2005

Solution – General Physics II

a) This is a slight modification of the 'rocket equation' of elementary mechanics. We know that during the burn, the mass is $m/m_0 = 1 - (2/3)(t/90s) = 1 - t/135s$. Since the burn rate is constant we have $\dot{m}/m = 1/(t - 135s)$ which means that the acceleration during the burn is $a(t) = -g - v_{ex}\dot{m}/m = -g - v_{ex}/(t - 135s)$. Integrating, we get $v(t) = -gt - v_{ex}\ln(1 - t/135s)$. So we start at altitude h_0, we rise $\int_0^{t_b} v(t)dt$ during the burn and then we coast an additional $v(t_b)^2/(2g)$. Altogether this is

$$h = h_0 + [g\ t_b + v_{ex}\ln(1 - t_b/135s)]^2/2g - g\ t_b^2/2 + v_{ex}(135s - t_b)\ln(1 - t_b/135s) + v_{ex}t_b$$

$$h = h_0 + v_{ex}t_b + 135s\ v_{ex}\ln(1 - t_b/135s) + v_{ex}^2\ln^2(1 - t_b/135s)/2g$$

which is only somewhat horrible.

b) We now need to solve for v_{ex} giving h=10^5m for a 75s burn. Re-arrange as a quadratic equation

$$v_{ex}^2[\ln^2(1 - t_b/135s)/2g] + v_{ex}[135s\ v_{ex}\ln(1 - t_b/135s) + t_b] + [h_0 - h] = 0$$

We want a numerical answer for $t_b = 75$s, $h - h_0 = 85$km so the coefficients are

$$v_{ex}^2[0.03355\text{s}^2/\text{m}] + v_{ex}[-34.476\text{s}] + [-8.5 \times 10^4\text{m}] = 0$$

The + root of the solution is physical, giving an exhaust speed of 2186m/s.

c) Well, this is just plugging into a) with v_{ex} from b) and $t_b = 90$s. Interestingly, we find $h(t_b = 90s) = 1.82 \times 10^5$m=182km or nearly 2× as high. That last little bit of burn against the lightest m contributes the most Δp.

d) Since subsonic blunt object drag force goes as ρv^2, our problem is to estimate the density scale height of the atmosphere. We are assuming an isothermal N_2 atmosphere with $\langle m \rangle = 28m_p$ and $T = 300$K. So $P = \rho kT/\langle m \rangle$ and hydrostatic equilibrium is

$$dP/dz = (kT/\langle m \rangle)d\rho/dz = -\rho g.$$

Thus $\rho = \rho_0 e^{-(g\langle m \rangle/kT)z}$. Plugging in, we get $\rho = \rho_0 e^{-z/9030}$, which is reasonable. So with a scale height of 9km, at 15km, the atmospheric density and drag force goes down to $e^{-15/9} \approx 0.18\times$ the sea level value. Actually, the atmospheric temperature increases above the tropopause and the true drag force is down by nearly another factor of two at 15km.