2005 E + M

Answers

1) a) To satisfy the conditions of $E_\parallel = 0$ and $dE_\perp/dn = 0$ at conducting surfaces, E_z must be $E_z = E_0 \sin(n\pi x/2a)\sin(m\pi y/a)\cos(p\pi z/L)\sin(\omega t+\phi)$, where n, m, and p are arbitrary integers and ϕ is an arbitrary phase factor. To satisfy the condition that $E_z = 0$ at x = a and y = a/2, at least one of n and m must be even. Maxwell's equations require $\nabla^2 E_z + (\omega/c)^2 = 0$, yielding $\left[(n/2a)^2 + (m/a)^2 + (p/L)^2\right] = (\omega_{nmp}/\pi c)^2$. The lowest frequency will occur for the smallest values of n, m, and p. To avoid a trivial solution, only p can be zero. The smallest legitimate solution occurs when n = 2, m =1, and p = 0. Thus, the lowest frequency is

$f_{210} = \omega_{210}/2\pi = (c/2)\left[(2/2a)^2 + (1/a)^2\right]^{1/2} = c/a\sqrt{2}$.

Since $a(cm) = \sqrt{4.5} = 3/\sqrt{2}$, f_{210} = (3 10^{10} cm/sec)/3cm = 10 GHz.

b) From Maxwell's equation $\nabla \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$, the electric field can have no transverse components. This is because B_z = zero by definition, and there are no field variations in the z direction because p was chosen to be zero above. From $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$, $\frac{\partial B_x}{\partial t} = -\frac{\partial E_z}{\partial y}$ and $\frac{\partial B_y}{\partial t} = \frac{\partial E_z}{\partial x}$. These lead to:

$$E_x = E_y = B_z = 0$$

$$E_z = E_0 \sin(\pi x/a)\sin(\pi y/a)\sin(\omega_{210}t+\phi)$$

$$B_x = \frac{\pi}{a\omega_{210}} E_0 \sin(\pi x/a)\cos(\pi y/a)\cos(\omega_{210}t+\phi)$$

$$B_y = -\frac{\pi}{a\omega_{210}} E_0 \cos(\pi x/a)\sin(\pi y/a)\cos(\omega_{210}t+\phi)$$

2) An electron can only exchange energy with the cavity field if sees an electric field. Thus it must go through with x ? 0. Furthermore, if all of the electrons go through at the same x, just as many will see one sign of the oscillating field as will see the other. Thus the average exchange will be zero. The first pass beam will be deflected by B_y and this deflection will be converted to a displacement on the second pass. Since B_y is oscillating at ω, x on the second pass will also. Depending on the time taken for an electron to return to the cavity, the average energy exchange can be positive, negative, or zero. Let's assume the exchange is positive-that is, the beam gives energy to the cavity. The rate of energy exchange is power delivered to the cavity by the beam (P_{beam}) and is clearly proportional to the product of B and E, it is also proportional to the energy (U) stored in the cavity. It also must be proportional to the number of electrons interacting with the fields/unit time, or the current I. We can write P_{beam} = K U I, where K is a constant representing the strength of coupling between the cavity fields and the beam. Losses in the cavity (P_{cavity}) dissipate the stored energy while the beam power (P_{beam}) adds energy.

$\frac{dU}{dt} = P_{beam} - P_{cavity} = KIU - \frac{\omega U}{Q} = U(KI - \omega/Q)$. Therefore $U = U_0 e^{(KI-\omega/Q)t}$ $U = U_0 e^{(KI-\omega/Q)t}$

Obviously, $I_{threshold}$ = ω/KQ