnetic permeability is μ_0

- a) Use Gauss's law and the conservation of charge to show that any free-charge density ρ_f decays in a time $\tau = \epsilon_0/\sigma$. Take $\sigma = 10^6 \text{ (ohm-m)}^{-1}$, typical of a nonmagnetic metal, and evaluate this decay time τ . For waves with $\omega\tau \ll 1$, why can you assume that $\rho_f = 0$?
- b) Write down the remaining three Maxwell's equations relating \mathbf{E} and \mathbf{B} . Combine them to obtain the following modified wave equation for \mathbf{E} :

$$\nabla^2 \mathbf{E} = \mu_0 \sigma \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}, \quad (1)$$

and find the corresponding equation for \mathbf{B} . Discuss the physical interpretation of the two terms on the right-hand side.

- c) Study a plane wave of the form $\mathbf{E}(z, t) = E_0 \hat{\mathbf{x}} e^{i(kz - \omega t)}$. Find the detailed form of the dispersion relation between k and ω . Throughout the rest of the problem, assume $\omega\tau \ll 1$. Show that $k = (1 + i)/\delta$, and find the explicit form for the "skin depth" δ . Discuss the form of the propagating electromagnetic wave in the conducting medium. Discuss how δ varies with the frequency ω . Take $\omega = 3 \times 10^{10} \text{ rad/s}$, which is a typical microwave frequency, and use σ from (a). Evaluate δ and also find the wavelength λ_v of this microwave wave if it were in vacuum. Find the ratio δ/λ_v .
- d) Use Maxwell's equations to find the corresponding propagating magnetic field $\mathbf{B}(z, t) = B_0 \hat{\mathbf{y}} e^{i(kz - \omega t)}$ and the corresponding ratio cB_0/E_0 . Discuss how this ratio differs from the familiar result for a plane wave in vacuum.
- e) Study the reflection and transmission of a plane wave in vacuum ($z < 0$) that it incident normally on the semi-infinite metallic region ($z > 0$). Show that the ratio of the reflected amplitude E_r to the incident amplitude E_i is $E_r/E_i = (k - k')/(k + k') \approx -1 + 2k/k'$, where k is the wave number in vacuum and k' is the corresponding complex wave number in the metal [given in (c) above in terms of the skin depth δ]. Discuss the physics of this result, especially the conservation of energy.

Electromagnetism Solutions

- a) For this simple model, Gauss's law says $\nabla \cdot \mathbf{E} = \rho_f/\epsilon_0$, and the conservation of charge says $\nabla \cdot \mathbf{j}_f = \sigma \nabla \cdot \mathbf{E} = -\partial \rho_f/\partial t$. Combining these two equations yields $\partial \rho_f/\partial t = -\rho_f/\tau$, where $\tau = \epsilon_0/\sigma$. Hence any free charge density decays exponentially in a time $\approx \tau$. With the numbers given, τ is of order 10^{-17} s, and $\omega\tau \ll 1$ for visible light and all longer wavelengths.
- b) For this conducting medium, Maxwell's equations are $\nabla \cdot \mathbf{E} = 0$ (no free-charge density), $\nabla \cdot \mathbf{B} = 0$ (no magnetic monopoles), $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$ (Faraday's law), and $\nabla \times \mathbf{B} = \mu_0 \mathbf{j}_f + \mu_0 \epsilon_0 \partial \mathbf{E}/\partial t$ (Ampère's law). The two terms on the right are the conduction current and the displacement current, respectively.

Take the curl of Faraday's law and use Ampère's law plus a vector identity to obtain the specified wave equation for \mathbf{E} (and similarly for \mathbf{B} , which obeys the same equation). The term involving σ arises from the conduction current and will dominate over the displacement current in the case considered here of a good conductor with $\omega\tau \ll 1$. This arises because the displacement current involves a second time derivative, whereas the conduction current involves only a single time derivative. The resulting equation is effectively a diffusion equation (or that for temperature in a good thermal conductor).

- c) The wave equation for \mathbf{E} has plane-wave solutions as indicated, with the dispersion relation $k^2 = i\omega\mu_0\sigma + \omega^2\epsilon_0\mu_0$. If $\omega\tau \ll 1$, the second term (the displacement current) is negligible, and the conduction current dominates. To find the form of the wave number, take the square root of $i = e^{i\pi/2}$ which is $\pm(1+i)/\sqrt{2}$, and take the plus sign. In this way, $k = (1+i)/\delta$, where $\delta = \sqrt{2/(\mu_0\sigma\omega)}$.

Thus the wave has two factors, $E(z, t) \propto \exp[i(z/\delta - \omega t)] \exp(-z/\delta)$. Consider a snapshot at fixed time. The first factor is a wave with wavelength $2\pi\delta$ as its periodicity length. In contrast, the second factor decreases very rapidly and exponentially attenuates by a factor $e^{-2\pi}$ in this same periodicity length. Thus the amplitude is highly confined in a surface layer of width of order the skin depth δ . Note that δ varies with frequency like $\omega^{-1/2}$. For $\omega = 3 \times 10^{10}$ rad/sec and $\sigma = 10^6$ (ohm-m) $^{-1}$, this gives $\delta = 8 \mu\text{m}$. For comparison, a plane wave in free space with the same microwave frequency would have a vacuum wavelength $\lambda_v = 6$ cm, so that δ/λ_v is of order 10^{-4} .

- d) Faraday's law readily yields the ratio of the amplitude B_0 of the \mathbf{B} field (polarized along $\hat{\mathbf{y}}$) to E_0 : $cB_0/E_0 = c(1+i)/(\omega\delta) = \lambda_v(1+i)/(2\pi\delta)$, which is very large for a typical metallic conductivity. As a result the magnetic effects totally dominate the physics. In addition, B_0 is 45° out of phase relative to E_0 . For a plane wave in vacuum, the ratio is simply $cB_0/E_0 = ck/\omega = 1$, with both \mathbf{E} and \mathbf{B} in phase.
- e) The electric field has three amplitudes all polarized along $\hat{\mathbf{x}}$

$$\text{incident:)} E_i e^{i(kz - \omega t)}; \quad \text{reflected:)} E_r e^{i(-kz - \omega t)}; \quad \text{and transmitted:)} E_t e^{i(k'z - \omega t)}, \quad (1)$$

where $k = \omega/c$ and $k' = (1+i)/\delta$. The tangential component of \mathbf{E} is continuous so that $E_i + E_r = E_t$. Similarly, the tangential component of \mathbf{H} (along $\hat{\mathbf{y}}$) is continuous, and Faraday's law gives

$$\text{incident:)} H_i = kE_i/(\mu_0\omega); \quad \text{reflected:)} H_r = -kE_r/(\mu_0\omega); \quad \text{and transmitted:)} H_t = k'E_t/(\mu_0\omega). \quad (2)$$

When expressed in terms of the E amplitudes, we have $E_i - E_r = k'E_t/k$, and some algebra yields

$$\frac{E_r}{E_i} = \frac{k - k'}{k + k'} \approx -1 + \frac{2k}{k'}, \quad (3)$$

where the last form arises because k/k' is small of order δ/λ_v .

Note that the reflection coefficient $R = |E_r/E_i|^2$ is not unity because of the small (complex) correction. Still most of the energy is reflected (we know that a silvered mirror reflects most of the energy). The remaining small amount of energy is dissipated in the metal in the surface layer of order δ in thickness.

Classical Mechanics

Klingon engineers have constructed a fiendish trap for unsuspecting, passing spaceships. It consists of a spherical attractive well that can be approximated by the potential

$$V(r) = \frac{-K^2 m}{r^4}$$

where m is the mass of the spaceship (assumed to be much smaller and lighter than the trap).

- a) Show that the period of a circular orbit of radius r is $\pi r^3/K$ but that circular orbits are unstable as the Klingon engineers had anticipated.
- b) The Starship Enterprise approaches the trap with speed v and shuts down its engines in order to avoid detection. What is Captain Kirk's minimum prudent impact parameter in order to avoid capture?
- c) Now assume that the trap is moving through space at a constant velocity. It acts as a cosmic "vacuum cleaner", sweeping up interstellar dust which can be regarded as having uniform density ρ and negligible random motion. Calculate the dust mass collection rate.

Classical Mechanics Solutions

- a) The effective potential per unit mass is

$$V_{\text{eff}}(r) = \epsilon - \frac{\dot{r}^2}{2} = \frac{\ell^2}{2r^2} - \frac{K^2}{r^4},$$

where ϵ is the energy per unit mass and ℓ is the angular momentum per unit mass. The circular orbit has $dV_{\text{eff}}/dr = 0$ or $r_{\text{circ}} = 2K/\ell$. Hence the period is $P = 2\pi r^2/\ell = \pi r^3/K$. V_{eff} has a maximum at r_{circ} and so the equilibrium is unstable.

- b) A spaceship with $\epsilon < V_{\text{circ}} = \ell^4/16K^2$ will be repelled by the effective potential and so capture will be avoided if

$$\frac{1}{2}v^2 = \epsilon < V_{\text{circ}} = \frac{\ell^4}{16K^2} = \frac{v^4 b^4}{16K^2}$$

where b is the impact parameter. Hence

$$b > \left(\frac{8K^2}{v^2} \right)^{1/4}$$

to avoid capture.

- c) Just treat the object as having a scattering cross-section of b . Then the mass capture rate is $\pi b^2 \times \rho v = 2\sqrt{2}\pi K \rho$.