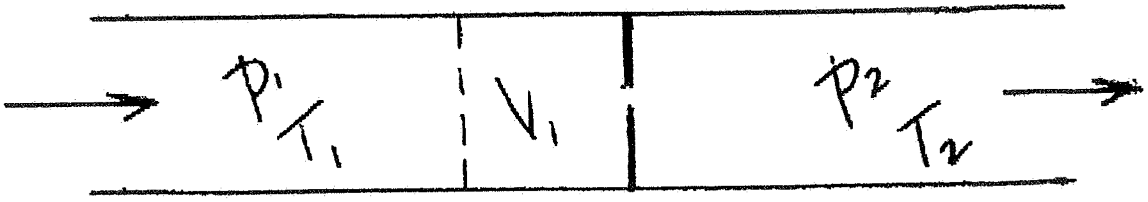
2003
Day 1
(drafts ?)

GENERAL PHYSICS

In a helium refrigerator the initial stages of cooling are achieved by adiabatic expansion of the gas. The final stage of cooling, however, is achieved by Joule-Thomson expansion, a process in which the steady state gas flow is throttled in a valve (or an orifice) as indicated below.



a) Consider N helium atoms occupying the volume V just to the left of the orifice. Assuming that no heat is added to or lost from these atoms as they pass to the right of the orifice, show that the enthalpy $H \equiv U + PV$ remains constant in the process.

For an ideal gas the enthalpy is a function of T alone and thus in a Joule-Thomson expansion the temperature does not change. For a real gas such as helium, however, the temperature can change, and this change is given by the Joule-Thomson coefficient

$$\mu \equiv \left(\frac{\partial T}{\partial P}\right)_H$$

b) We want an expression for the Joule-Thomson coefficient in terms of measurable parameters of the gas. Since the expansion proceeds at constant enthalpy, let us obtain an expression for $dH(S, P)$, and then write this in terms of T and P, the measurable independent variables in the throttling process. Show that

$$\mu = \frac{T(\partial V/\partial T)_P - V}{c_P}$$

c) For a van der Waals gas such as helium the equation of state can be written

$$(P + a/V^2)(V - b) = NkT$$

To first order in the parameters a and b write an expression for V in terms of T and P, again the measurable variables in the throttling process.

d) Calculate the Joule-Thomson coefficient for our van der Waals gas and comment on the temperature dependence, both mathematically and physically. Also comment on the temperature change of the gas after expansion.

Qualifying Exam - 2002, E & M

November 5, 2002

A particle of mass m and charge e moves along an equilibrium circular orbit of radius r_0 in the horizontal plane of a magnetic gap (shown in the figure below) in which the magnetic field falls off along the radius such that:

$$B_z(r) = \frac{A}{r^n}; \qquad 0 < n < 1. \tag{1}$$

The orbital center coincides with the symmetry axis formed by the z-direction.

a) Determine the frequency of the vertical oscillations of the particle for small deviations from the horizontal plane.

b) Determine the frequency of radial oscillations of the particle for small deviations from the equilibrium orbit.

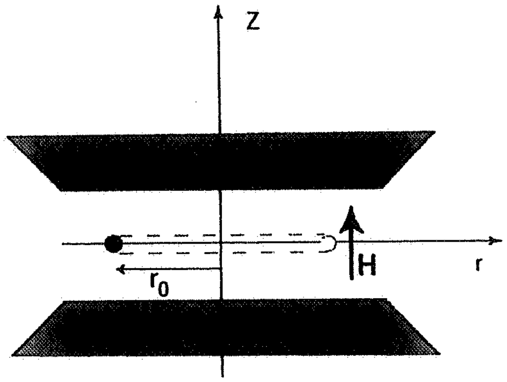
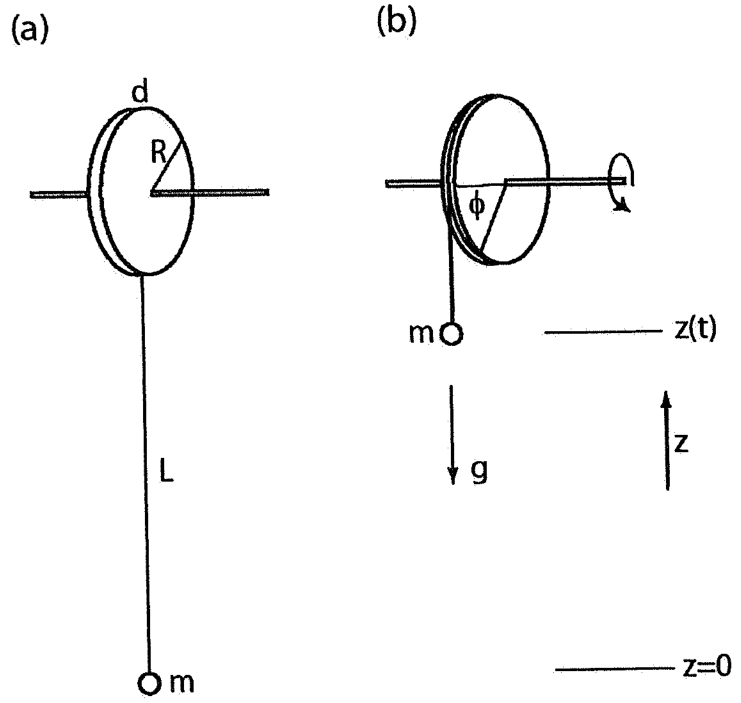


Figure 1: The mass m is circulating in the plane z=0. The dashed line represents the trajectory. The direction r represent the radial distance in the two-dimensional $x-y$ plane.



Physics Qualifying Exam 2002-2003
Mechanics

Consider a system consisting of a mass m attached by a massless thread of length L to a point on the circumference of a disk of radius R, as shown in Figure (a). The disk is held at constant height and free to rotate about an axis through its center. Originally the thread is wound around the disk and unwinds as the mass m is accelerated by gravity from its original height $z = L$, as shown in Figure (b). Neglect friction and assume $L \gg R$, i.e. neglect the displacement in the horizontal plane.

i) Calculate the moment of inertia I of the disk having uniform density ρ, thickness d, and mass M.

ii) Using energy conservation, calculate the maximum angular frequency of rotation ω_0 of the disk.

iii) Write down the equations of motion for the angle $\phi(t)$ of the disk and the position $z(t)$ of the mass m.

iv) Find the solutions $\phi(t), z(t)$, and check them against the result obtained under ii). What is the period T of the motion, i.e. the time for the system to exactly return to the initial state?

v) Sketch the results $z(t)$ and $\phi(t)$ over more than one period. Comment on the change of linear and angular momentum, especially near the turning points.

vi) If you want to determine the earth's gravitational acceleration g from a measurement of the period T, what is the uncertainty δg if the relative uncertainties are $\delta T/T = 5 \times 10^{-3}$ and $\delta L/L = 10^{-2}$, and all other uncertainties are negligible?

1

Quantum Mechanics ~~V~~

(October 23, 2002)

The hamiltonian for a particle of mass m in a one-dimensional harmonic oscillator potential is

$$H_0 = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}, \tag{1}$$

where x and $p = -i\hbar(\partial/\partial x)$ are the position and momentum operators in the position representation.

(a) **(2 points)** The ladder operators a and $a^\dagger$ are defined as follows:

$$a = \frac{1}{\sqrt{2}}\left(\frac{x}{d} + i\frac{d}{\hbar}p\right) \quad \text{and} \quad a^\dagger = \frac{1}{\sqrt{2}}\left(\frac{x}{d} - i\frac{d}{\hbar}p\right), \tag{2}$$

where $d = \sqrt{\hbar/(m\omega)}$ is a characteristic length. Show that these operators obey the commutation relations $[a, a^\dagger] = 1$, $[a, a] = [a^\dagger, a^\dagger] = 0$. Hence show that the Hamiltonian can be rewritten as $H_0 = \hbar\omega(a^\dagger a + \frac{1}{2})$.

(b) **(2 points)** Assume that the hermitian operator $a^\dagger a$ obeys an eigenvalue equation $a^\dagger a|\lambda\rangle = \lambda|\lambda\rangle$, with eigenstate $|\lambda\rangle$ and eigenvalue λ. Show that all the eigenvalues are non-negative by considering $\langle\lambda|a^\dagger a|\lambda\rangle$. Use the commutation relations to show that $[a^\dagger a, a] = -1$. Hence show that $a|\lambda\rangle$ is an eigenstate of $a^\dagger a$ with eigenvalue $\lambda - 1$. Conclude that $a^\dagger a$ must have a lowest eigenstate $|0\rangle$ with eigenvalue $\lambda = 0$. Use these relations to show that $a|n\rangle = \sqrt{n}\,|n-1\rangle$. Repeat the arguments to show that the state $a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$. Finally, conclude that these states $|n\rangle$ are eigenstates of the hamiltonian H_0 with eigenvalues $\hbar\omega(n + \frac{1}{2})$.

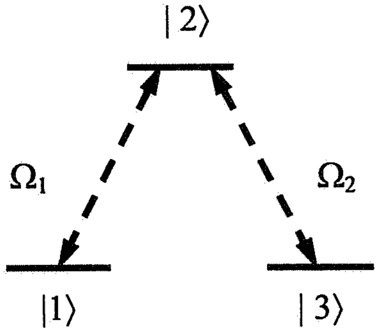
(c) **(2 points)** Use the relation $a|0\rangle = 0$ to obtain the (unnormalized) ground-state wave function $\psi_0(x) \propto \exp(-\frac{1}{2}x^2/d^2)$. Discuss the physical meaning of the length d.

(d) **(3 points)** Suppose that the particle carries a charge e. If the system is subjected to a static electric field $\boldsymbol{\mathcal{E}} = \mathcal{E}\hat{x}$, why is $H_1 = -e\mathcal{E}x$ the perturbation hamiltonian? Use first- and second-order perturbation theory to show that the ground-state energy is shifted by an amount $\Delta E_0 \approx -\frac{1}{2}(e^2/m\omega^2)\,\mathcal{E}^2$. Hence conclude that the charged harmonic oscillator has a polarizability $e^2/m\omega^2$.

(e) **(1 point)** Construct an argument to demonstrate that the energy shift ΔE_0 given above is, in fact, *exact* to all orders of perturbation theory (namely, that there are no corrections of higher order than $\mathcal{E}^2$).

Quantum mechanics question #2

Suppose we have a three level atom with energies $E_1 = 0 = E_3$ and $E_2 = \hbar\omega$. We put the atoms in two optical fields with matrix elements $\Omega_1 = \langle 2 | H_{EM} | 1 \rangle$, $\Omega_2 = \langle 2 | H_{EM} | 3 \rangle$ and $\langle 1 | H_{EM} | 3 \rangle = 0$.

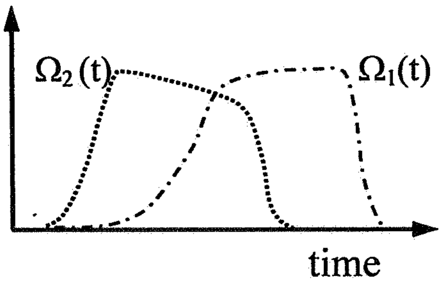


a) Write down the Hamiltonian for this system in matrix form, defining explicitly your representation for $|1\rangle$, $|2\rangle$ and $|3\rangle$. (e.g. $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = |1\rangle$, etc.)

b) With the interaction H_{EM} turned on, find the new eigenstates for the system and label them $|1'\rangle$, $|2'\rangle$ and $|3'\rangle$. Show that one of the new eigenstates has energy $E = 0$ for any value of Ω_1 and Ω_2, and that this state is not coupled to the original state $|2\rangle$. Show explicitly that the other two eigenstates are orthogonal to each other and to the uncoupled state.

c) Let Ω_1 and Ω_2 be functions of time. Write down the differential equations that describe the time dependence of the eigenstates.

d) Suppose $d\Omega_1/dt$ and $d\Omega_2/dt$ vary slowly with respect to the probability amplitudes a(t), b(t) and c(t) of the states $|1'\rangle$, $|2'\rangle$ and $|3'\rangle$, respectively. In this case, the system is said to "adiabatically" follow the steady state solution. What is the final state of the system if it is originally in state $|1\rangle$ if the shape of the $\Omega_1(t)$ and $\Omega_2(t)$ are shown below. Explain your reasoning.



QUALIFIER EXAM 2002-3 STATISTICAL MECHANICS QUESTION

A Schwarschild black hole is characterized by a mass M. Such an object has an *event horizon*, a spherical surface within which nothing ever escapes, classically. The area of the event horizon is given, in 3 spatial dimensions, by $A = aM^2$ where a is a positive dimensionful constant.

Beckenstein and Hawking discovered that black holes behave like thermal systems. Their thermodynamic properties, in 3 spatial dimensions, are summarized by saying that the energy of such a black hole is given by $E = bM$ and the entropy by $S = cA$ where b and c are positive dimensionful constants.

1. Using the above thermodynamic data, calculate the temperature of the Schwarschild black hole.
2. Calculate its heat capacity (holding no other thermodynamic variable fixed; i.e. don't worry about P and V, which do not appear here).

Another kind of black hole is the A-S black hole. Again, denote its mass by M. The area of the event horizon of the A-S black hole, in 2 spatial dimensions, is given by $A = a'\sqrt{M}$. Its thermodynamic properties, in 2 spatial dimensions, are summarized by the relations $E = b'M$ and $S = c'A$. Here a', b' and c' are positive, dimensionful constants.

3. Using the above thermodynamic data, calculate the temperature of the A-S black hole.
4. Calculate its heat capacity (holding no other thermodynamic variable fixed).

A negative heat capacity for a thermal system indicates that it is in an *unstable* thermal equilibrium.

Let us now try to understand qualitatively the stability of the thermal equilibrium for the two types of black holes described above. Black holes, like other thermal objects, emit radiation according to the blackbody laws[1].

---

[1] This is a slight oversimplification. Don't worry about it.

In particular the total flux emitted per unit area, J, is given by the Stefan-Boltzmann law, $J = \sigma T^{D+1}$ where σ is a dimensionful constant and D is the spatial dimension.

Imagine each black hole in turn is immersed in a gas of radiation at the temperature of the black hole, which serves as a heat bath. Now imagine that a thermal fluctuation makes the mass of each black hole momentarily decrease, $M \to M - \delta M$.

5. Discuss what happens qualitatively to each type of black hole after this fluctuation and comment on the stability of each.

Now we will follow in greater detail the "evaporation" of these black holes. Remove the radiation bath surrounding each one, so that each radiates away some part of its energy ($\sim M$) during every unit of time.

6. Write a differential equation describing the behavior of M as a function of time for each type of black hole: Schwarschild in 3 dimensions and A-S in 2 dimensions.
7. Solve these equations to find the time at which the mass has dropped to zero, and the black hole has completely "evaporated." Sketch the behavior of M and the temperature T as a function of time for each type of black hole.

Qualifying Exam 2002/03

Special Relativity

(1) Bob and John are two twin brothers. They have heard about the twin paradox and decide to test it. Bob stays at home, and John, immediately after celebrating his 20th birthday, flies together with his friends to the nearby star with speed $v = 0.6c$ and returns back with the same speed. At the time when he returns, Bob is 10 years older. How old will John be when they meet?

(2) Draw the space-time diagram that shows John's travel. This is the graph that shows John's trajectory in Bob's reference frame, using Bob's coordinate system (x,t).

(3) John thinks that Bob is flying with respect to him. Unlike Bob, he expects that he will be older than Bob when they meet (this is the essence of the paradox). To check what is actually going on, the two brothers decide to send each other radio messages containing video records of their birthday parties. How many times will Bob see his happy brother during the first five years of waiting (Bob's time)? How many times will Bob see his brother during the last five years of waiting (Bob's time)? How many times will John see Bob during the last 5 years of his trip (John's time)? (Assume for all these questions that upon arrival on Earth they don't send each other another transmission, but rather congratulate in person, which is not being counted; i.e. John will see on his *whole* trip Bob celebrate 9 times, although he is gone for 10 years (in Bob's time).)

(4) During the last couple of years waiting for his brother, Bob noticed, watching the live video transmission, that John is rapidly aging, but his voice sounds like a happy voice of a young baby. How much did the frequency of his voice change?

General Physics

The definition of viscosity of a fluid is η is as follows: Take two parallel plates of area A separated by distance D moving with relative velocity V parallel to the plates. The force between them is given by

$$F = \frac{\eta A V}{D}.$$

The viscosity of a gas depends on the mass of the gas molecules m, the average velocity of the molecules v, the number density of molecules ρ and the mean free path λ of the molecules. For a low density gas the viscosity is proportional to the density ρ.

a) Up to dimensionless constants of order unity, find the formula for the viscosity.

b) Suppose that instead of the mean free path of the molecules and their velocity you are given the scattering cross section and the temperature. Find the viscosity (again, up to dimensionless constants of order unity). What happens to the viscosity as the cross section goes to zero, that is as the molecules become free particles with no interactions? Does this make sense? Explain.

c) Estimate the viscosity of air at room temperature. Note that Boltzmann's constant k_B is on the order of 10^{-23} Joules per Kelvin, that the radius of an air molecule is about 10^{-10} meters and that the density of air at room temperature is about 1 kilogram per cubic meter.