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Default and information

Kay Giesecke*

Department of Management Science and Engineering, Stanford University, Stanford, CA 94305-4026, USA

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Abstract

In a traditional structural model of default it is implicitly assumed that the information used to calibrate and run the model is publicly available. In reality, model inputs and parameters are unobservable. In this article we analyze the role of information in structural models, which we specify through a model definition of the default time and a model filtration. The model definition relates the default of a firm to its assets and liabilities. The model filtration describes the information of investors relative to the model definition. It parameterizes a family of default models for a given default time. An important situation is when the default is not observable with respect to the model filtration. Examples include models with incomplete information about firm assets and models with incomplete information about the liability-dependent barrier that triggers default. Here the default time is typically totally inaccessible, as in the intensity-based, reduced-form models of default. In this case the model admits generalized reduced-form security pricing formulae in terms of the trend, which is the cumulative intensity. The trend can be explicitly characterized through the conditional default probability given the model filtration. If the trend is absolutely continuous with respect to the Lebesgue measure, then its density is the intensity and our formulae simplify to the classical intensity-based formulae. Not every incomplete information model admits an intensity. A model in which investors cannot observe the default barrier is a first example.

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*Tel.: + 1 650 723 9265; fax: + 1 650 723 1614.
E-mail address: giesecke@stanford.edu.
URL: http://www.stanford.edu/~giesecke.

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1. Introduction

Recently a number of accounting scandals shocked the business world. Management at Enron, WorldCom, and Tyco misrepresented the level of assets and liabilities on corporate statements. Auditors, possibly knowingly, approved the incorrect statements that were then released to the public. Misled investors suffered huge losses from the ensuing bankruptcies, which are among the largest ever in U.S. corporate history.

In this article we analyze the role of the time-dependent revelation of investor information in structural models of default, which relate a firm's assets to its liabilities. There are many economically plausible structural definitions of default. The classical definition advanced by Black and Scholes (1973) and Merton (1974), postulates that a firm defaults at debt maturity if assets are not sufficient to pay off the debt. Another definition proposed by Black and Cox (1976) takes as premise that a firm may default at any time before debt maturity. This is described by defining the default event as the first time the firm's assets fall to some lower barrier. The asset level which triggers default can be imposed exogenously [Black and Cox (1976), Longstaff and Schwartz (1995)] or endogenously by having the shareholders optimally liquidate the firm [e.g. Leland (1994), Leland and Toft (1996), Anderson and Sundaresan (1996)].

Absent from the discussion in these papers is the concept of information and how it is revealed over time. In this article, we model the evolution of investor information explicitly through a filtration $\mathbb{G}$. A default model is specified by two components: a *default stopping time* τ and a *model filtration* $\mathbb{F} \subseteq \mathbb{G}$. In a structural model, τ is given by a model definition of the default event, such as first passage of assets to a barrier. The associated model filtration $\mathbb{F}$ describes the information of investors with respect to the inputs to the model definition. For a first passage definition, this can include direct, noisy or lagged observations of the firm assets or the liabilities obtained from corporate statements or other public information services. For a given model definition of default, we thus obtain a class of default models that is parameterized by the model filtration.

In the structural models mentioned above it is implicitly assumed that investors can observe the inputs to the model definition of default. The associated model filtration $\mathbb{F}$ is generated by the firm assets and the default barrier. This creates a *complete information* model. Moreover, if assets cannot jump, investors can anticipate a default: τ is *predictable*. We prove that any default model with predictable default time implies that model credit spreads must go to zero as maturity goes to zero. In a complete information model, there is no short term credit risk so short dated corporate bonds have the same yields as Treasuries. Empirical short term spreads are, however, significantly positive, which explains why a complete information model is difficult to fit to market data.

The resolution of this caveat lies in the informational assumptions. In reality, investors have *incomplete information* about the inputs to the model definition of default. The asset value of a firm is hard to observe directly. Accounting statements that purportedly contain the information required to deduce the correct default trigger level are difficult to interpret. The associated model filtration $\mathbb{F}$ may be generated by noisy or lagged observations of the asset value or the default barrier, for example. It is too coarse to distinguish the states where the firm defaults. Being unable to observe the true distance to default, usually investors cannot anticipate the default: τ is *totally inaccessible*. Investors face short term credit risk and thus demand a premium over Treasury yields for holding short dated corporate bonds.

Incomplete information models are true hybrids: they share the model definition of default with the structural models and the inaccessibility of the default time with the reduced-form models of default. Reduced-form or intensity-based models lack the model definition of default that is central to the structural models. Instead, the information described by the model filtration $\mathbb{F}$ is used to directly model the intensity, or conditional rate of default. The intensity is identified with the short term credit spread. It leads to tractable formulae for default probabilities and prices of credit sensitive securities, see for example Artzner and Delbaen (1995), Jarrow and Turnbull (1995), Duffie and Singleton (1999), Duffie et al. (1996), and Lando (1998).

In the context of a first passage model with incomplete asset observation, Duffie and Lando (2001) show that the commonality between incomplete information models and reduced-form models extends to the pricing formulae. Duffie and Lando give conditions for the existence of an intensity and calculate it in terms of the conditional asset density. Surprisingly, this fails in other situations: there are models with inaccessible default time but no intensity. We give a first example, a first passage model in which investors cannot observe the default barrier. The associated model filtration $\mathbb{F}$ is generated by the asset process. With a time-invariant random barrier, investors observe the barrier's upper bound as the minimum asset process. When assets exceed their minimum to date, the short term credit spread is zero unless the asset value can suddenly drop. When assets are at their minimum, the short term spread is infinite since investors face the risk that the firm defaults in the next instant. This dichotomy is not consistent with an intensity.

Using an approach that is different from that of Duffie and Lando (2001), we show that most incomplete information models can be analyzed from a common vantage point. We introduce the *trend* of a default model and prove that under mild technical conditions, all incomplete information models lead to *generalized* reduced-form security pricing formulae in terms of their trend. The trend is a nondecreasing, $\mathbb{F}$-predictable process. If stopped at default, it equals the compensator of the nondecreasing process that indicates default. If the trend has absolutely continuous paths, then the associated density is the intensity and our pricing formulae simplify to the classical intensity-based formulae. Hence the trend can be interpreted as the cumulative intensity.

The trend can be explicitly constructed in terms of the conditional default probability given the information described by the model filtration $\mathbb{F}$. This construction is very powerful: it requires neither assumptions on the distribution of assets and default barrier nor on the specific structure of $\mathbb{F}$. It is based on arguments from the theory of enlargement of filtrations. We give sufficient conditions for the existence of an intensity in terms of the $\mathbb{F}$-conditional distribution of assets. Table 1 provides a summary.

Table 1
Type of information vs. default time properties, intensity, and trend for first passage default models

	Information			
	Complete	Incomplete		
		Assets	Barrier	Both
Default	Predictable	Inaccessible	Inaccessible	Inaccessible
Intensity	No	Yes	No	Yes
Trend	No	Yes	Yes	Yes

The remainder of this article is organized as follows. In Section 2, we provide a definition of a default model in terms of a stopping time and a model filtration $\mathbb{F}$. We introduce a class of first passage default models that is parameterized by $\mathbb{F}$. The case of complete information is discussed in Section 3. Here $\mathbb{F}$ is generated by the asset process and the default barrier. Section 4 contains the analysis of the incomplete information case. In Section 5 we introduce the trend of a default model, discuss its analytic properties and provide a generalized reduced-form security pricing formula in terms of the trend. In Section 6, we explicitly characterize the trend for the class of first passage models with incomplete information and give conditions under which it admits an intensity. Section 7 contains an explicit case study. We analyze the effects of incomplete information on the term structure of credit spreads. Assuming that the evolution of assets is driven by a Brownian motion, we obtain closed-form expressions for the trend, default probability, bond price and credit spread.

2. A first passage default model

The uncertainty of investors is modeled by the complete probability space $(\Omega, \mathcal{G}, \mathrm{P})$. Here Ω is a set that represents the states of the world. The σ-algebra $\mathcal{G}$ determines the resolution to which investors can distinguish different states of the world $\omega \in \Omega$. The symbol P denotes the probability measure that gives the likelihood of any event in $\mathcal{G}$. We assume that investors are risk-neutral, so P is a martingale measure with respect to the numéraire security with value $\alpha_t = \exp(\int_0^t r_s \, \mathrm{d}s)$ at t. Here r denotes the risk-free rate of interest, which we assume is deterministic.

The evolution of information available to investors is modeled by the *investor filtration* $\mathbb{G}$. This is a sequence $(\mathcal{G}_t)_{t \geqslant 0}$ of sub σ-algebras of $\mathcal{G}$ indexed by time t. Here $\mathcal{G}_t$ stands for the set of events which can be distinguished at time t, or the information available at t. Since information is accumulated over time, a natural requirement is that the family $\mathbb{G}$ is nondecreasing, $\mathcal{G}_s \subset \mathcal{G}_t$ for $s \leqslant t$. We impose two additional technical conditions, often called the 'usual conditions.' The first is that $\mathbb{G}$ is right continuous. The second is that $\mathcal{G}_0$ contains all P-null sets, meaning that one can always identify a sure event. Every filtration we introduce is assumed to satisfy these conditions.

Definition 2.1. A *default model* is specified by a pair $(\tau, \mathbb{F})$ consisting of a $\mathbb{G}$-stopping time τ that designates the time of a firm's default and a model filtration $\mathbb{F} = (\mathscr{F}_t)_{t \geqslant 0} \subseteq \mathbb{G}$.

In a structural default model, τ is given by a model definition of default. We consider a standard specification, see for example Black and Cox (1976), Leland (1994), or Longstaff and Schwartz (1995). We start with a stochastic process V, called the *asset process*, where V_t is a sufficient statistic for the firm's future cash flows as seen from time t. The running minimum asset process M is defined by $M_t = \min\{V_s : 0 \leqslant s \leqslant t\}$. It describes the evolution of the *historical asset low* over time.

The firm is (partly) financed by a bond, issued at time 0. We assume that the firm's management chooses to default on the bond if V falls to a level $D < V_0$. The *default barrier D* is a time-invariant random variable that is independent of V. The firm's *default time* τ is given by

$$\tau = \inf\{t > 0 : V_t \leqslant D\}. \tag{1}$$

The default indicator process N is defined by $N_t = 1_{\{t \geqslant \tau\}}$. That is, N is zero before default and jumps to one at default. Note that $\{\tau \leqslant t\} = \{M_t \leqslant D\}$, meaning that default by time t is equivalent to the historical asset low at t being below the default barrier.

A default model $(\tau, \mathbb{F})$ based on definition (1) is called a *first passage default model*. The model filtration $\mathbb{F}$ describes the information available to investors relative to the definition (1). We obtain a family of first passage models by varying the specification of $\mathbb{F}$. Below we consider some of these specifications.

3. Complete information

Suppose the firm updates investors continuously through time about its true financial condition. Investors observe the firm's asset value V as it evolves over time and, right at bond issuance at time 0, the firm's default barrier D. The corresponding model filtration $\mathbb{F}$ is generated by the σ-algebras

$$\sigma(V_s : s \leqslant t) \vee \sigma(D). \tag{2}$$

This model filtration creates a complete information model $(\tau, \mathbb{F})$. Here investors observe the distance of the firm to default, i.e. the nearness of V to D. It follows that τ is an $\mathbb{F}$-stopping time. We may then define the investor filtration $\mathbb{G} = \mathbb{F}$. Moreover, in familiar examples τ is *predictable*. This means that there is an increasing sequence (τ_n) of stopping times, strictly smaller than τ, which converges to τ almost surely. Intuitively, investors can foretell the default event by observing a succession of pre-default events, such as V falling dangerously close to D.

The existence of an *announcing sequence* (τ_n) depends on the analytic properties of the asset process V. If V is a continuous process,[1] then $\tau_n = \inf\{t > 0 : V_t \leqslant D + 1/n\}$

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[1] We use the phrase *continuous process* to designate a process whose paths are continuous almost surely. A process whose paths are monotone almost surely is called a *monotone process*.

defines an announcing sequence. If we allow for jumps in V as for example in Zhou (2001), then assets can continuously 'diffuse' to the default barrier or cross it with a sudden jump. In this situation the times τ_n just defined converge to τ only with a probability strictly less than one, so that τ is not predictable any more.

The probabilistic properties of τ govern the properties of the model credit spread term structure. The *credit spread* $S(t,\varepsilon)$ on a zero bond issued by the firm is the difference between the yield at time t on a corporate zero bond with zero recovery and that on a credit risk-free zero bond, both maturing at $t+\varepsilon$. With deterministic interest rates and unit face values, for each $t<\tau$ we have

$$S(t,\varepsilon) = -\frac{1}{\varepsilon}\log(1-q(t,\varepsilon)), \quad t\geqslant 0,\ \ \varepsilon>0 \tag{3}$$

almost surely. Here $q(t,\varepsilon)$ is the conditional default probability at t for a term $\varepsilon\geqslant 0$:

$$q(t,\varepsilon) = \mathrm{P}[\tau\leqslant t+\varepsilon \,|\, \mathscr{G}_t]. \tag{4}$$

The term structure of credit spreads at t is the schedule of $S(t,\varepsilon)$ against the term ε. The *short spread* $\lim_{\varepsilon\downarrow 0} S(t,\varepsilon)$ at t gives the excess yield over the risk-free yield demanded by investors for assuming the risk of default over an infinitesimal term.

Proposition 3.1. *Let* $(\tau,\mathbb{F})$ *be a default model such that, for each* $t<\tau$, *the limit* $\lim_{\varepsilon\downarrow 0}(1/\varepsilon)q(t,\varepsilon)$ *exists and is finite almost surely. Then, for each* $t<\tau$ *we have*

$$\lim_{\varepsilon\downarrow 0} S(t,\varepsilon) = \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\, q(t,\varepsilon)$$

almost surely.

Proof. If, for each $t<\tau$, $\lim_{\varepsilon\downarrow 0}(1/\varepsilon)q(t,\varepsilon)$ exists and is finite almost surely, then for each $t<\tau$, $q(t,\varepsilon)$ tends to 0 as ε tends to 0, almost surely. Now the statement follows from the definition (3) by Taylor's theorem. □

Consider the dyadic decomposition $Z_n=\{k2^{-n}\,|\,k=0,1,\ldots\}$ for $n\geqslant 1$. For a given n and fixed time t, pick a t_i from Z_n such that $t_i<t\leqslant t_i+2^{-n}=t_{i+1}$. We analyze the spread $S(t_i,2^{-n})$ when τ is predictable. We show that $S(t_i,2^{-n})$ tends to 0 with n tending to ∞ almost surely with respect to $\mathrm{P}\times\mathrm{Leb}$ where Leb denotes Lebesgue measure.

Proposition 3.2. *In any default model* $(\tau,\mathbb{F})$ *for which* τ *is* $\mathbb{G}$*-predictable and where for each* $t<\tau$, $\lim_{n\uparrow\infty} q(t,2^{-n})=0$ *almost surely, for* $t<\tau$ *we have*

$$\lim_{n\uparrow\infty}\sum_{t_i\in Z_n} S(t_i,2^{-n})\,1_{\{t_i<t\leqslant t_{i+1}\}} = 0$$

almost surely $\mathrm{P}\times\mathrm{Leb}$.

Proof. Let X be a nonnegative, right continuous supermartingale. To X corresponds a unique finite measure P^X on the σ-field $\mathscr{P}$ of predictable sets in $\bar{\Omega}=\Omega\times(0,\infty]$

such that for any stopping time T

$$\mathrm{P}^X[B \times (T,\infty]] = \mathrm{E}[X_T 1_B], \quad B \in \mathscr{G}_T,$$

see Föllmer (1972). Here we write $B \times (T,\infty] = \{(\omega,t) \in \bar{\Omega} : \omega \in B, T(\omega) < t\}$.

Let P^M be the measure on $\mathscr{P}$ associated to the nonnegative, right continuous supermartingale $M = 1 - N$. Since $\mathrm{P}^M[\bar{\Omega}] = \mathrm{E}[M_0] = 1$, P^M is a probability measure.

For the default stopping time τ we have

$$\mathrm{P}^M[\Omega \times (\tau,\infty]] = \mathrm{E}[1 - N_\tau] = \mathrm{P}[\tau > \tau] = 0.$$

Since τ is predictable, there exists a sequence (τ_n) of stopping times such that $\tau_n \uparrow \tau$ almost surely. For an announcing time τ_n we have

$$\mathrm{P}^M[\Omega \times (\tau_n,\infty]] = \mathrm{E}[1 - N_{\tau_n}] = \mathrm{P}[\tau > \tau_n] = 1.$$

Let $S_n = \Omega \times (\tau_n,\infty] \backslash \Omega \times (\tau,\infty]$ and note that $\mathrm{P}^M[S_n] = 1$. It follows that the measure P^M is concentrated on the set

$$S = \bigcap_{n=1}^{\infty} S_n = \{(\omega,t) \in \bar{\Omega} : \tau(\omega) = t\}.$$

Let P^Y be a measure on $\mathscr{P}$ such that

$$\mathrm{P}^Y[B \times (t,s]] = (s-t)\mathrm{P}[B], \quad 0 \leqslant t \leqslant s, \ B \in \mathscr{G}_t.$$

But $\mathrm{P}^Y[S] = 0$ so the measures P^M and P^Y are singular on $\mathscr{P}$.

Consider the Radon–Nikodym density

$$\mathscr{D}^Y M = \frac{\mathrm{d}\mathrm{P}^M}{\mathrm{d}\mathrm{P}^Y}$$

of the absolutely continuous part of P^M with respect to P^Y, see Airault and Föllmer (1974). This is a predictable process which is uniquely determined up to a set of P^Y-measure 0. Since P^M and P^Y are singular, $\mathscr{D}^Y M = 0$ on $\mathscr{P}$.

Theorem 2.2 in Airault and Föllmer (1974) identifies the density $\mathscr{D}^Y M$ P^Y-almost surely with the *dyadic derivative*

$$\mathscr{D}^Y M(\omega,t) = \lim_{n\uparrow\infty} \sum_{t_i \in Z_n} \frac{\mathrm{E}[M_{t_i} - M_{t_{i+1}} \mid \mathscr{G}_{t_i}]}{t_{i+1} - t_i}(\omega)\, 1_{\{t_i < t \leqslant t_{i+1}\}}.$$

By the definition of M and q, on $\{\tau > t\}$ we have

$$\begin{aligned}
\mathscr{D}^Y M(t) &= \lim_{n\uparrow\infty} \sum_{t_i \in Z_n} \frac{1}{2^{-n}}\, q(t_i, 2^{-n})\, 1_{\{t_i < t \leqslant t_{i+1}\}} \\
&= -\lim_{n\uparrow\infty} \sum_{t_i \in Z_n} \frac{1}{2^{-n}}\, (\log(1 - q(t_i, 2^{-n})))\, 1_{\{t_i < t \leqslant t_{i+1}\}} \\
&= \lim_{n\uparrow\infty} \sum_{t_i \in Z_n} S(t_i, 2^{-n})\, 1_{\{t_i < t \leqslant t_{i+1}\}}
\end{aligned}$$

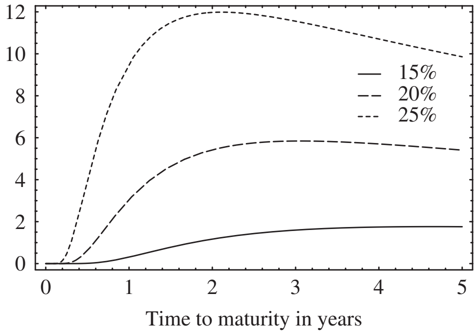

Fig. 1. Term structure of credit spreads with complete information. We plot $S(t, \varepsilon)$ (in percent) as a function of term ε (in years), for firms with asset volatilities $\sigma = 15\%$, 20%, and 25%. The assumptions underlying this graph are discussed in Section 7.

almost surely P^Y. The second line follows from Taylor's theorem and the third line follows from the definition of the spread. This gives the result. $\square$

Zero short spreads imply that bond investors do not demand compensation for bearing short-term credit risk. In our illustration in Fig. 1, this includes maturities of up to 3 months, depending on the riskiness of the firm. This is economically hardly plausible, since it corresponds to investors with perfect foresight over short horizons. Empirical studies such as Sarig and Warga (1989) find that in general, credit spreads remain bounded away from zero, suggesting that investors do face short term credit risk. The predictability of defaults is not consistent with the presence of short term risk.

4. Incomplete information

In reality, bond investors' access to inside firm information is usually limited. Typically, investors are not kept fully informed by firm management about the true assets of the firm and the true asset level which triggers default.

Example 4.1 (*Lambrecht and Perraudin, 1996; RiskMetrics Group, 2002; Giesecke and Goldberg, 2004a*). Investors observe firm assets, but not the default barrier. The model filtration $\mathbb{F}$ is generated by the σ-algebras $\sigma(V_s : s \leqslant t)$.

Example 4.2 (*Duffie and Lando, 2001*). Investors observe the default barrier and receive at times $t_1 < t_2 < \cdots < t_m$ a noisy asset report $Y_{t_k} = V_{t_k} + U_{t_k}$, where U_{t_k} is some independent noise random variable. The model filtration $\mathbb{F}$ is generated by the σ-algebras $\sigma(D) \vee \sigma(Y_s, s \leqslant t, s \in \{t_1, \ldots, t_m\})$. The variance of U_{t_k} can be interpreted as a measure of the degree of asset noise at time t. The U_{t_k} can be serially correlated, reflecting persistence of noise in time, or correlated with the asset value V_{t_k}.

Example 4.3 (*Kusuoka, 1999*). Investors observe the default barrier and receive noisy asset reports continuously through time by observing some process Y whose drift $\mu = f(V_t, t)$ is modulated by the asset process V for some smooth function f. The model filtration $\mathbb{F}$ is generated by the σ-algebras $\sigma(D) \vee \sigma(Y_s, s \leqslant t)$.

Example 4.4. Bond investors observe neither firm assets nor its default barrier. The model filtration $\mathbb{F}$ is generated by the trivial σ-algebra.

Here the model filtration $\mathbb{F}$ is too coarse to discern the distance of the firm to default: the default time τ is not an $\mathbb{F}$-stopping time. This creates an incomplete information model $(\tau, \mathbb{F})$. For a given $\mathbb{F}$, the investor filtration $\mathbb{G}$ is generated by the σ-algebras

$$\mathscr{G}_t = \{B \in \mathscr{G} : \exists B_t \in \mathscr{F}_t,\ B \cap \{\tau > t\} = B_t \cap \{\tau > t\}\}. \tag{5}$$

The filtration $\mathbb{G}$ so constructed contains $\mathbb{F}$ and makes τ a stopping time. Moreover, τ is *totally inaccessible*: we have $\mathrm{P}[\tau = T < \infty] = 0$ for all predictable stopping times T. An announcing sequence does not exist. Intuitively, investors can never be sure about the distance of the firm to default so the default hits the market as a complete surprise.

The inaccessibility of defaults is consistent with the empirical observation that prices of credit sensitive securities drop sharply at the bankruptcy announcement, see Beneish and Press (1995) and Duffie et al. (2003). The predictability of defaults is not consistent with the presence of jumps, since it implies that prices converge continuously to their default-contingent values.

5. Trend and security prices

The model filtration $\mathbb{F}$ parameterizes a family of first passage default models $(\tau, \mathbb{F})$ described in Sections 3 and 4. We provide a generalized reduced-form representation of *all* models $(\tau, \mathbb{F})$ that are based on incomplete information and satisfy a technical condition.

We describe the default indicator process N in terms of the model filtration $\mathbb{F}$. Consider the *conditional survival probability*[2]

$$L_t = \mathrm{E}[1 - N_t | \mathscr{F}_t] = \mathrm{P}[\tau > t | \mathscr{F}_t]. \tag{6}$$

Assume that there is always a chance that the firm defaults: for each $t > 0$, we have $L_t > 0$ almost surely and $\mathrm{E}[L_t] > 0$.

The process L is a supermartingale in the model filtration $\mathbb{F}$: for $s < t$, we have $L_s \geqslant \mathrm{E}[L_t \mid \mathscr{F}_s]$. The Doob–Meyer decomposition theorem states that there exists a unique nondecreasing $\mathbb{F}$-predictable process K starting at zero such that the process $L + K$ becomes an $\mathbb{F}$-martingale, see Dellacherie and Meyer (1982). The process K negates the downward tendency in L; it is therefore called the compensator to L.

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[2] To define the process L uniquely up to indistinguishability, we can take L to be the *optional projection* of $(1 - N)$ onto the filtration $\mathbb{F}$. Then Eq. (6) holds true for each t almost surely. See Dellacherie and Meyer (1982) for details about this construction.

Definition 5.1. Let $L_{t-}=\lim_{s\uparrow t}L_s$ and $L_{0-}=1$. The *trend* of the default model $(\tau,\mathbb{F})$ is the nondecreasing $\mathbb{F}$-predictable process A defined by the Stieltjes integral

$$A_t=\int_0^t \frac{\mathrm{d}K_s}{L_{s-}}. \tag{7}$$

The default indicator process N is nondecreasing, hence a submartingale in the investor filtration $\mathbb{G}$: we have $N_s\leqslant \mathrm{E}[N_t\,|\,\mathscr{G}_s]$ for $s\leqslant t$. The Doob–Meyer decomposition states that there exists a unique nondecreasing $\mathbb{G}$-predictable compensator C starting at zero such that the process $N-C$ becomes an $\mathbb{G}$-martingale. The following result, due to Jeulin and Yor (1978), identifies C as the trend stopped at τ, denoted A^τ. It forms the basis for the reduced-form price representations we develop below.

Proposition 5.2. *The process $N-A^\tau$ is a martingale in the investor filtration $\mathbb{G}$.*

If τ is an $\mathbb{F}$-stopping time, then $L=1-N$. The assumption $L_t>0$ is violated. If τ is predictable, then the default indicator N is its own $\mathbb{G}$-compensator.

If τ is not an $\mathbb{F}$-stopping time, then L, the trend A and the corresponding $\mathbb{G}$-compensator to N are non-trivial. The analytic properties of A are governed by the properties of L.

Proposition 5.3. *The trend A is a continuous process if and only if the conditional survival probability L satisfies $\mathrm{E}[L_{S-}]=\mathrm{E}[L_S]$ for every $\mathbb{G}$-predictable stopping time S.*

Proof. We observe that (7) is a Stieltjes integral because K is a monotone process and hence has paths of finite variation almost surely. It follows that A is a continuous process if and only if K is a continuous process. As shown in Dellacherie and Meyer (1982), the compensator of a càdlàg supermartingale L that admits a Doob–Meyer decomposition is a continuous process if and only if L satisfies $\lim_{n\uparrow\infty}\mathrm{E}[L_{S_n}]=\mathrm{E}[L_S]$ for every announcing sequence of stopping times $(S_n)\uparrow S$. Noting that L is bounded by 1, the result follows from the dominated convergence theorem. □

Proposition 5.4. *Assume the trend A is a continuous process. For a fixed time T, let $Z\in\mathscr{G}_T$ be an integrable random variable. If the path of the process Y given by*

$$Y_t=\mathrm{E}[Z\mathrm{e}^{A_t-A_T}\,|\,\mathscr{G}_t],\quad t\leqslant T,$$

is almost surely continuous at τ, then for each $t<\tau$ we have

$$\mathrm{E}[Z(1-N_T)\,|\,\mathscr{G}_t]=\mathrm{E}[Z\mathrm{e}^{A_t-A_T}\,|\,\mathscr{G}_t]$$

almost surely, $t\leqslant T$.

Proof. Define $H_t=\mathrm{E}[Z\mathrm{e}^{-A_T}\,|\,\mathscr{G}_t]$ so that $Y=\mathrm{e}^A H$. The product rule gives $\mathrm{d}Y_t=\mathrm{e}^{A_t}\,\mathrm{d}H_t+Y_{t-}\,\mathrm{d}A_t$. Denote by $\Delta W_t=W_t-W_{t-}$ the jump of some càdlàg process W at t. Defining $U_t = (1 - N_t)Y_t$, the product rule yields

$$\begin{aligned} \mathrm{d}U_t &= -Y_{t-}\,\mathrm{d}N_t + (1 - N_{t-})\,\mathrm{d}Y_t + \Delta(1 - N_t)\Delta Y_t \\ &= (1 - N_{t-})\mathrm{e}^{A_t}\,\mathrm{d}H_t - Y_{t-}(\mathrm{d}N_t - (1 - N_{t-})\,\mathrm{d}A_t) \\ &= (1 - N_{t-})\mathrm{e}^{A_t}\,\mathrm{d}H_t - Y_{t-}(\mathrm{d}N_t - \mathrm{d}A_t^\tau). \end{aligned} \tag{8}$$

Integration of both sides of (8) gives

$$U_T - U_t = \int_t^T (1 - N_{s-})\mathrm{e}^{A_s}\,\mathrm{d}H_s - \int_t^T Y_{s-}\,\mathrm{d}(N_s - A_s^\tau).$$

Note that $(H_t)_{0 \leqslant t \leqslant T}$ and $N - A^\tau$ are $\mathbb{G}$-martingales. Since the integrands are bounded and predictable, U is a $\mathbb{G}$-martingale so $U_t = Y_t(1 - N_t) = \mathrm{E}[U_T \mid \mathcal{G}_t] = \mathrm{E}[Z(1 - N_T) \mid \mathcal{G}_t]$. $\square$

If the trend is a continuous process, then Proposition 5.4 yields generalized reduced-form formulae for default probabilities and prices of credit sensitive securities in the investor filtration $\mathbb{G}$. A *credit sensitive security* is specified by a pair (T, X), where T is a maturity date and $X \in \mathcal{G}_T$ is an integrable random variable. The security pays $X1_{\{\tau > T\}}$ at T, so its price C_t at time $t \leqslant T$ is given by $C_t = \mathrm{E}[\mathrm{e}^{-\int_t^T r_s\,\mathrm{d}s} X(1 - N_T) \mid \mathcal{G}_t]$.

Corollary 5.5. *If the trend A is a continuous process and the path of the process Y with*

$$Y_t = \mathrm{E}[X\mathrm{e}^{-\int_t^T r_s\,\mathrm{d}s + A_t - A_T} \mid \mathcal{G}_t], \quad t \leqslant T, \tag{9}$$

is almost surely continuous at τ, then for each $t < \tau$ we have $C_t = Y_t$ almost surely, $t \leqslant T$, and

$$q(t, \varepsilon) = 1 - \mathrm{E}[\mathrm{e}^{A_t - A_{t+\varepsilon}} \mid \mathcal{G}_t]$$

almost surely, $\varepsilon \geqslant 0$.

Corollary 5.5 implies that valuing a credit sensitive security (T, X) with terminal payoff $X1_{\{\tau > T\}}$ using the numéraire $\alpha_t = \exp(\int_0^t r_s\,\mathrm{d}s)$ is equivalent to valuing a security with terminal payoff X using the adjusted numéraire $\alpha_t \exp(A_t)$. A special case of this result is at the center of the *intensity-based*, reduced-form approach to default modeling described in, among others, Artzner and Delbaen (1995), Jarrow and Turnbull (1995), Duffie and Singleton (1999), Duffie et al. (1996), and Lando (1998). Unlike structural models, intensity-based models do not rely on a model definition of default. Instead, the conditional survival probability L of an intensity-based model $(\tau, \mathbb{F})$ is directly modeled in terms of information described by the model filtration $\mathbb{F}$.

Definition 5.6. A default model $(\tau, \mathbb{F})$ is called *intensity-based* if there exists a bounded, nonnegative $\mathbb{F}$-predictable process λ such that for each $t \geqslant 0$

$$L_t = \exp\left(-\int_0^t \lambda_s\,\mathrm{d}s\right) \tag{10}$$

almost surely. The process λ is called the *intensity* of the model $(\tau, \mathbb{F})$.

An alternative definition of an intensity-based model relies on the trend. This definition is closer to the classic definition of the intensity in the theory of point processes, see Daley and Vere-Jones (2003) for example.

Definition 5.7. A default model $(\tau, \mathbb{F})$ is called *intensity-based* if there exists a bounded, nonnegative $\mathbb{F}$-predictable process λ such that for each $t \geqslant 0$

$$A_t = \int_0^t \lambda_s \, \mathrm{d}s \tag{11}$$

almost surely. The process λ is called the *intensity* of the model $(\tau, \mathbb{F})$.

We demonstrate that Definition 5.6 is stronger than Definition 5.7.

Proposition 5.8. *A default model* $(\tau, \mathbb{F})$ *that is intensity-based in the sense of Definition* 5.6 *is also intensity-based in the sense of Definition* 5.7. *A default model* $(\tau, \mathbb{F})$ *that is intensity-based in the sense of Definition* 5.7 *is also intensity-based in the sense of Definition* 5.6 *if the* $\mathbb{F}$*-compensator* K *to* L *is given by* $1 - L$.

Proof. If the model is intensity-based in the sense of Definition 5.6, then Eq. (10) defines a monotone and continuous process L. Since L is $\mathbb{F}$-adapted and continuous, it is $\mathbb{F}$-predictable. By the uniqueness of the Doob–Meyer decomposition, the $\mathbb{F}$-compensator K of L is given by $K = 1 - L$. Then A satisfies Eq. (11) since

$$A_t = -\int_0^t \frac{\mathrm{d}L_s}{L_s} = -\log L_t. \tag{12}$$

If the model is intensity-based in the sense of Definition 5.7, then A is a continuous process. With the hypothesis $K = 1 - L$, the definition of the trend (7) implies that

$$L_t = -\int_0^t L_{s-} \, \mathrm{d}A_s.$$

There exists a unique solution L to this equation given by $L_t = \exp(-A_t)$. This follows from Theorem 37 in Protter (2004, Chapter II), noting that A is a continuous process of bounded variation. This implies that L satisfies Eq. (10). ☐

Remark 5.9. Two sufficient conditions for $K = 1 - L$ are as follows.
(1) L is a monotone and $\mathbb{F}$-predictable process.
(2) L is a monotone process and every $\mathbb{F}$-martingale has a continuous modification.

Suppose a default model is intensity-based in the weak sense of Definition 5.7. Then Corollary 5.5 yields the classical reduced-form formula for the price of the credit sensitive security (T, X) proved in Duffie et al. (1996). In this case, the adjusted numéraire is given by $\exp(\int_0^t (r_s + \lambda_s) \, \mathrm{d}s)$.

Proposition 5.10. *In any default model* $(\tau, \mathbb{F})$ *that is intensity-based in the strong sense of Definition* 5.6 *with a right continuous intensity* λ*, for each* $t < \tau$ *we have*

$$\lim_{\varepsilon \downarrow 0} S(t, \varepsilon) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} q(t, \varepsilon) = \lambda_t \tag{13}$$

almost surely.

Proof of Proposition 5.10. Let X be a nonnegative and integrable random variable. Theorem 14 in Protter (2004, Chapter VI) implies that for each $t \geqslant 0$

$$\mathrm{E}[X \mid \mathscr{G}_t] = \frac{1}{L_t}\mathrm{E}[X1_{\{\tau > t\}} \mid \mathscr{F}_t] + X1_{\{\tau \leqslant t\}} \tag{14}$$

almost surely.[3] For each $t \geqslant 0$ and $\varepsilon > 0$, letting $X = 1_{\{\tau \leqslant t+\varepsilon\}}$ in Eq. (14) gives

$$q(t,\varepsilon) = \mathrm{P}[\tau \leqslant t+\varepsilon \mid \mathscr{G}_t] = \frac{1}{L_t}\,\mathrm{P}[t < \tau \leqslant t+\varepsilon \mid \mathscr{F}_t] + 1_{\{\tau \leqslant t\}}$$

almost surely. It follows that for each $t < \tau$

$$q(t,\varepsilon) = \frac{1}{L_t}\,\mathrm{E}[L_t - L_{t+\varepsilon} \mid \mathscr{F}_t] = \mathrm{E}[1 - \mathrm{e}^{-\int_t^{t+\varepsilon}\lambda_s\,\mathrm{d}s} \mid \mathscr{F}_t]$$

almost surely.

For each $t < \tau$ we get,

$$\begin{aligned}
\lim_{\varepsilon \downarrow 0}\frac{1}{\varepsilon}\,q(t,\varepsilon) &= \lim_{\varepsilon \downarrow 0}\frac{1}{\varepsilon}\,\mathrm{E}\left[\int_t^{t+\varepsilon}\lambda_s\,\mathrm{d}s + o\left(\int_t^{t+\varepsilon}\lambda_s\,\mathrm{d}s\right) \mid \mathscr{F}_t\right] && (15)\\
&= \lim_{\varepsilon \downarrow 0}\frac{1}{\varepsilon}\mathrm{E}\left[\int_t^{t+\varepsilon}\lambda_s\,\mathrm{d}s \mid \mathscr{F}_t\right] + \lim_{\varepsilon \downarrow 0}\frac{1}{\varepsilon}\,o(\varepsilon) && (16)\\
&= \lim_{\varepsilon \downarrow 0}\mathrm{E}\left[\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\lambda_s\,\mathrm{d}s \mid \mathscr{F}_t\right] && (17)\\
&= \mathrm{E}\left[\lim_{\varepsilon \downarrow 0}\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\lambda_s\,\mathrm{d}s \mid \mathscr{F}_t\right] && (18)\\
&= \mathrm{E}[\lambda_{t+} \mid \mathscr{F}_t] && (19)\\
&= \lambda_t && (20)
\end{aligned}$$

almost surely. Line (15) follows from Taylor's theorem. In lines (16) and (17) we use that for each t, the intensity λ_t is bounded almost surely. Then line (18) is due to the dominated convergence theorem. In line (20) we use the fact that λ is right continuous and $\mathbb{F}$-adapted.

Eq. (13) now follows from Proposition 3.1. □

6. Trend, intensity and information

In incomplete information models $(\tau, \mathbb{F})$, the trend and the security prices it implies are parameterized with the information contained in the model filtration $\mathbb{F}$. In Propositions 6.1–6.5 we characterize the trend for several informational scenarios and give conditions under which it can be used to estimate prices of credit sensitive securities via the generalized reduced-form formulae in Corollary 5.5. Under

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[3] Theorem 14 in Protter (2004, Chapter VI) represents a right continuous version of the martingale defined by $\mathrm{E}[X \mid \mathscr{G}_t]$ in terms of the optional projection Y of the process $X(1-N)$ onto the model filtration $\mathbb{F}$. This representation implies our Eq. (14) since the process $(\mathrm{E}[X(1-N_t) \mid \mathscr{F}_t])$ is a version of Y.

additional regularity conditions on the $\mathscr{F}_t$-conditional distribution of the asset minimum M_t, a model $(\tau, \mathbb{F})$ with incomplete asset information admits an intensity, regardless of whether the default barrier is observable or not. In this case the generalized reduced-form pricing formulae simplify to the classical intensity-based formulae.

Proposition 6.1. *Assume that the model filtration $\mathbb{F}$ is generated by the asset process V. Suppose investors form a prior distribution on the unobserved default barrier with distribution function G on $(-\infty, V_0)$. Then, for each $t \geqslant 0$ we have*

$$L_t = G(M_t)$$

almost surely. If V is a continuous process and G is a continuous function, then the default time is totally inaccessible in $\mathbb{G}$. The trend A is a continuous process and for each $t \geqslant 0$

$$A_t = -\log G(M_t)$$

almost surely. If G admits a density and the minimum M changes on a set of Lebesgue measure 0, *then an intensity in the weak sense of Definition* 5.7 *does not exist.*

Proof. Recall the definition of the default time (1). Since the minimum process M is $\mathbb{F}$-adapted and D is independent of the σ-algebra $\mathscr{F}_t$, we get

$$L_t = \mathrm{P}[\tau > t \mid \mathscr{F}_t] = \mathrm{P}[D < M_t \mid \mathscr{F}_t] = G(M_t).$$

If V is a continuous process, then so is M. From the continuity of G, it then follows that L is a continuous process. Since L is $\mathbb{F}$-adapted and continuous, it is $\mathbb{F}$-predictable. Since M is a monotone process and G is a monotone function, L is a monotone process. By the uniqueness of the Doob–Meyer decomposition, the $\mathbb{F}$-compensator K of L is given by $K = 1 - L$. Then the trend is a continuous process given by formula (12). Since $(t \wedge \tau)$ is a continuous process, the corresponding $\mathbb{G}$-compensator $A^\tau = -\log G(M_{\cdot \wedge \tau})$ to N is a continuous process. This implies that τ is a totally inaccessible $\mathbb{G}$-stopping time, see Dellacherie and Meyer (1982).

Definitions 5.6 and 5.7 are equivalent by Proposition 5.8. We consider the path regularity of the trend. If G has a density g, then

$$\mathrm{d}A_t = -\frac{g(M_t)}{G(M_t)}\,\mathrm{d}M_t.$$

Hence A is concentrated on the set $\{t \geqslant 0 : V_t = M_t\}$, which has Lebesgue measure 0 by hypothesis. This implies that A is singular: it does not admit the representation (11). □

Standard models for V include diffusion processes such as Brownian motion. For these models the associated minimum M decreases on a set of Lebesgue measure 0.

The reason for the absence of an intensity is investors' *learning* about the default barrier as time passes. Since investors observe the asset process V, they are also fully informed about the historical low of assets to date M. With a time-invariant default barrier, if the firm has not defaulted by time t investors know that the default barrier must lie *below* M_t at t.

Next we consider cases where investors cannot observe the firm's asset value perfectly after the bonds have been issued at time 0. The firm discloses at time 0 its initial asset value V_0. After issuance, investors may receive incomplete asset information, such as noisy or lagged asset reports. This is described by the model filtration $\mathbb{F}$; see Examples 4.2 and 4.3. Central to the analysis below is the $\mathbb{F}$-conditional distribution of the asset minimum.

Definition 6.2. For $t \geqslant 0$, the regular version of the $\mathscr{F}_t$-conditional distribution function of M_t is denoted $H(t, \cdot)$.

In Section 7, we compute $H(t, \cdot)$ explicitly for several informational scenarios. The path properties of $H(t, \cdot)$ play an important role in the sequel. We list some properties that we will refer to below.

Condition 6.3. (1) For each $t > 0$ and $x < V_0$, the variable $H(t, x) < 1$ almost surely.
(2) For each $x < V_0$, the process $H(\cdot, x)$ is continuous and monotone.
(3) For each $x < V_0$, the process $H(\cdot, x)$ is absolutely continuous[4] with a bounded, nonnegative, right continuous and $\mathbb{F}$-predictable density process $h(\cdot, x)$.

We first consider the case where investors observe the default barrier but cannot observe the firm's asset value perfectly.

Proposition 6.4. *Assume that the variables V_0 and D are $\mathscr{F}_0$-measurable. Then, for each $t \geqslant 0$ we have*

$$L_t = 1 - H(t, D)$$

almost surely. If Conditions 6.3(1) *and* (2) *are satisfied, then the default time is totally inaccessible in $\mathbb{G}$. The trend is a continuous process and for each $t \geqslant 0$ we have*

$$A_t = -\log(1 - H(t, D))$$

almost surely. If in addition Condition 6.3(3) *is satisfied, then there is an intensity λ in the strong sense of Definition* 5.6. *For each $t > 0$ we have*

$$\lambda_t = \frac{h(t, D)}{1 - H(t, D)}$$

almost surely.

Proof. Note that $H(0, x) = 0$ for $x < V_0$ so $L_0 = 1$. Since the random variable D is $\mathscr{F}_t$-measurable and independent of M_t, we have for $t > 0$

$$L_t = 1 - \mathrm{P}[M_t \leqslant D \mid \mathscr{F}_t] = 1 - H(t, D).$$

By the assumption $H(t, D) < 1$, the conditional probability $L_t > 0$ for $t > 0$. If $H(\cdot, D)$ is a continuous and monotone process, then so is L. In this case, L is $\mathbb{F}$-predictable so $K = 1 - L$ and A is given by formula (12). Since L is a continuous process, so is the $\mathbb{G}$-compensator A^τ to N, implying that τ is a totally inaccessible $\mathbb{G}$-stopping time.

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[4] Let X be a nondecreasing process. We say that X is an *absolutely continuous process* if the random measure on $\mathbb{R}_+$ associated to X is absolutely continuous with respect to Lebesgue measure almost surely.

By Proposition 5.8, Definitions 5.6 and 5.7 are equivalent. If $H(\cdot, D)$ is an absolutely continuous process, then so is the trend A. The intensity λ_t is given by the derivative of A_t with respect to time t, almost surely. $\square$

Duffie and Lando (2001) derive the intensity from the limit representation (13) under the assumptions that the default barrier D is equal to the constant $d < V_0$ almost surely and the asset process V evolves according to the stochastic differential equation $\mathrm{d}V_t = \mu(t, V_t)\,\mathrm{d}t + \sigma(t, V_t)\,\mathrm{d}B_t$. Here, B is a standard $\mathbb{G}$-Brownian motion under P and μ and σ satisfy certain conditions. Duffie and Lando establish conditions under which $\lim_{\varepsilon \downarrow 0}(1/\varepsilon)q(t, \varepsilon)$ is for each $t \in (0, \tau)$ equal to

$$\lambda_t = \tfrac{1}{2}\sigma^2(t,d)k_z(t,d) \tag{21}$$

almost surely. Here, $k(t,\cdot)$ is the conditional density of V_t given $\mathscr{F}_t$ and $\{\tau > t\}$, and $k_z(t,z)$ is the partial derivative of $k(t,z)$ with respect to z from the right. Proposition 6.4 provides an alternative representation of formula (21) in terms of the $\mathscr{F}_t$-conditional density of the asset minimum M_t. In the special case where the random barrier is governed by a Dirac distribution concentrated on d and Conditions 6.3(1) and (2) are satisfied, it yields the formula $A_t = -\log(1 - H(t,d))$ for the trend. If, in addition, Condition 6.3(3) is satisfied, then the intensity is given by $\lambda_t = h(t,d)/(1 - H(t,d))$. While less explicit, our result does not require a hypothesis on the dynamics of the asset process V. In the special case where V evolves according to the SDE mentioned above, arguments similar to those proposed by Duffie and Lando can be used to recover the formula (21) from our intensity representation.

Finally we consider the case where investors can neither observe the firm's asset value V after the bonds have been issued nor the firm's default barrier D.

Proposition 6.5. *Assume that the variable* V_0 is $\mathscr{F}_0$*-measurable but the variable D is never* $\mathscr{F}_t$*-measurable. Suppose investors form a prior on D with distribution function G on* $(-\infty, V_0)$*. Then, for each* $t \geqslant 0$ *we have*

$$L_t = 1 - \int_{-\infty}^{V_0} H(t,x)\,\mathrm{d}G(x) \tag{22}$$

almost surely. If Conditions 6.3(1) *and* (2) *are satisfied, then the default time is totally inaccessible in* $\mathbb{G}$*. The trend is a continuous process and for each* $t \geqslant 0$ *we have*

$$A_t = -\log\left(1 - \int_{-\infty}^{V_0} H(t,x)\,\mathrm{d}G(x)\right)$$

almost surely. If in addition Condition 6.3(3) *is satisfied, then there is an intensity* λ *in the strong sense of Definition* 5.6*. For each* $t > 0$ *we have*

$$\lambda_t = \frac{\int_{-\infty}^{V_0} h(t,x)\,\mathrm{d}G(x)}{1 - \int_{-\infty}^{V_0} H(t,x)\,\mathrm{d}G(x)} \tag{23}$$

almost surely.

Proof. Since $H(0,x) = 0$ for $x < V_0$, we get $L_0 = 1$. Noting that the variable D is independent of $\mathscr{F}_t$, we have $\mathrm{P}[D \leqslant x \mid \mathscr{F}_t] = G(x)$ so formula (22) is due to the fact that

$$L_t = 1 - \mathrm{P}[M_t \leqslant D \,|\, \mathscr{F}_t] = 1 - \mathrm{E}[H(t, D) \,|\, \mathscr{F}_t] = 1 - \int_{-\infty}^{V_0} H(t, x)\, \mathrm{d}\mathrm{P}[D \leqslant x \,|\, \mathscr{F}_t].$$

By the assumption $H(t,x) < 1$, we get $L_t > 0$. If $H(\cdot, x)$ is a continuous and monotone process for all $x < V_0$, then L is a continuous and monotone process. Thus $K = 1 - L$ and A is given by formula (12). Since L is a continuous process, so is the $\mathbb{G}$-compensator A^τ to N, implying that τ is a totally inaccessible $\mathbb{G}$-stopping time.

Noting Proposition 5.8, the intensity λ_t is given as the derivative of the trend A_t with respect to t for each $t \geqslant 0$, almost surely. For each $t > 0$ we have

$$\lambda_t = \frac{1}{L_t} \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_{-\infty}^{V_0} (H(t + \varepsilon, x) - H(t, x))\, \mathrm{d}G(x)$$

almost surely. Since $H(\cdot, x)$ is an absolutely continuous process for each $x < V_0$ with bounded and right continuous density $h(\cdot, x)$, formula (23) follows from the dominated convergence theorem. □

7. Term structure of credit spreads

We illustrate the implications of information on the term structure of credit spreads in the family of first passage default models $(\tau, \mathbb{F})$. We analyze four cases: complete information, no information about the barrier, no information about the assets and no information about the barrier and the assets. Each situation implies qualitatively different term structures. These differences suggest that we might be able to infer the information situation of investors from the shapes of the spread curves observed in the market. The empirical study of Yu (2005) provides first steps in this direction.

We specialize in the general setup of the previous sections by making concrete assumptions about the distribution of the assets and the default barrier. This allows us to obtain *closed-form expressions* for trends, default probabilities, bond prices, and credit spreads from the general results we derived above. We suppose that the total market value Z of the firm follows a geometric Brownian motion with drift given by the riskless rate r and volatility $\sigma > 0$. That is, $Z_t = Z_0\, \mathrm{e}^{V_t}$ with initial value $Z_0 > 0$. Here $V_t = mt + \sigma B_t$ is a Brownian motion with drift $m = r - \frac{1}{2}\sigma^2$ and B is a standard $\mathbb{G}$-Brownian motion under P. In the sequel we take V to be our 'asset process' in the sense of Section 2. Then the distribution function $\Psi(t, x) = \mathrm{P}[M_t \leqslant x]$ of the historical asset low M_t is

$$\Psi(t, x) = \Phi\left(\frac{x - mt}{\sigma\sqrt{t}}\right) + \exp\left(\frac{2mx}{\sigma^2}\right) \Phi\left(\frac{x + mt}{\sigma\sqrt{t}}\right), \tag{24}$$

for $x \leqslant 0$ and $t > 0$. Here Φ is the standard normal distribution function. Unless noted otherwise, our base case parameters for calculations are $r = 6\%$ and $\sigma = 20\%$. Furthermore, the prior default barrier distribution with respect to Z is assumed to be uniform on $(0, Z_0)$. This implies that investors' barrier prior with respect to the asset process V is represented by the distribution function $G(x) = g(x) = \mathrm{e}^x$ for $x < 0$.

7.1. Complete information

Suppose the model filtration $\mathbb{F}$ is generated by the process V and the random variable D, see Section 3. The investor filtration $\mathbb{G} = \mathbb{F}$. The conditional default probability at time $t < \tau$ for the term ε is then given by

$$q(t, \varepsilon) = \mathrm{P}[\tau \leqslant t + \varepsilon \,|\, \mathcal{G}_t] = \mathrm{P}[\tau \leqslant t + \varepsilon \,|\, \mathcal{F}_t] = \Psi(\varepsilon, D - V_t)$$

almost surely. Since $V_t > D$, we have $\lim_{\varepsilon \downarrow 0} q(t, \varepsilon) = 0$ almost surely so the short spread is zero by Proposition 3.2. Fig. 1 shows the spread term structure for a fixed distance to default of $V_t - D = 0.4$ and varying asset volatilities σ.

7.2. No information about the default barrier

With perfectly observable assets the model filtration $\mathbb{F}$ is generated by the process V. Proposition 6.1 implies that $A = -M$. According to Corollary 5.5, the conditional default probability at time $t < \tau$ for the term ε is then given by

$$q(t, \varepsilon) = 1 - \mathrm{E}[\mathrm{e}^{M_{t+\varepsilon} - M_t} \,|\, \mathcal{G}_t] = p(\varepsilon, V_t - M_t) \tag{25}$$

almost surely. Here, for $\varepsilon > 0$ and $v \geqslant 0$, we define

$$p(\varepsilon, v) = \int_{-\infty}^{-v} \Psi(\varepsilon, y) \mathrm{e}^{y+v} \, \mathrm{d}y.$$

Defining the constants $\nu = m + \sigma^2$, $\gamma = 1 + 2m/\sigma^2$, $\delta = m - \gamma\sigma^2$, and $\beta = -m\gamma + \gamma^2\sigma^2/2$, integration by parts gives the following formula for $p(\varepsilon, v)$:

$$\Phi\left(\frac{-v - m\varepsilon}{\sigma\sqrt{\varepsilon}}\right) - \mathrm{e}^{v + r\varepsilon} \Phi\left(\frac{-v - \nu\varepsilon}{\sigma\sqrt{\varepsilon}}\right) + \frac{\mathrm{e}^{(1-\gamma)v}}{\gamma} \Phi\left(\frac{m\varepsilon - v}{\sigma\sqrt{\varepsilon}}\right) - \frac{\mathrm{e}^{v + \varepsilon\beta}}{\gamma} \Phi\left(\frac{\delta\varepsilon - v}{\sigma\sqrt{\varepsilon}}\right). \tag{26}$$

For the short-spread, we get

$$\lim_{\varepsilon \downarrow 0} S(t, \varepsilon) = \begin{cases} 0, & V_t > M_t, \\ \infty, & \text{else} \end{cases}$$

almost surely. Spread curves for varying distance $V_t - M_t$ are shown in Fig. 2. Note that the rate at which the spread converges to 0 in case $V_t = M_t$ is much smaller than in the complete information case.

With incomplete barrier information, default probabilities, bond prices, and credit spreads depend on the path of the asset value through the historical asset low M_t. In the complete information case discussed above, these quantities depend only on the current asset value V_t. In an empirical implementation, Giesecke and Goldberg (2004a) show that the path-dependence translates into 'early warning:' the incomplete information model is more reactive than the complete information

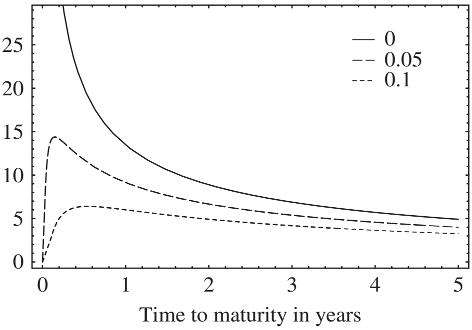

Fig. 2. Term structure of credit spreads with perfect asset observation but unobservable default barrier. We plot $S(t, \varepsilon)$ (in percent) as a function of term ε (in years), for $V_t - M_t = 0$, 0.05, and 0.1.

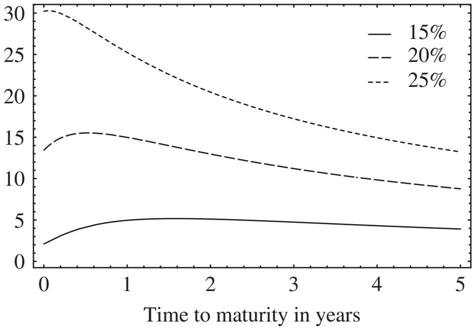

Fig. 3. Term structure of credit spreads when the default barrier is observable but assets are not. We plot $S(t, \varepsilon)$ (in percent) as a function of ε (in years), for $\sigma = 15\%$, 20%, and 25%. We set $t = 0.5$ years.

model. It detects a deterioration in the equity market-implied credit quality earlier than the complete information model.

7.3. No information about the assets

Suppose the default barrier is revealed to investors at $t = 0$, but no information about assets will be available. The corresponding model filtration $\mathbb{F}$ is generated by the random variable D. Since $\Psi(t, x)$ satisfies Conditions 6.3(1) and (2), Proposition 6.4 implies that $A_t = -\log(1 - \Psi(t, D))$ almost surely. From Corollary 5.5, the conditional default probability at time $t < \tau$ for the term ε is

$$q(t, \varepsilon) = \frac{\Psi(t + \varepsilon, D) - \Psi(t, D)}{1 - \Psi(t, D)}$$

almost surely. Fig. 3 shows the corresponding spreads curves, for which we set $D = -0.3$. The term structure is slightly hump shaped. The short credit spread is positive. Since $\Psi(t, D)$ satisfies Condition 6.3(3), for $t > 0$ we have $\lim_{\varepsilon \downarrow 0} S(t, \varepsilon) = \lambda_t = (\partial/\partial t)\Psi(t, D)/(1 - \Psi(t, D))$ almost surely by Propositions 5.10 and 6.4. Here the

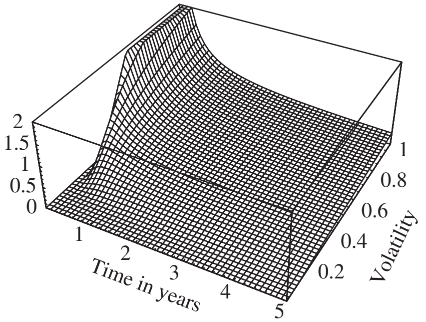

Fig. 4. Default arrival intensity when the default barrier is observable but assets are not. We plot λ_t (in events per year) as a function of t (in years) and σ (in percent).

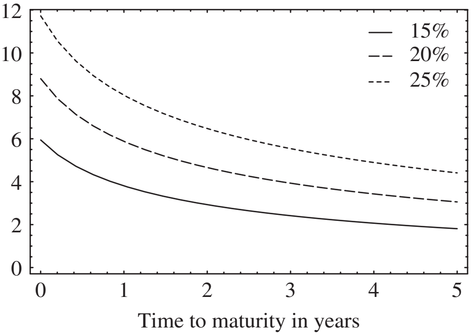

Fig. 5. Term structure of credit spreads when both assets and default barrier are unobservable. We plot $S(t, \varepsilon)$ (in percent) as a function of ε (in years), for $\sigma = 15\%$, 20%, and 25%. We set $t = 0.5$ years.

partial derivative of $\Psi(t, \cdot)$ is easily obtained from formula (24). In Fig. 4 the intensity is plotted as a function of time t and asset volatility σ. The intensity is increasing in the degree of business risk, as proxied by asset volatility σ.

7.4. No information about the barrier and the assets

If no information about assets and barrier is available, then the model filtration $\mathbb{F}$ is the trivial completed filtration. Investors observe only the default. Since $\mathrm{p}(t, 0)$ satisfies Conditions 6.3(1) and (2), Proposition 6.5 implies that the trend is a deterministic function of time given by $A_t = -\log(1 - \mathrm{p}(t, 0))$ where $\mathrm{p}(t, 0)$ is given by the formula (26). According to Corollary 5.5, the conditional default probability at time $t < \tau$ for the term ε is

$$q(t, \varepsilon) = \frac{\mathrm{p}(t + \varepsilon, 0) - \mathrm{p}(t, 0)}{1 - \mathrm{p}(t, 0)}$$

almost surely. Spread curves are shown in Fig. 5, where we vary the asset volatility. As expected, the higher the volatility the higher the spread. Spreads are bounded

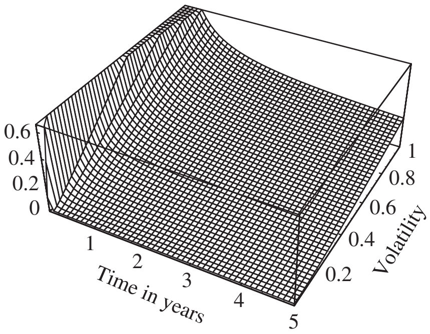

Fig. 6. Default arrival intensity when both assets and default barrier are unobservable. We plot λ_t (in events per year) as a function of t (in years) and σ (in percent).

away from zero for all maturities. Noting that $\mathrm{p}(t,0)$ satisfies Condition 6.3(3), for $t>0$ the short credit spread satisfies $\lim_{\varepsilon\downarrow 0} S(t,\varepsilon)=\lambda_t=(\partial/\partial t)\mathrm{p}(t,0)/(1-\mathrm{p}(t,0))$ almost surely. In Fig. 6 we graph λ_t as a function of time t and asset volatility σ.

8. Conclusion

First passage models postulate that a firm defaults when its assets fall below some liability-dependent barrier. In these models, it is typically assumed that the information used to calibrate and run the model is publicly available. In practice, however, asset value, volatility and growth rate are difficult to observe directly. Corporate statements are difficult to interpret, making it hard to deduce the correct value of the default barrier.

We analyze the role of investor information in first passage models. We specify a default model through a stopping time and a model filtration. The latter describes investors' information about the inputs to the model definition of default. For a given default definition the model filtration parameterizes a family of default models.

In incomplete information models the model filtration is too coarse to make the default observable. Examples include models with incomplete asset information such as the one described by Duffie and Lando (2001), and models with incomplete information about the default barrier. Here the default time is typically totally inaccessible, as in the classical reduced-form models of default. In reduced-form models, the information in the model filtration is used to describe the intensity, or conditional rate of default. The intensity leads to convenient security pricing formulae.

The commonality between incomplete information models and reduced-form models extends to the pricing formulae. We introduce the *trend* of a default model and show that under mild conditions, all incomplete information models with inaccessible default time admit *generalized* reduced-form pricing formulae in terms of their trend. If the trend is absolutely continuous, then the associated density is the intensity and our formulae simplify to the classical intensity-based formulae.

There are incomplete information models that do not admit an intensity. We provide a first example, a model in which investors cannot observe the default barrier. Here investors learn over time about the location of the barrier: it must lie below the observable historical low of assets to date if the firm has not defaulted. *Learning* has interesting empirical implications, which are analyzed in Giesecke and Goldberg (2004a). We find that the incomplete information model reacts more quickly to changes in the asset value, since it takes account of the whole history of asset values, not just current values. It detects credit quality changes earlier than the corresponding model with complete information. Learning has also important implications in the context of multiple firms. As described in Giesecke (2004) and Giesecke and Goldberg (2004b), it leads to information-based *contagion* effects between firm defaults.

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References

Airault, H., Föllmer, H., 1974. Relative densities of semimartingales. Inventiones Mathematicae 27, 299–327.

Anderson, R., Sundaresan, S., 1996. Design and valuation of debt contracts. Review of Financial Studies 9, 37–68.

Artzner, P., Delbaen, F., 1995. Default risk insurance and incomplete markets. Mathematical Finance 5, 187–195.

Beneish, M., Press, E., 1995. Interrelation among events of default. Contemporary Accounting Research 12, 299–327.

Black, F., Cox, J.C., 1976. Valuing corporate securities: some effects of bond indenture provisions. Journal of Finance 31, 351–367.

Black, F., Scholes, M., 1973. The pricing of options and corporate liabilities. Journal of Political Economy 81, 81–98.

Daley, D., Vere-Jones, D., 2003. An Introduction to the Theory of Point Processes, vol. I. Springer, New York.

Dellacherie, C., Meyer, P.A., 1982. Probabilities and potential. North-Holland, Amsterdam.

Duffie, D., Lando, D., 2001. Term structures of credit spreads with incomplete accounting information. Econometrica 69 (3), 633–664.

Duffie, D., Singleton, K.J., 1999. Modeling term structures of defaultable bonds. Review of Financial Studies 12, 687–720.

Duffie, D., Schroder, M., Skiadas, C., 1996. Recursive valuation of defaultable securities and the timing of resolution of uncertainty. Annals of Applied Probability 6, 1075–1090.

Duffie, D., Pedersen, L.H., Singleton, K.J., 2003. Modeling sovereign yield spreads: a case study of Russian debt. Journal of Finance 58 (1), 119–159.

Föllmer, H., 1972. The exit measure of a supermartingale. Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete 21, 154–166.

Giesecke, K., 2004. Correlated default with incomplete information. Journal of Banking and Finance 28, 1521–1545.

Giesecke, K., Goldberg, L., 2004a. Forecasting default in the face of uncertainty. Journal of Derivatives 12 (1), 1–15.

Giesecke, K., Goldberg, L., 2004b. Sequential defaults and incomplete information. Journal of Risk 7 (1), 1–26.

Jarrow, R.A., Turnbull, S.M., 1995. Pricing derivatives on financial securities subject to credit risk. Journal of Finance 50 (1), 53–86.

Jeulin, T., Yor, M., 1978. Grossisment d'une filtration et semimartingales: formules explicites in: Séminaire de Probabilités XII, Lecture Notes in Mathematics, vol. 649, Springer, Berlin, pp. 78–97.

Kusuoka, S., 1999. A remark on default risk models. Advances in Mathematical Economics 1, 69–82.

Lambrecht, B., Perraudin, W., 1996. Creditor races and contingent claims. European Economic Review 40, 897–907.

Lando, D., 1998. On Cox processes and credit risky securities. Review of Derivatives Research 2, 99–120.

Leland, H.E., 1994. Corporate debt value, bond covenants, and optimal capital structure. Journal of Finance 49 (4), 1213–1252.

Leland, H.E., Toft, K.B., 1996. Optimal capital structure endogenous bankruptcy and the term structure of credit spreads. Journal of Finance 51 (3), 987–1019.

Longstaff, F.A., Schwartz, E.S., 1995. A simple approach to valuing risky fixed and floating rate debt. Journal of Finance 50 (3), 789–819.

Merton, R.C., 1974. On the pricing of corporate debt: the risk structure of interest rates. Journal of Finance 29, 449–470.

Protter, P., 2004. Stochastic Integration and Differential Equations. Springer, New York.

RiskMetrics Group, 2002. Creditgrades. Technical Document, New York.

Sarig, O., Warga, A., 1989. Some empirical estimates of the risk structure of interest rates. Journal of Finance 44, 1351–1360.

Yu, F., 2005. Accounting transparency and the term structure of credit spreads. Journal of Financial Economics 75 (1), 53–84.

Zhou, C., 2001. The term structure of credit spreads with jump risk. Journal of Banking and Finance 25, 2015–2040.