QUALIFIER 2004–5 QUANTUM MECHANICS QUESTION SOLUTION

Use $\sigma_i\sigma_j = i\ \epsilon_{ijk}\sigma_k + \delta_{ij}I$.

- To find the eigenvalues of A, B, C, D square each operator. For example,
$$A^2 = (\sigma_x^1)^2(\sigma_y^2)^2(\sigma_y^3)^2 = (I)^3 = I$$
Others work similarly. So the eigenvalues of all operators are ± 1.

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$$\begin{aligned} ABCD &= (\sigma_x^1\sigma_y^1\sigma_y^1\sigma_x^1)((\sigma_y^2\sigma_x^2\sigma_y^2\sigma_x^2)(\sigma_y^3\sigma_y^3\sigma_x^3\sigma_x^3) \\ &= (I)((-i)^2I)(I) \\ &= -I \end{aligned}$$

- The first four equations are just the statements $A^2 = B^2 = C^2 = D^2 = I$. The fifth was derived above.

  Now write the equations on values:
$$\begin{aligned} 1 &= v(A)v(\sigma_x^1)v(\sigma_y^2)v(\sigma_y^3) \\ 1 &= v(B)v(\sigma_y^1)v(\sigma_x^2)v(\sigma_y^3) \\ 1 &= v(C)v(\sigma_y^1)v(\sigma_y^2)v(\sigma_x^3) \\ 1 &= v(D)v(\sigma_x^1)v(\sigma_x^2)v(\sigma_x^3) \\ -1 &= v(A)v(B)v(C)v(D) \end{aligned}$$

  Now multiply all the LHS and all the RHS together. The LHS product gives -1. Each operator occurs twice on the RHS and $v = \pm 1$. So the RHS product gives $+1$. Hence we arrive at the equation $-1 = +1$, a contradiction. No value function v can exist.

This is the simplest known proof of the Kochen-Specker theorem, one of the classic arguments against hidden variable theories. It was found by N. D. Mermin in 1990. See N.D. Mermin, Phys. Rev. Lett. (65) 1990, 3373.