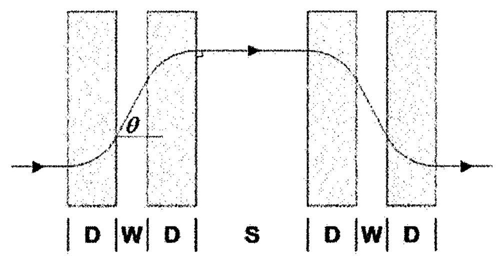
General Physics #1

1) An electron with kinetic energy T travels a distance L in a field-free region.

a) If the electron is ultrarelativistic, and it takes 100 ns ($= 10^{-7}$ s) for the electron to cover the distance, what is L?

b) Another electron, with a kinetic energy 1% less than T, travels the same path. Estimate the difference in travel time if (i) $T = 10$ keV, (ii) $T = 10$ GeV. (In the spirit of estimating the answer, feel free to use 500 keV/c^2 as the rest mass of the electron.)

2) For this part of the problem, the electron with kinetic energy T travels the same distance L as above, but now it encounters four rectangular bending magnets arranged as shown in the figure. The four magnets are identical in all respects (dimensions and field strength) and can be considered "hard-edged" (i.e., the field is uniform inside the magnets and zero outside.) The electron's trajectory (indicated in the figure) is in the plane of the paper and in the mid-plane of the magnets.



a) If the polarity of the first magnet is assumed to be +, what are the polarities of the other three magnets? (+, ?, ?, ?)

b) Suppose that the *momentum* of the electron is 100 MeV/c and that the magnet width D is 1 m. What is the strength of the magnetic field if the bend angle θ is 30°?

c) Electrons with different kinetic energies will follow different paths through the magnets, and the paths can be expected to have different lengths. Calculate the first-order path length difference with momentum as a function of D, W, S, and θ as shown in the figure. (Don't worry about making the path length equal to L.)

d) Now make the length of the trajectory equal to L, but simplify by setting $S = 0$, and take the limit as D becomes arbitrarily small. Assume that $\theta = 30°$. If two electrons differ in energy by 1% at 10 GeV, calculate the difference in their transit times through the four-magnet system.

Mechanics: The Dark Side of Gravity

A. Imagine that you are studying planar orbits in a central potential $V(\mathbf{r}) = V(r)$. Write down the Lagrangian for a particle of mass m in terms of the dynamical variables $r(t)$, $\theta(t)$ describing the polar coordinates of the particle in the plane, with center at the origin of the force. Assuming that the angular momentum is ℓ, write down the energy as a function of $r(t)$ and $\dot{r}(t)$.

Observations of the orbital velocities of stars or gas clouds at different distances from the galactic center have revealed that in most galaxies, the velocities are constant over some substantial range of radial distances from the galactic center. For the rest of this question, you may assume that the "galaxy" has some spherically symmetric mass distribution and that the stars move in circular orbits.

B. Is this observed behavior what you would expect in a galaxy in which most of the mass is concentrated at the center? How does the velocity behave as a function of the radius of the orbit in such a galaxy, assuming that the radius is outside the concentration of mass at the center?

C. Use the observed behavior to infer the distribution of mass $M(r)$ in the galaxy over the regime in which the constant velocities are observed. Here $M(r)$ denotes the total mass contained within the radius r.

D. Certain spiral galaxies have bulges (protrusions) within which the observed velocities behave like $v(r) \sim r$. Given only this information, what is a simple guess for the mass per unit volume $\rho(r)$ within the bulge portion of such a galaxy?

Quantum Mechanics #1

A reasonable model for a diatomic molecule (in which the vibrational degrees of freedom of the molecule are ignored) is a rigid rotator with moment of inertia I.

(i) Using this model, state the energy eigenvalues and eigenfunctions for this system. (An explicit representation is not required.) What is the degeneracy of each energy level?

(ii) Suppose the molecule is a polar molecule with a magnetic moment given by $\mu_0 \mathbf{L}$, where $\mathbf{L}$ is the angular momentum of the molecule. Find the energy eigenvalues and eigenfunctions when the molecule is placed in a uniform magnetic field $\mathbf{B}$.

(iii) Suppose instead that the molecule has a permanent electric dipole moment $\mathbf{d}$. Using perturbation theory, calculate the energy levels of the molecule in the presence of a uniform electric field $\mathbf{E}$ *to first order in* $\mathbf{E}$. Does the perturbation separate the degenerate eigenstates?

Special Relativity: Time Dilation in a Gravitational Field

Special relativity allows us to understand many features of the more complicated science, general relativity. How about understanding the physics of black holes?

Perhaps you know that if you throw your friend toward a black hole, you will never see him reach the black hole: time "stops" when he approaches the "event horizon." (Well, time does not stop for him—he will fall to the singularity—but you will never see him fall through the event horizon.)

1)

a) To understand what happens, consider first a clock on a ceiling in a room on Earth. The clock works by emitting light with a frequency f_0. The time we see is proportional to the number of oscillations. What is the frequency f of the light that you see standing on the floor a distance h below the clock? How does it affect the time shown by the clock?

Hint: Use the Principle of Equivalence. It says that an observer in a gravitational field with gravitational acceleration g will see everything as if he were accelerating with acceleration $-g$ in the absence of any gravitational field. (You feel "heavy" in a rapidly accelerating elevator, and you feel weightless in a freely falling elevator.) Using this principle, imagine that you are accelerating towards the ceiling, and calculate the frequency f of the light as you see it. When the wavefront emitted by the clock in this imaginary frame hits you, you will be moving toward it with some speed v. This results in a Doppler shift. Perform your calculations to first order in v/c.

b) What will happen if the clock is below you, in a well of a depth h?

2) On basis of this discovery, one may have an idea to keep Sleeping Beauty young forever by lowering her down into a very deep well. How deep would such a well have to be so that we would never see her growing older?

3) If you are attracted by a star, the gravitational acceleration g is a function of distance r from the star, and the potential gh (equal to potential energy mgh per unit mass m) is replaced by the gravitational energy $GM(\frac{1}{r_1} - \frac{1}{r_0})$. Assume that you are an observer infinitely far away from the star ($r_1 \to \infty$) and that the clock emitting light with a frequency f_0 is at a distance $r_0 = r$ from the star. Find the value of r such that the frequency f that you measure becomes zero (i.e., "time stops"). This is the event horizon. (In this calculation, we will miss a factor of 2 because our approximation $v/c \ll 1$ fails near the event horizon.)

General Physics #2

A system of N classical electrons (i.e., they are spinless and do not Coulomb-interact with each other) is confined to a cubic box of size L. The sides of the cube are oriented along the X, Y and Z axes, and the center of the box is at the origin of the coordinate system. There is a magnetic field of strength B in the Z direction. The box of electrons is thermally isolated and at temperature T.

a) Describe the motion of a typical electron at different positions in the box. Do this for electrons both near and far from the walls of the box.

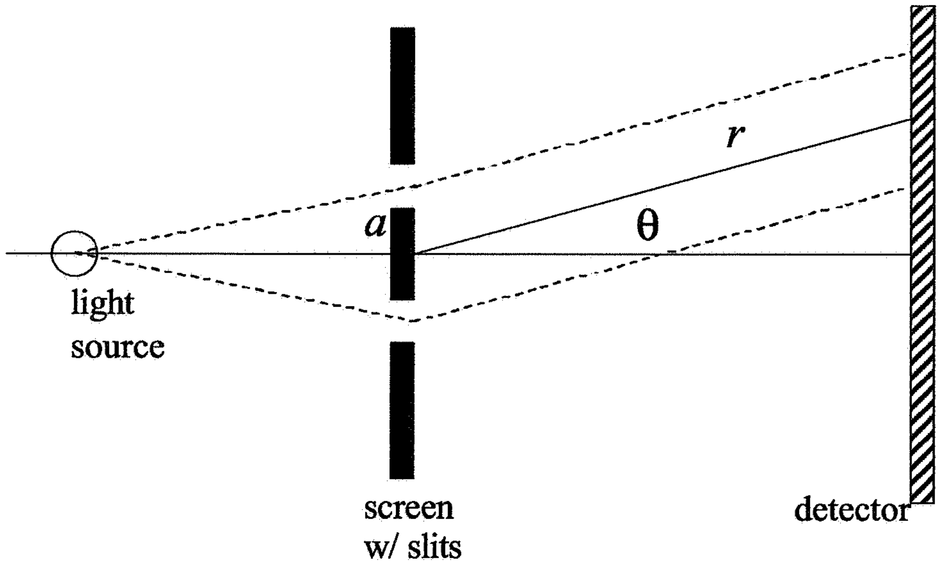
b) Compute the total energy of the electrons.

c) What is the magnetic moment of the system of electrons?

d) Replace the magnetic field with an electric field of strength E, also along the Z axis. Compute the energy and the electric dipole moment of the system.

Quantum Mechanics #2

Consider a two-slit diffraction experiment. A light source emits unpolarized light, which passes through two slits separated by $2a$ and is then detected.



<u>Part I</u>

Derive the expression for the intensity of detected light:

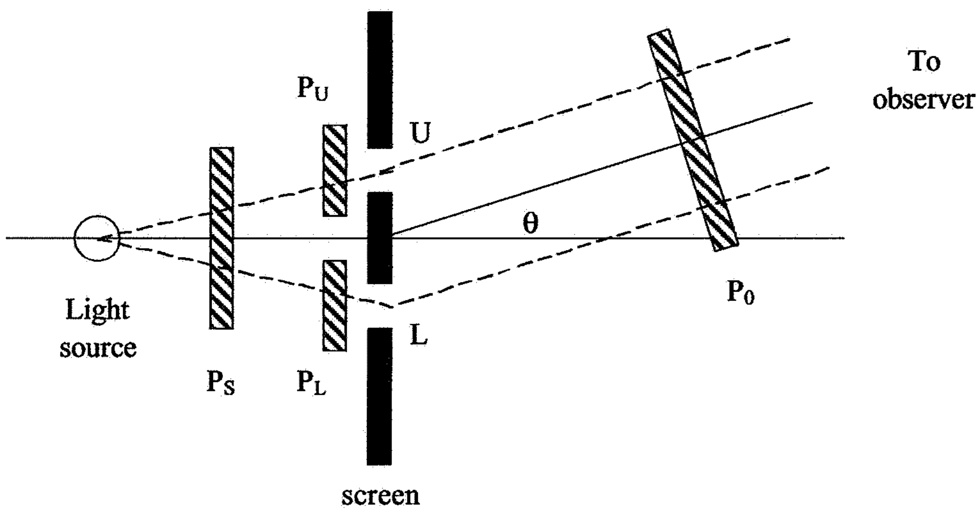
$$I(r,\theta)=|A|^2=4I_0(r,\theta)\cos^2\left(\frac{2\pi a}{\lambda}\sin\theta\right),$$

where $I_0(r,\theta)=|A_0|^2$ is the intensity observed from only one of the two slits at a time, λ is the wavelength of the light, and θ is the angle of observation as shown.

(Assume that $a \gg$ size of slit.)

Part II

Now, we place polarization filters in several locations: after the light source (P_S), in front of the slits (P_U and P_L), and in front of the observer (P_O), as shown below.



For each combination below, find an expression for the detected intensity, analogous to the expression for the unfiltered experiment. Again assume that the light source itself emits unpolarized light and that the slits are not sensitive to the state of the polarization. [The term "horizontal" ("vertical") refers to a filter that lets through only light polarized in the horizontal (vertical) direction, and "circular" refers to a filter which lets through only right-circularly polarized light.]

(a) P_S = absent $\quad$ P_U = absent $\quad$ P_L = horizontal $\quad$ P_O = absent

(b) P_S = absent $\quad$ P_U = horizontal $\quad$ P_L = vertical $\quad$ P_O = absent

(c) P_S = circular $\quad$ P_U = horizontal $\quad$ P_L = vertical $\quad$ P_O = circular

(d) P_S = circular $\quad$ P_U = horizontal $\quad$ P_L = horizontal $\quad$ P_O = circular

(e) P_S = circular $\quad$ P_U = horizontal $\quad$ P_L = vertical $\quad$ P_O = absent

Statistical Mechanics

Part 1.

Consider N decoupled spins described by variables $s_i = \pm 1,\ i = 1, \ldots, N$. The energy of each spin configuration is given by

$$E = -h \sum_i s_i$$

where h is proportional to the magnetic field.

1. Compute the partition function, the free energy, and the magnetization

$$M = \left\langle \sum_i s_i \right\rangle$$

for this system.

2. Show that $M \to 0$ as $h \to 0$ for any temperature T. This shows that this system never undergoes a magnetic ordering phase transition.

Part 2.

Now we will study a simple model system that does undergo a magnetic ordering phase transition. Consider the spins discussed in Part 1. Now let each spin interact with every other spin in a pairwise manner with strength J/N. The energy function is now

$$E = -\frac{J}{N} \sum_{i,j=1}^{N} s_i s_j - h \sum_i s_i$$

1. Show for the simple spin configuration $s_i = 1,\ i = 1, \ldots, N$ that the energy E is extensive (that is, proportional to N) at large N.

2. Show that E can be written

$$E = -\frac{J}{N} \left(\sum_i s_i \right)^2 - h \sum_i s_i \ .$$

3. Write down the partition function Z for this system, which you will not yet be able to compute.

4. Use the following identity

$$C \int_{-\infty}^{\infty} dx \exp(-x^2/2 + xy) = \exp(y^2/2)$$

to rewrite Z as an integral over x (C is an irrelevant constant). You can now do the sums over the s_i. Write down the answer.

5. Rescale x to put your answer in the form

$$Z = C' \int dx \exp(-N f(x))$$

where C' is another irrelevant constant and you are to explicitly compute $f(x)$.

6. In the limit of large N, such an integral is dominated by the lowest minima of f. From now on, assume that $h = 0$. Sketch $f(x)$ for large T and for small T. Notice the difference in the minima pattern. Show that

$$f(x_{\text{min}}(T)) = \beta F(T)/N$$

where $F(T)$ is the free energy and $x_{\text{min}}(T)$ is the location of the minimum that dominates at temperature T. (If more than one minimum dominates, the left hand side should contain a sum over the minima.) Sketch $F(T)$.

7. The phase transition temperature T_c is the temperature at which F is not smooth. This is also the point at which the minimum at $x = 0$ stops dominating the integral. Calculate T_c.

Electromagnetism

a) A simple magnetic-hydrodynamic generator consists of a parallel-plate capacitor with plate area S and separation d that is placed into a flowing conducting liquid with conductivity σ. The liquid is moving parallel to the plates with a constant velocity $\vec{v}$. The capacitor is placed in a magnetic field $\vec{B}$ that is directed perpendicular to the velocity of the liquid (see Figure 1).

How much power will be dissipated in an external circuit containing a resistance R and the capacitor?

b) Assume now that a compass is placed a distance r below the lower horizontal wire of the circuit as shown in the figure. (The needle does not feel the magnetic field $\vec{B}$ in which the liquid is placed but only the field from the current in the lower horizontal wire, which can be treated as an infinite wire.) Estimate the current in the circuit needed to lift the compass needle off its peg. Assume that the residual magnetic moment density in the needle steel is equal to its saturation induction B_0 and that the steel's mass density is ρ.

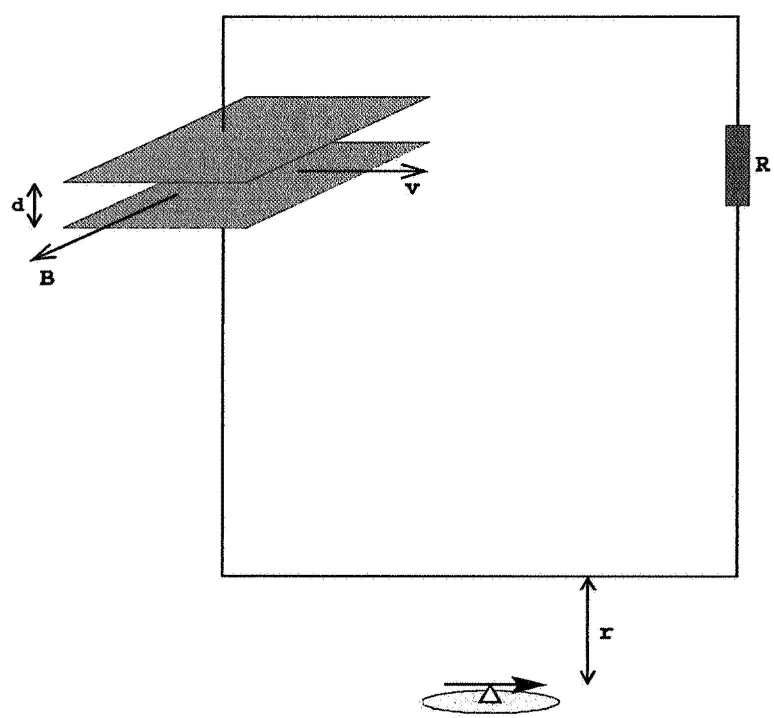


Figure 1: A simple magnetic-hydrodynamic generator.