DEPENDENT EVENTS AND CHANGES OF TIME*

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Abstract

Meyer (1971) showed that any point process whose compensator has continuous paths that increase to ∞ can be time-scaled to a standard Poisson process. In this article we consider the converse to this result. We construct a time change with continuous paths increasing to ∞ that transforms a standard Poisson process into a general point process with totally inaccessible arrivals and compensator given by the time change. The time change generates path-dependent or self-affecting point processes whose dynamics depend on the information generated by the arrivals of the process as well as other observable information describing the state of the random environment. The classical Hawkes and doubly stochastic processes are special cases. Time-changed Hawkes processes are shown to combine the best features of these classical families is a flexible and tractable way. We conclude by introducing time-change techniques to multi-name credit modeling. We describe the economy-wide default process as a time-changed Poisson process, whose arrivals are temporally clustered due to contagion and depend on the economic environment. Random thinning generates sub-models for individual firms and portfolios of firms.

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1 Introduction

Point processes on the real line provide the probabilistic tools to analyze the dynamics of events observed over time, such as earthquakes, insurance claims, corporate defaults, component failures in a system or trades on the stock exchange. The stochastic structure of a point process is encoded in its *compensator*, which describes the dynamics of event arrivals relative to the available information.

In 1971, Paul-André Meyer proved that any point process whose compensator has continuous paths that increase to ∞ can be time-scaled to a standard Poisson process. The scaling or *time change* is given by the inverse to the compensator. Thus, a large class of point processes with totally inaccessible event times can be viewed as standard Poisson processes in the appropriate time scale. This is analogous to Dubins & Schwarz's (1965) result that any continuous local martingale whose quadratic variation increases to ∞ can be time-scaled to a standard Brownian motion. Here the time change is given by the inverse to the quadratic variation.

Meyer's powerful result immediately yields statistical tests for the fit of a point process model. Observe a sequence of arrivals, transform the sequence with the model compensator. If the model is correctly specified, then the transformed sequence is a realization of a standard Poisson process. Vice versa, a parametric point process model can be estimated by choosing the parameters such that the model compensator gives the correct scaling.

In this paper we study the converse approach to Meyer's. We construct a time change with continuous paths increasing to ∞ that transforms a standard Poisson process into a general point process with totally inaccessible arrivals and compensator given by the time change. This construction immediately provides the recipe for simulating the resulting process, starting with a simple Poisson process. Meyer's result implies fitness tests for the resulting process.

Our approach generalizes the well-known construction of a time-inhomogeneous Poisson process from a homogeneous Poisson process by a deterministic change of time, which is recalled in Section 2. In Section 3, we construct a time change to generate path-dependent or *self-affecting* point processes whose dynamics depend on the information generated by the arrivals of the process as well as other observable information describing the state of the random environment. In this general case the time change is stochastic. At any time, it depends on the path of the resulting process and the random environment up to that time.

As a by-product, our construction provides a representation of the compensator of a self-affecting process in a filtration typically larger than the filtration generated by the process itself. This extends the results of Chou & Meyer (1975), who consider the filtration generated by the process itself. Our construction is also related to the methods in Jeulin & Yor (1978), who describe the compensator of a one-point process in the context of enlargement of filtrations.

Absolutely continuous time changes lead to intensity based point processes, which are very flexible and thus dominant in applications. The associated intensity describes the conditional event arrival rate relative to the observable information. Special cases include the Hawkes (1971) process and the doubly stochastic process, both discussed in Section 4. The Hawkes process is the prototypical example of a self-affecting process. Its intensity at any time depends only on the information generated by the arrivals observed up to that time, such as the arrival times and their "marks." This models the temporal clustering in arrivals. The doubly stochastic process is on the other end of the spectrum in that it does not have the self-affecting property. Its intensity depends on the information generated by the state of the random environment, but not on the arrival history.

In Section 5 we study the class of *time-changed Hawkes processes*, which are obtained through the composition of the two absolutely continuous time changes that generate the Hawkes and the doubly stochastic processes. The resulting processes generalizes both classical families and inherit their best features. Their intensities depend in a flexible way on both the arrival history and externally observable factors. Simulation is as easy and efficient as in the doubly stochastic case.

We conclude in Section 6 by proposing a time-change based approach to *multi-name credit modeling*. In the "top-down" framework of Giesecke & Goldberg (2005), the economy-wide default process is described by a time-changed Poisson process, whose arrivals are temporally clustered due to contagion and depend on the economic environment. The technique of random thinning consistently generates sub-models for individual firms or portfolios of firms. Applications include the pricing and risk management of multi-name credit products such as collateralized debt obligations, tranches, default baskets and options on these instruments.

Our time-change strategy for generating realistic stochastic process models for credit derivatives is conceptually analogous to the time-change approach in equity derivatives modeling that was initiated by Clark (1973) and further developed by Geman & Ané (1996), Madan, Carr & Chang (1998), Geman, Madan & Yor (2001) and others. Here a Brownian motion is time-changed to generate a general semimartingale asset price process whose diffusive volatility is stochastic and whose jumps arrive according to a stochastic intensity. This is motivated by Monroe's (1978) result that every semimartingale can be written as a time-changed Brownian motion.

2 Time-Changing the Poisson Process: Basic Idea

Let $V_1, V_2, \ldots$ be a sequence of independent, identically distributed standard exponential random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose V_i represents the waiting time between the $(i-1)$st and ith event, so $S_n = V_1 + V_2 + \ldots + V_n$ represents the arrival time of the nth event. Since S_n is a sum of independent exponentials, the counting process N^0 defined by

$$N^0_t = \sum_n 1_{\{S_n \leq t\}}$$

is a standard *Poisson process* in the filtration $\mathbb{H} = (\mathcal{H}_t)_{t \geq 0}$ generated by the times $(S_n)_{n \geq 1}$. For $s < t$, the distribution of the number of arrivals $N^0(s,t]$ in the interval $(s,t]$ is Poisson with parameter $t - s$, i.e.

$$\mathbb{P}[N^0(s,t] = n] = \frac{(t-s)^n}{n!} e^{-(t-s)} \quad n = 0, 1, 2, \ldots \tag{1}$$

so $\mathbb{E}[N^0(s,t]] = \mathrm{Var}[N^0(s,t]] = t - s$. The compensated Poisson process $(N^0_t - t)$ is an $\mathbb{H}$-martingale. In other words, the process (t) is the *compensator* to N^0. Hence the intensity, or conditional event arrival rate, at time t is 1, almost surely:

$$\lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \mathbb{E}[N^0(t, t+\epsilon] \,|\, \mathcal{H}_t] = 1.$$

The assumption of unit arrival rates underlying the standard Poisson process is too restrictive in many situations of practical interest. Suppose we want to generate a *time-inhomogeneous* Poisson process N with arrival times (T_n), whose conditional event arrival rate $\lambda(t)$ is a positive, deterministic function of time so as to capture the basic time variation in the arrivals. A simple way to do this is to let G be the continuous, strictly increasing deterministic function defined by

$$G(t) = \int_0^t \lambda(s) ds$$

and then *re-scale* the arrival times (S_n) of the standard Poisson process N^0 with G by setting

$$T_n = G^{-1}(S_n) = \inf\{t > 0 : G(t) \geq S_n\}. \tag{2}$$

The counting process N associated with the times (T_n) is given as the standard Poisson process N^0, *time-changed* by the deterministic function G:

$$N_t = \sum_{n=1}^{\infty} 1_{\{T_n \leq t\}} = \sum_{n=1}^{\infty} 1_{\{S_n \leq G(t)\}} = N^0_{G(t)}.$$

In other words, we can view the inhomogeneous Poisson process N as the Poisson process N^0 in the deterministic time scale generated by the function G.

For $s < t$, the distribution of the number of arrivals $N(s,t]$ in the interval $(s,t]$ is Poisson with parameter $G(t) - G(s)$:

$$\mathbb{P}[N(s,t] = n] = \frac{(G(t) - G(s))^n}{n!} e^{-(G(t)-G(s))} \quad n = 0, 1, 2, \ldots \tag{3}$$

so $\mathbb{E}[N(s,t]] = \mathrm{Var}[N(s,t]] = G(t) - G(s)$. It is easy to check that the process $N - G$ is an martingale in the filtration $\mathbb{G}$, defined by $\mathcal{G}_t = \mathcal{H}_{G(t)}$. Hence the function G is the compensator to N in the time-changed filtration $\mathbb{G}$. It follows that almost surely

$$\lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \mathbb{E}[N(t, t+\epsilon] \,|\, \mathcal{G}_t] = \lambda(t).$$

The inhomogeneous Poisson process captures the deterministic variation in event arrival intensities, for example the known seasonality in the arrival of insurance claims. It does not account for the information generated by its arrivals, neither does it account for other information revealed over time, for example about the state of the stochastic environment. Below we extend the time change argument to generate general counting processes N, whose arrival intensities vary stochastically over time. This variation may come from the response in the intensity on arrivals of N, or from the dependence on the state of the random environment.

3 Self-affecting Time-Changed Poisson Processes

We provide a rigorous construction of the *stochastic* time change transforming a standard Poisson process into a general counting process. Let $(\Omega, \mathcal{H}, \mathbb{P})$ be a probability space equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ containing all $\mathbb{P}$-null sets of $\mathcal{H}$. We take as given a standard marked Poisson process $N^0 = (N^0_t)_{t \geq 0}$ with respect to the filtration $\mathbb{H} = (\mathcal{H}_t)_{t \geq 0}$, which contains $\mathcal{F}_\infty$ for all t. This implies that the process N^0 is independent of $\mathbb{F}$. Let the arrival times of N^0 be denoted $0 < S_1 < S_2 < \ldots$ and the marks be denoted $Z_n \in \mathcal{M}$, where $\mathcal{M}$ is a general mark space.

3.1 Construction

We define a point process through measurable transformations of the inter-arrival times of the Poisson process. The transformation of $S_n - S_{n-1}$ will be given by the inverse of an adapted process G_n, which we refer to as the nth *time change*. To allow transformations back and forth between the Poisson process and the process we create, we require all time changes to be continuous one-to-one mappings from the non-negative real line onto itself. Thus, for every n, G_n is continuous and strictly increasing, with $G_n(0) = 0$, $G_n(t) < \infty$ for all t and $\lim_{t\to\infty} G_n(t) = \infty$, almost surely. Note that G_n^{-1} exists, and also satisfies these conditions.

Define the filtration $\mathbb{G}^0 = \mathbb{F}$, and for $n \geq 1$:

- Let G_n be a stricty increasing, continuous, $\mathbb{G}^{n-1}$-adapted process on $\mathbb{R}_+$, with the properties above.

  At a time t after the $(n-1)$st arrival, $G_n(t)$ depends on information in $\mathbb{F}$ up to time $T_{n-1} + t$, the arrival times $T_1, \ldots, T_{n-1}$, and their marks $Z_1, \ldots, Z_{n-1}$.

- Let $T_n = T_{n-1} + G_n^{-1}(S_n - S_{n-1}) = \sum_{k=1}^{n} G_k^{-1}(S_k - S_{k-1})$.

- Let the filtration $\mathbb{G}^n$ be the right continuous version of $(\mathcal{G}^n_{T_n+t})_{t\geq 0}$, where

  $$\mathcal{G}^n_u = \mathcal{F}_u \vee \sigma(1_{\{T_k \leq s\}}, Z_k 1_{\{T_k \leq s\}} : s \leq u, k \leq n).$$

  Let the definition of the first n times (T_k) be given. At time $T_n + t$, the σ-algebra $\mathcal{G}^n_{T_n+t}$ represents the information described by the information in $\mathbb{F}$ up to time $T_n + t$ and the information generated by the times $T_1, \ldots, T_n$, their marks $Z_1, \ldots, Z_n$ and the Poisson times $S_1, \ldots, S_n$.

This construction clearly indicates how to simulate the resulting process. To do this, we first need to simulate the arrival times S_n of the standard Poisson process N^0, and then the processes G_n^{-1}, which may depend on the Poisson arrivals. The G_n's may also depend on the information in $\mathbb{F}$, which may make the simulation of their inverses computationally intensive. Below, we consider a general, but more tractable, form for G_n that simplifies simulation.

3.2 Counting Process

The counting process N associated with the event times (T_n) is defined by

$$N_t = \sum_{k=1}^{\infty} 1_{\{T_k \leq t\}}. \tag{4}$$

We wish to ensure that the process N is non-exploding, which is guaranteed so long as $\lim_{n\to\infty} T_n = \infty$ almost surely. Since this is not guaranteed by our construction, we provide sufficient conditions on the transformations G_n.

Proposition 3.1. *If there exists a subsequence n_k, an $\epsilon > 0$, and a sequence of bounds $(M_k)_{k\geq 0}$ such that $\sum_{k=1}^{\infty} e^{-M_k} = \infty$ and $G_{n_k}(\epsilon) < M_k$ almost surely for all k, then the process N is non-exploding.*

Proposition 3.2. *If there exists a non-exploding point process $\hat{N}$, constructed from N^0 with time change functions $(\hat{G}_n)_{n\geq 0}$, and such that $G_n(t) \leq \hat{G}_n(t)$ for all t, then N is non-exploding.*

In applications, we are often dealing with a finite number of arrivals. If we are interested in at most the first n_0 arrivals, we can take $G_n(t) = t$ for all $n > n_0$ without loss of generality. This will yield a non-exploding point process N with the desired dynamics for the first n_0 arrivals, and we can then focus on $N_{t\wedge T_{n_0}}$. In the following, we assume the process N is non-exploding.

3.3 Compensator and Distribution

Let the σ-algebra $\mathcal{G}_t$ represent all the information available at time t. The corresponding filtration $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ is the right continuous version of the filtration generated by the σ-algebras

$$\bigvee_{n=1}^{\infty} \mathcal{G}_t^n \supset \mathcal{F}_t \vee \sigma(1_{\{T_n \leq s\}}, Z_n 1_{\{T_n \leq s\}} : s \leq t, n \geq 1).$$

Under the additional assumption that $\mathbb{E}[N_t] < \infty$ for all t, the counting process N defined in (4) is a right continuous $\mathbb{G}$-submartingale. It admits a unique Doob-Meyer decomposition into the sum of a $\mathbb{G}$-local martingale and a non-decreasing, $\mathbb{G}$-predictable process starting at zero called the $\mathbb{G}$-*compensator* of N.

We construct the compensator from the time changes G_n. Define

$$G(t) = \sum_{k=1}^{n-1} G_k(T_k - T_{k-1}) + G_n(t - T_{n-1}) \quad \text{on} \quad \{T_{n-1} \leq t < T_n\}. \tag{5}$$

By the properties of the G_n's, and since $T_n \to \infty$ almost surely, the process G is continuous and $\mathbb{G}$-adapted, hence predictable. It is also strictly increasing, with $G(0) = 0$, $G(t) < \infty$ almost surely, and $\lim_{t\to\infty} G(t) = \infty$ almost surely.

Furthermore, G defines a change of time, and we have that for all n,

$$S_n = \sum_{k=1}^{n} G_k(T_k - T_{k-1}) = G(T_n). \tag{6}$$

Since G is strictly increasing, for any n and t, we have $T_n = G^{-1}(S_n)$ and the events $\{T_n \le t\} = \{S_n \le G(t)\}$. It follows that

$$N_t = N^0_{G(t)}. \tag{7}$$

Theorem 3.3. *N and G are $\mathbb{G}$-adapted, and for all n, the stopped process $(N_{t\wedge T_n} - G(t\wedge T_n))$ is a $\mathbb{G}$-martingale. If $T_n \to \infty$ almost surely, $N-G$ is a $\mathbb{G}$-local martingale, with localizing sequence (T_n). Thus G is the $\mathbb{G}$-compensator of N.*

Remark 3.4. Under the additional assumption that $\mathbb{E}[N_t] < \infty$ for all $t \ge 0$, we have $\mathbb{E}[[M, M]_t] < \infty$ for $M = N - A$. By Corollary 3 in Protter (2004, page 73), the process M is a martingale with $\mathbb{E}[M_t^2] < \infty$ for all $t \ge 0$. □

Remark 3.5. Since the compensator G to N is continuous almost surely, the arrival times (T_n) of N are totally inaccessible stopping times in the filtration $\mathbb{G}$, see Dellacherie & Meyer (1982, Chapter VII, Theorem 14). □

Remark 3.6. Theorem 3.3 is a partial converse to Meyer's (1971) time change theorem for point processes. Let N be a $\mathbb{G}$-adapted counting process with continuous compensator A, such that $\lim_{t\to\infty} A_t = \infty$ almost surely. If A^{-1} is the right-continuous inverse of A, then $N_{A^{-1}}$ is a standard Poisson process with respect to the filtration $(\mathcal{G}_{A_t^{-1}})_{t\ge 0}$. Meyer transforms a general counting process N with totally inaccessible arrival times into a Poisson process, whereas we start with a Poisson process and transform it into a general process N with totally inaccessible arrival times. In both cases, the transformation is given by the compensator to N. □

The distribution of the inter-arrival times can be expressed in terms of the G_n's.

Theorem 3.7. *For $t \ge s$ and $n \ge 1$, the following equalities hold almost surely*

$$\mathbb{P}(T_n - T_{n-1} > t | \mathcal{G}^{n-1}_{T_{n-1}+s}) = \mathbb{E}[e^{-G_n(t)} | \mathcal{G}^{n-1}_{T_{n-1}+s}] \tag{8}$$

$$\mathbb{P}(T_n - T_{n-1} > t | \mathcal{G}_{T_{n-1}+s}) = \mathbb{E}[e^{-[G_n(t)-G_n(s)]} | \mathcal{G}_{T_{n-1}+s}] \quad on \quad \{T_n - T_{n-1} > s\}. \tag{9}$$

Remark 3.8. Taking $t = s$ in equation (8) gives

$$G_n(t) = -\log \mathbb{P}(T_n - T_{n-1} > t \,|\, \mathcal{G}^{n-1}_{T_{n-1}+t}) \tag{10}$$

almost surely for each t. Thus, the time changes G_n can be specified through the conditional distribution of the inter-arrival times. By letting Z_n be the *optional projection* of $(1_{\{T_n - T_{n-1} > t\}})$ onto the filtration $\mathbb{G}^{n-1}$ and setting $G_n = -\log Z_n$ we define the process G_n up to indistinguishability such that (10) holds almost surely for each t. For details see Dellacherie & Meyer (1982, Chapter VI, Theorem 43). □

Remark 3.9. Chou & Meyer (1975) construct the compensator of a general counting process N with respect to the minimal completed filtration generated by N. For point processes N with continuous compensators that increase strictly to infinity, Theorem 3.3 and Remark 3.8 extend their construction to a filtration $\mathbb{G}$ that may be larger than the filtration generated by N. In our case, the "extra information" in $\mathbb{G}$ is represented by the filtration $\mathbb{F}$ and the information about the marks of N. □

3.4 Intensity

To study the properties of the time-changed Poisson process N, it is instructive to consider time changes that are absolutely continuous. Suppose G is almost surely absolutely continuous with respect to the Lebesgue measure. If $T_n \to \infty$ almost surely, this is equivalent to G_n being absolutely continuous for all n. Then $G(t) = \int_0^t \lambda_s ds$ and $G_n(t) = \int_0^t g_n(s)ds$, almost surely. Since G is the compensator to N, the density λ is the *intensity*, or conditional arrival rate of N. It is given by

$$\lambda_t = g_n(t - T_{n-1}) \quad \text{on} \quad \{T_{n-1} \leq t < T_n\}. \tag{11}$$

In the absolutely continuous case the process N is called intensity based.

Define, for $n \geq 1$ and $t \geq 0$, the following *impact processes*:

$$\begin{aligned} \nu_0(t) &= g_1(t) \\ \nu_n(t) &= g_{n+1}(t) - g_n(t + T_n - T_{n-1}). \end{aligned} \tag{12}$$

The intensity of the process N at time $T_n + t$ (the post-T_n intensity) is

$$\lambda_{T_n+t} = g_{n+1}(t)$$

and at time $t \in [T_{n-1}, T_n)$, the pre-T_n intensity process is

$$\lambda_t = g_n(t - T_{n-1})$$

Thus, $\nu_n(t)$ is the difference between the post-T_n and pre-T_n intensity functions evaluated at time $T_n + t$. This means that $\nu_n(t)$ represents the impact of the nth arrival on the intensity after time t has passed since the event, and ν_0 represents the arrival intensity of the first event. In particular, the jump in λ at time T_n is

$$\Delta\lambda_{T_n} = g_{n+1}(0) - g_n(T_n - T_{n-1}) = \nu_n(0).$$

Now, by summing over the ν_n's, we get

$$g_n(t - T_{n-1}) = \nu_0(t) + \sum_{k=1}^{n-1} \nu_k(t - T_k). \tag{13}$$

As a result, the intensity of N has the special form:

$$\lambda_t = \nu_0(t) + \sum_{T_k<t} \nu_k(t - T_k). \tag{14}$$

The kth arrival contributes the term $\nu_k(t - T_k)$ to the intensity λ. This term may depend on T_i and Z_i for $i \le k$ and so the arrival intensity of the $(k+1)$st event depends on the timing of the previous k events and their associated marks. When the impact processes are positive, the arrivals cluster together, or are overdispersed relative to a reference process with intensity ν_0. Conversely, when the impact processes are negative, the arrivals are spread out, or are underdispersed relative to this reference process. Since the past arrivals affect the timing of future arrivals, N can be described as path-dependent, or *self-affecting*.

The second consequence of (13) is that

$$G_n(t) = \int_0^t \nu_0(s + T_{n-1})ds + \sum_{k=1}^{n-1} \int_0^t \nu_k(s + T_{n-1} - T_k)ds.$$

Since G_n is a $\mathbb{G}^{n-1}$-adapted, continuous, one-to-one, mapping of the non-negative real line onto itself, this yields conditions on the sequence of impact processes.

Proposition 3.10. *For any $n \ge 0$, the impact processes ν_n satisfy the following conditions:*

(1) ν_n is $\mathbb{G}^n$-adapted since g_{n+1} is $\mathbb{G}^n$-adapted.

(2) $\nu_n(t) > -\sum_{k=0}^{n-1} \nu_k(t + T_n - T_k)$ almost surely for all $t \ge 0$.

(3) $\int_0^t \nu_n(s)ds < \infty$ almost surely since $G_{n+1}(t)$ is finite almost surely for all $t \ge 0$.

(4) ν_n must be such that $\lim_{t\to\infty} G_{n+1}(t) = \infty$, almost surely for all $t \ge 0$.

Furthermore, we see that

$$\nu_n(t) \equiv 0 \quad \Leftrightarrow \quad G_{n+1}(t) = G_n(t + T_n - T_{n-1}) - G_n(T_n - T_{n-1})$$

Example 3.11 (Positive Impact Processes). Let $\nu_0(t) \equiv \nu_0 > 0$ be constant, and take $\nu_n(t) = a_n e^{-b_n t}$, for $a_n \geq 0$ and $b_n > 0$. These impact processes satisfy the conditions in Proposition 3.10, even if a_n and b_n are functions of $T_1, \dots, T_n$ and $Z_1, \dots, Z_n$. □

Example 3.12 (Negative Impact Processes). Let $\nu_0(t) \equiv \nu_0 > 0$ be constant, and choose a sequence $p_n \in (0, 1)$. Take $\nu_n(t) = -\nu_0(1 - p_n)\Pi_{k=1}^{n-1} p_k$. Then the conditions in Proposition 3.10 are satisfied. In particular, condition (2) holds since for any $n \geq 1$ and any $t \geq 0$,

$$-\sum_{k=0}^{n-1} \nu_k(t + T_n - T_k) = -\nu_0 \prod_{k=1}^{n-1} p_k < \nu_n(t).$$

In addition, p_n can be a function of $T_1, \dots, T_n$ and $Z_1, \dots, Z_n$. □

4 Two Classical Families as Special Cases

In the intensity based case, we can specify the time changes via the filtration $\mathbb{F}$ and the impact processes (ν_n), based on our view of how the intensity of the process will respond to its arrivals and the information described in $\mathbb{F}$. Following this strategy, below we characterize two classical families of intensity based processes, the Hawkes process and the doubly stochastic process. This provides some understanding of the class of processes that can be obtained by time-changing the Poisson process.

4.1 Hawkes Processes

Assume the filtration $\mathbb{F}$ is trivial: $\mathcal{F}_t = \mathcal{F}_0 = \sigma(Z_0) \vee \mathcal{N}$ for all t, where $Z_0 \in \mathcal{M}$ is some initial mark and $\mathcal{N}$ is the set of $\mathbb{P}$-null sets of $\mathcal{H}$. Therefore, the impact variable $\nu_n(t)$ is measurable with respect to the σ-algebra $\sigma(T_1, \dots, T_n) \vee \sigma(Z_0, \dots, Z_n)$ for all t. Since this σ-algebra does not depend on t, $\nu_n(t)$ is a deterministic function of t, the times $T_1, \dots, T_n$ and the marks $Z_0, \dots, Z_n$. Then the intensity of the resulting point process N is given by $\lambda = \mu$ with

$$\mu_t = \nu_0(t) + \sum_{T_n < t} \nu_n(t - T_n) \tag{15}$$

and N is called a *Hawkes process*. It was proposed by Hawkes (1971) for the special case ν_0 a constant and $\nu_n(t) = \nu(t, Z_n)$ for $n \geq 1$, where $\nu : \mathbb{R}_+ \times \mathcal{M} \to \mathbb{R}_+$ is a non-negative deterministic function. With this choice an arrival always increases the intensity, which suggests to call the process N "self-exciting." See Kwiecinski & Szekli (1996) for a rigorous analysis of this property.

Versions of the Hawkes process have been widely used in seismology, neurophysiology, epidemiology and reliability; see Daley & Vere-Jones (2003) for many examples. In finance, variants of the Hawkes process have recently been proposed by Giesecke & Goldberg (2005) as probabilistic models for the arrival of dependent firm defaults in a bond portfolio. Here the idea is that an individual default can trigger further defaults through contagion. Bowsher (2003) introduces the Hawkes process as an econometric model for the frequency of stock market trades, which are often temporarily clustered.

Simulation of the arrival times (T_n) corresponding to a choice of impact processes (ν_n) can be done as follows:

(1) Simulate $Z_0 \in \mathcal{M}$, and let $T_0 = 0$.

(2) For $n \geq 0$: given $T_0, \ldots, T_n$ and $Z_0, \ldots, Z_n$, simulate T_{n+1} as the first arrival time of an inhomogenous Poisson process with intensity at time t given by

$$\nu_0(t) + \sum_{k=1}^{n} \nu_k(t - T_k),$$

which is a deterministic function of t given the available information. Given T_{n+1}, simulate $Z_{n+1} \in \mathcal{M}$.

The second step in the above procedure can be done by thinning a homogeneous Poisson process, or by the inverse compensator method, which is equivalent to deterministic transformations of the inter-arrival times. Moller & Rasmussen (2004) propose alternative simulation methods that are based on the Poisson cluster process representation of N established in Hawkes & Oakes (1974).

4.2 Doubly Stochastic Processes

Hawkes processes are based on the assumption that $\mathbb{F}$ is trivial. In this case, the only randomness in the intensity is from the realized arrival times and marks. We consider the case when $\mathbb{F}$ is non-trivial, but the impact processes ν_n are identically zero for $n \geq 1$. This implies the randomness in the intensity is due solely to the arrival of information described in $\mathbb{F}$. Such processes are not self-affecting.

Suppose $\Gamma(t) = \int_0^t \gamma_s ds$ is a strictly increasing, absolutely continuous, $\mathbb{F}$-adapted process. For concreteness, we assume that $X \in \mathbb{R}^d$ is a stochastic process describing the state of the random environment, $\mathbb{F}$ is the filtration generated by X, and $\gamma_t = h(X_t)$, where $h : \mathbb{R}^d \to \mathbb{R}_+$ is a positive measurable function. For $n \geq 1$, choose

$$\nu_0(t) = \gamma_t, \qquad \nu_n(t) = 0.$$

This yields a point process N with intensity $\lambda = \gamma$, and compensator $G = \Gamma$. Using Proposition A.10, the strong Markov property of the Poisson process, and the fact that Γ is $\mathbb{F}$-adapted, we almost surely get, for $t > s$

$$\begin{aligned}
\mathbb{P}(N_t - N_s = k \,|\, \mathcal{G}_s \vee \mathcal{F}_t) &= \mathbb{P}(N^0_{\Gamma(t)} - N^0_{\Gamma(s)} = k \,|\, \mathcal{G}_s \vee \mathcal{F}_t) \\
&= \mathbb{E}[\mathbb{P}(N^0_{\Gamma(t)} - N^0_{\Gamma(s)} = k \,|\, \mathcal{H}_{\Gamma(s)}) \,|\, \mathcal{G}_s \vee \mathcal{F}_t] \\
&= \frac{1}{k!}\mathbb{E}[(\Gamma(t) - \Gamma(s))^k e^{-[\Gamma(t)-\Gamma(s)]} \,|\, \mathcal{G}_s \vee \mathcal{F}_t] \\
&= \frac{1}{k!}(\Gamma(t) - \Gamma(s))^k e^{-[\Gamma(t)-\Gamma(s)]}.
\end{aligned}$$

In other words, conditional on the σ-algebra $\mathcal{G}_s \vee \mathcal{F}_t$, the random variable $N_t - N_s$ has the Poisson distribution with parameter $\Gamma(t) - \Gamma(s)$; compare with (3). Thus N is a *doubly stochastic process*, or Cox process driven by $\mathbb{F}$; see Section 3 in Giesecke & Goldberg (2005) for a rigorous analysis. As outlined in Daley & Vere-Jones (2003), the doubly stochastic process is a versatile probabilistic model with applications in seismology and reliability, to name a few fields. In finance, it was recently used by Das, Duffie & Kapadia (2004) to model the arrival of defaults in a security portfolio. The idea is that dependence between default events arises from the dependence on the economic environment, whose state is described by the process X.

The time change implies that

$$T_n = \Gamma^{-1}(S_n) = \inf\{t > 0 : \Gamma(t) \geq S_n\} \tag{16}$$

for the Poisson arrivals (S_n). Hence, a doubly stochastic arrival time T_n can be simulated by finding the time that Γ first reaches the Poisson arrival time S_n. Here, S_n and Γ are generated independently of each other.

5 A Class of Time-Changed Hawkes Processes

We propose a broad subclass of intensity based, time-changed Poisson processes that are a combination of the two classical families covered in the preceding section. Their intensities will depend both on information in $\mathbb{F}$ and on the realization of

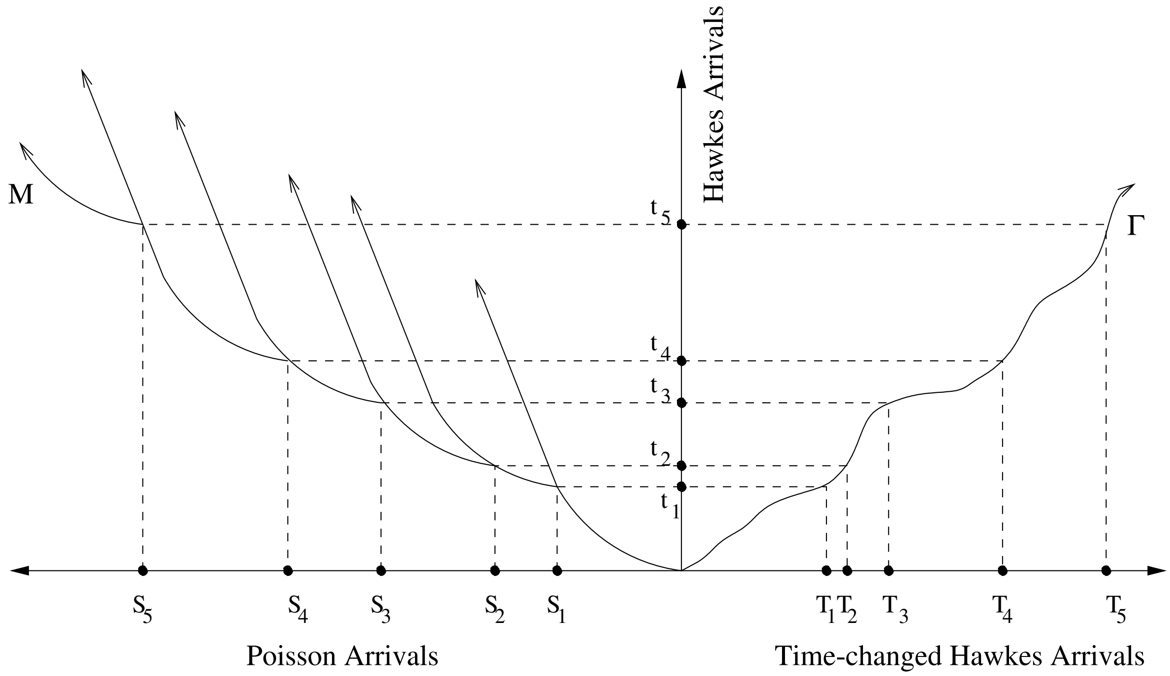

Figure 1: Construction of the time-changed Hawkes process.

arrivals and their marks. Furthermore, processes in this subclass are more tractable and easier to simulate than general time-changed Poisson processes.

The subclass is constructed via a stochastic time change that is the composition of two time changes, where the first is of the form described in Section 4.1 and the second is of the form in Section 4.2. Furthermore, the two time changes are independent, given $\mathcal{F}_0$. The first time change models the dependence of the process on its arrival history, and the second one represents the effect of externally observable factors as described by the filtration $\mathbb{F}$.

5.1 Construction

The construction is summarized in Figure 1. As above, let (S_n) be the arrival times of the standard Poisson process N^0 in the filtration $\mathbb{H}$. In a first step we time-change N^0 to obtain a Hawkes process N^1, whose arrival times we denote (t_n). As described in Section 4.1, this requires a time change process defined by $M(t) = \int_0^t \mu_s ds$, where

μ is given by

$$\mu_t = \nu_0(t) + \sum_{t_n < t} \nu_n(t - t_n)$$

for impact processes (ν_n). The counting process N^1 associated with the arrival times (t_n) has intensity μ and compensator M. By (6), we know that $t_n = M^{-1}(S_n)$.

In a second step, we time-change N^1 to obtain our target process N with arrival times (T_n), using the time change process $\Gamma(t) = \int_0^t \gamma_s ds$. Here, γ is given by

$$\gamma_t = h(X_t)$$

for a positive, measurable function h and a stochastic process X that generates the filtration $\mathbb{F}$, as described in Section 4.2. By construction, the time change M is conditionally independent of the time change Γ given $\mathcal{F}_0$.

The process N so constructed is called a *time-changed Hawkes process*. The associated time change is given by the composition of M and Γ:

$$G(t) = M(\Gamma(t)) = \int_0^{\Gamma(t)} \mu_s ds. \tag{17}$$

Note that both the Hawkes process and the doubly stochastic process are special cases. The time change G can be decomposed into the inter-arrival time changes G_n, where $G_n(t) = G(t + T_{n-1}) - G(T_{n-1})$. In addition, the mark Z_n which was previously associated with the arrival t_n, is now associated with T_n.

5.2 Intensity

From equation (17), we see that the compensator G of the time-changed Hawkes process N is absolutely continuous. The corresponding intensity is given by

$$\lambda_t = \gamma_t \cdot \mu_{\Gamma(t)}. \tag{18}$$

The first factor is the intensity of the corresponding doubly stochastic process, and reflects information contained in $\mathbb{F}$. The second factor, which is not found in doubly stochastic models, is the self-affecting component of the intensity.

Since Γ is a continuous, strictly increasing process almost surely, $\mu \circ \Gamma$ possesses many of the same path properties as μ. For example, μ is discontinuous at τ if and only if $\mu \circ \Gamma$ is discontinuous at $\Gamma(\tau)$. Similarly, μ is decreasing (increasing) at τ if and only if $\mu \circ \Gamma$ is decreasing (increasing) at $\Gamma(\tau)$. Most importantly, $\mu \circ \Gamma$ jumps at T_n if and only if μ jumps at t_n, and the jumps are of the same size.

Thus, if γ is continuous, the jump in the intensity of N at the nth arrival time T_n depends on both the specification of the self-affecting properties (through ν_n) and on information in $\mathbb{F}$ (through γ):

$$\Delta\lambda_{T_n} = \gamma_{T_n}\nu_n(0).$$

For identifiability of the factors in equation (18), we must make assumptions on γ and μ. For interpretation, it is most convenient to take $\nu_0 \equiv 1$ or $\nu_0(t) \to 1$ as $t \to \infty$. If $\nu_0 \equiv 1$, then γ is the intensity of the first arrival. If we only assume $\nu_0(t) \to 1$, then λ_t approaches γ_t as more and more time passes without any arrivals. In that case, we can think of γ as the "base" intensity, or the intensity of the process in the absence of excitation. We may wish to choose $\nu_0(t) \to 1$ to include contributions to the intensity from events that happened before $t = 0$.

In applications, one must choose a "base" intensity γ, as well as the impact processes (ν_n). The choice of impact processes determines how the intensity λ of the time-changed Hawkes process responds to arrivals, where the response is always measured as a factor of the "base" intensity.

5.3 Simulation

Due to the interactions that take place in even the simplest of self-affecting processes, simulation is typically required to evaluate quantities of interest. When the filtration $\mathbb{F}$ is non-trivial, simulation becomes even more computationally intensive, since the paths of $\mathbb{F}$-adapted processes are often needed. The particular form of the time-changed Hawkes process simplifies the simulation procedure, as compared with a general time-changed Poisson process. It also enables us to change the dependence of a process on its history, and re-simulate the results without having to start over.

The arrival times of the time-changed Hawkes process N are given through equation (6) by

$$T_n = G^{-1}(S_n) = \Gamma^{-1}(M^{-1}(S_n)) = \Gamma^{-1}(t_n). \tag{19}$$

Since Γ is finite and increases to infinity almost surely, the process N is nonexploding if and only if the threshold process is nonexploding. Equation (19) implies that T_n can be simulated as the first time that the process Γ reaches the arrival time t_n of the Hawkes process:

$$T_n = \inf\{t > 0 : \Gamma(t) \geq t_n\}. \tag{20}$$

Here, Γ and t_n are generated independently of each other given the information contained in $\mathcal{F}_0$. This is analogous to simulating the arrivals of a doubly stochastic process, compare equation (16). The only difference is that in (20), the "thresholds" (t_n) are the arrivals of a Hawkes process, while in (16) the thresholds (S_n) are the arrivals of a standard Poisson process. In the former case, the distances between thresholds may be neither independent nor exponential.

Suppose we are given m different parameterizations of the impact processes (ν_n) for the general Hawkes process. Each parametrization represents a different assumption about the dependence of N on its history. These parameterizations will lead to m different threshold sequences $(t_n^1)_{n\geq 1}, \ldots, (t_n^m)_{n\geq 1}$. We can then simulate arrivals from the corresponding processes $N^1, \ldots, N^m$ by simulating the process Γ only once, and setting $T_n^i = \Gamma^{-1}(t_n^i)$.

This method is useful when comparing different assumptions on the impact processes, since it is significantly more efficient than simulating Γ for each different parametrization. This is especially true when Γ needs to be simulated using a fine time discretization. Furthermore, any differences between the counting processes is due only to the choice of impact processes, and not to different realizations of Γ, a fact that makes it easier to compare choices.

6 Towards Multi-Name Credit Modeling

We conclude by proposing time-changed Poisson processes to the reduced form modeling of multi-name credit risk. We specialize in Giesecke & Goldberg's (2005) "top-down" modeling framework. Here the portfolio default process and its compensator are modeled first. The technique of random thinning consistently generates the default process and the associated compensator of individual firms.

The default times in a portfolio of n_0 firms are modeled by the increasing sequence (T_n) of totally inaccessible $\mathbb{G}$-stopping times, where the filtration $\mathbb{G}$ describes the evolution of the information available to investors. The dynamics of the associated counting process N depend on the firm-specific default risks and the dependence among individual defaults. The default dependence generates *temporal clusters* in the sequence (T_n). It has two broad sources. The first is "contagion:" a default can trigger financial distress at other firms via explicit linkages between firms. The information revealed at a default can also lead investors to update their assessment of the credit quality of similar surviving firms, which generates information based contagion. The second source is firms' dependence on common economic factors, whose evolution over time is described by the filtration $\mathbb{F} \subseteq \mathbb{G}$.

We take account of the clustering and the dependence on the risk factors generating $\mathbb{F}$ by modeling the default process N as a self-affecting, time-changed Poisson process with $\mathbb{G}$-compensator G. We can then write $N = N^0_G$, where N^0 is a standard Poisson process in the filtration $\mathbb{G}$. When time is measured according to the default time scale G, defaults arrive at a rate of one per unit time. Conditional default probabilities given the σ-algebra $\mathcal{G}_t$ can be calculated using the formulae established in Theorem 3.7. These formulae extend the single-name reduced-form formulae of Duffie, Schroder & Skiadas (1996) to the multi-name case.

The mark Z_n associated with T_n denotes the loss at the nth default. The compound point process L defined by

$$L_t = \sum_{n=1}^{N_t} Z_n = \sum_{n=1}^{n_0} Z_n 1_{\{T_n \leq t\}}$$

describes the cumulative portfolio losses over time. Portfolio credit derivatives such as collateralized debt obligations, tranches and default baskets can be viewed as derivative securities written on the compound point process L.

Consider a *tranche* $i \in$ {Equity, Mezzanine, Senior, Super Senior}, which is specified by a lower attachment point K^i_L, an upper attachment point K^i_U and a maturity date T, see Figure 2. The difference $K^i = K^i_U - K^i_L$ is the tranche notional. The portfolio losses L are allocated to the tranches according to a "waterfall:" tranche i absorbs the losses that exceed K^i_L up to K^i_U. At time t, the cumulative losses allocated to tranche i, denoted L^i_t, are given by

$$L^i_t = (L_t - K^i_L)^+ - (L_t - K^i_U)^+$$

so that $L^i_t \in [0, K^i]$ almost surely.

The tranche investor pays the losses allocated to the tranche as they occur: at time $t \leq T$, the payment is $L^i_t - L^i_{t-}$. The value of these payments at time 0 is

$$D^i = E\left[\int_0^T e^{-rt} dL^i_t\right] = e^{-rT} E[L^i_T] + r \int_0^T e^{-rt} E[L^i_t] dt,$$

where we assume that the underlying probability measure P is a pricing measure and $r > 0$ is a constant risk-free interest rate. We recognize the term $e^{-rt}E[L^i_t]$ as the price at time 0 of a call spread on L with strikes K^i_L and K^i_U and maturity t.

For agreeing to pay the losses allocated to the tranche, the investor receives a periodic coupon payment proportional to the non-defaulted tranche notional:

$$S^i (K^i - L^i_t)^+, \quad t \in \{t_1, t_2, \ldots, t_m = T\}.$$

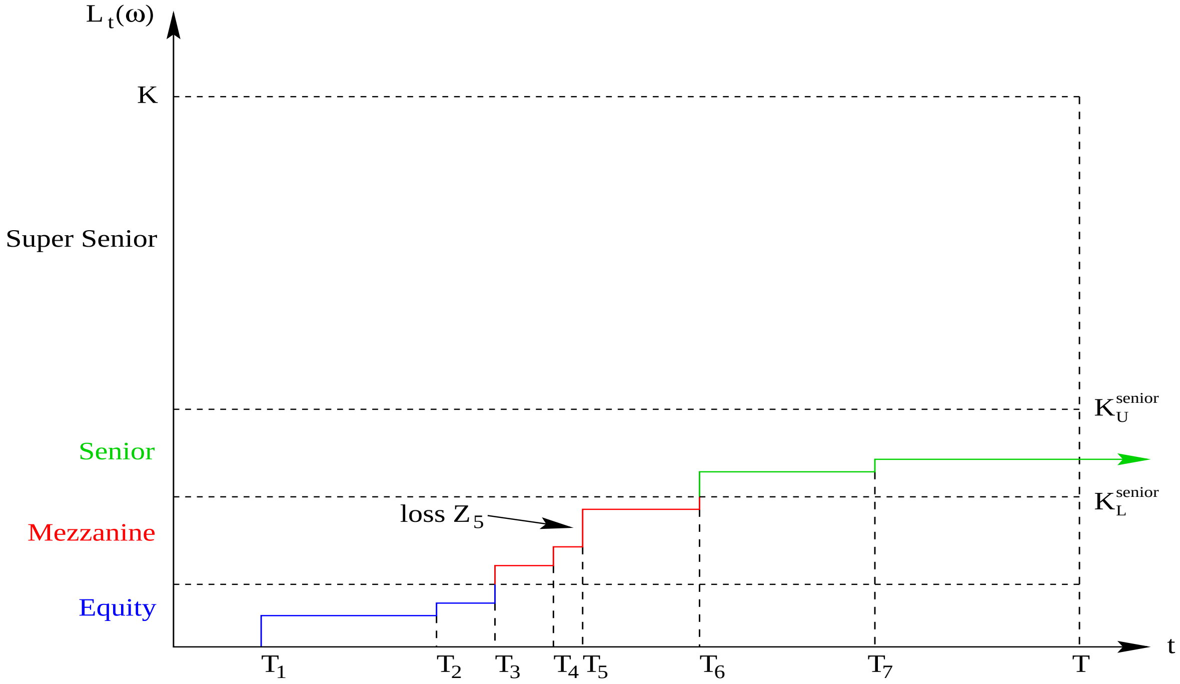

Figure 2: A sample path of the cumulative portfolio losses L and the allocation of losses to tranches.

The market value of these coupons at time 0 is given by

$$F^i(S^i) = E\left[\sum_{k=1}^{m} e^{-rt_k} S^i (K^i - L^i_{t_k})^+\right] = S^i \sum_{k=1}^{m} e^{-rt_k}(K^i - E[L^i_{t_k}]).$$

The tranche spread S^i is chosen such that at time 0, the market value of the investor's position is 0. That is, S^i satisfies the equation $D^i = F^i(S^i)$. The tranche legs D^i and $F^i(S^i)$ can be priced by exploiting the option characterization of these positions. Options on the compound point process L are most conveniently valued when the characteristic function of L_t is available. We explore this in Errais & Giesecke (2005), using the time change representation of the default counting process $N = N^0_G$ driving L.

Our approach to multi-name credit based on time-changed Poisson processes provides an alternative to the standard industry approach based on copula functions, see Li (2000), Gregory & Laurent (2003) and Andersen, Sidenius & Basu (2003). In a copula-based approach, the focus is on the *unordered* sequence of default stopping times (τ_n). The marginal distribution of each τ_n is specified, as well as a copula that fully describes the dependence between the times. Together, the marginals and the copula completely specify the joint distribution of $\tau_1, \dots, \tau_n$.

In our point process approach the fundamental object is the default order statistics, i.e. the distribution of the ordered default times (T_n) where $T_1 = \min_k\{\tau_k\}$ and $T_{n_0} = \max_k\{\tau_k\}$. The copula approach models the distribution of the unordered sequence (τ_n), which must then be transformed into the order statistics. While possible, this can be a complex task, see Maurer & Margolin (1976). Simplifications are possible in the exchangeable case where the marginals have the same distribution and the copula is symmetric, but this is often too restrictive.

The time-change approach allows a flexible and intuitive specification of the default dependence structure embedded in the order statistics. The dependence structure is *dynamically* updated with the information that is revealed over time. In the copula approach, the default dependence structure is fixed through the copula at time 0 and remains invariant. Moreover, the choice of copula is often arbitrary.

A Proofs

We begin by establishing the following proposition.

Proposition A.1. *Suppose we are given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a filtration $\mathbb{H} = (\mathcal{H}_t)_{t\geq 0}$ that satisfies the usual hypotheses, an $\mathbb{H}$-local martingale $M^0 = (M^0_t)_{t\geq 0}$ with localizing sequence $(S_n)_{n\geq 1}$, and a continuous process $G = (G(t))_{t\geq 0}$, increasing to infinity almost surely such that for every $t \geq 0$, $G(t)$ is an $\mathbb{H}$-stopping time with $\mathbb{P}(G(t) < \infty) = 1$. Then the time changed process M given by $M_t = M^0_{G(t)}$ is a local martingale with respect to $(\mathcal{H}_G(t))_{t\geq 0}$. The localizing sequence is given by $(T_n)_{n\geq 0}$ where $T_n = \inf\{t : G(t) > S_n\}$.*

Proof. Clearly M is adapted. Let $T_n = \inf\{t : G(t) > S_n\}$. Since (S_n) is a localizing sequence $S_n \to \infty$ almost surely. Furthermore, since G is finite and increases to infinity almost surely, $\mathbb{P}(T_n < \infty) = 1$ and $T_n \to \infty$ almost surely.

By localization, $M^0_{t\wedge S_n}$ is a uniformly integrable martingale. The Optional Sampling Theorem implies, for any $t \geq s \geq 0$,

$$\mathbb{E}[M^0_{G(t)\wedge S_n} | \mathcal{H}_{G(s)\wedge S_n}] = M^0_{G(s)\wedge S_n}.$$

But $G(t) \wedge S_n = G(t) \wedge G(T_n)$, so

$$\mathbb{E}[M^0_{G(t\wedge T_n)} | \mathcal{H}_{G(s)\wedge G(T_n)}] = M^0_{G(s\wedge T_n)}.$$

Applying the result of Protter (2004, Corollary on p.10) gives

$$\begin{aligned} M^0_{G(s\wedge T_n)} &= \mathbb{E}[M^0_{G(t\wedge T_n)}|\mathcal{H}_{G(s)\wedge G(T_n)}] \\ &= \mathbb{E}[\mathbb{E}[M^0_{G(t\wedge T_n)}|\mathcal{H}_{G(T_n)}]|\mathcal{H}_{G(s)}] \\ &= \mathbb{E}[M^0_{G(t\wedge T_n)}|\mathcal{H}_{G(s)}] \end{aligned}$$

But since $M_t = M^0_{G(t)}$,

$$\mathbb{E}[M_{t\wedge T_n}|\mathcal{H}_{G(s)}] = M_{s\wedge T_n}.$$

Hence M is a local martingale with respect to $(\mathcal{H}_G(t))_{t\geq 0}$ with localizing sequence $(T_n)_{n\geq 0}$. □

Consider the situation described in Section 3.1, in which we are given a probability space $(\Omega, \mathcal{H}, \mathbb{P})$, a filtration $\mathbb{F}$ containing all $\mathbb{P}$-null sets of $\mathcal{H}$. We are also given a standard marked Poisson process N^0 with respect to the filtration $\mathbb{H}$, which contains $\mathcal{F}_\infty$ for all t. This implies that the process N^0 is independent of $\mathbb{F}$. Let the arrival times of this process be denoted $0 < S_1 < S_2 < \ldots$ and the marks be denoted $Z_n \in \mathcal{M}$, where $\mathcal{M}$ is a general mark space. Suppose we have constructed the $\mathbb{G}^{n-1}$-adapted time-changes G_n as in Section 3.1.

We have filtrations $\mathbb{H}$, $\mathbb{G}^n$ and $\mathbb{G}$ that satisfy the usual hypotheses, where:

$$\mathcal{H}_t \supset \mathcal{F}_\infty \vee \sigma(1_{\{S_k\leq s\}}, Z_k 1_{\{S_k\leq s\}} : s \leq t, k \geq 1) \tag{21}$$

$$\mathcal{G}^n_t = \bigcap_{u>t} \mathcal{F}_u \vee \sigma(1_{\{T_k\leq s\}}, Z_k 1_{\{T_k\leq s\}} : s \leq u, k \leq n) \tag{22}$$

$$\mathcal{G}_t = \bigcap_{u>t} \mathcal{F}_u \vee \sigma(1_{\{T_k\leq s\}}, Z_k 1_{\{T_k\leq s\}} : s \leq u, k \geq 1) \tag{23}$$

Since $\mathbb{F}$ contains all $\mathbb{P}$-null sets of $\mathcal{F}$, so will any filtration containing $\mathcal{F}_0$.

Define $H_n(t) = \min(T_n - T_{n-1}, (t - T_{n-1})^+)$. These processes are continuous, linear between T_{n-1} and T_n, and constant elsewhere. Thus,

$$G_n(H_n(t)) = \begin{cases} 0, & t < T_{n-1} \\ G_n(t - T_{n-1}), & T_{n-1} \leq t < T_n \\ G_n(T_n - T_{n-1}), & T_n \leq t \end{cases}$$

Then, we can re-write $G(t)$ as:

$$
\begin{aligned}
G(t) &= \sum_{k=1}^{n-1} G_k(T_k - T_{k-1}) + G_n(t - T_{n-1}) \quad \text{on} \quad \{T_{n-1} \le t < T_n\} \\
&= \sum_{k=1}^{\infty} G_k(H_k(t))
\end{aligned} \tag{24}
$$

Remark A.2. If $T_n \to \infty$ almost surely, then almost surely, G is continuous, strictly increasing, with $G(0) = 0$, $G(t) < \infty$, and $\lim_{t\to\infty} G(t) = \infty$. □

Remark A.3. If $\mathbb{F} = (\mathcal{F}_t)_{t\ge0}$ and $\mathbb{G} = (\mathcal{G}_t)_{t\ge0}$ are any filtrations, $\mathcal{F}_t \subset \mathcal{G}_t$, and $\mathbb{G}$ satisfies the usual hypotheses, then the right continuous completion of $\mathbb{F}$ is contained in $\mathbb{G}$. That is $\bigcap_{\epsilon>0} \mathcal{F}_{t+\epsilon} \subset \mathcal{G}_t$. □

Remark A.4. For any $n \ge 1$, $s \ge 0$, $H_n(s)$ is a stopping time for $\mathbb{G}^n$, and $(s-T_{n-1})^+$ is a stopping time for $\mathbb{G}^{n-1}$.

Proof. They are both non-negative and for $t \ge 0$,

$$
\begin{aligned}
\{(s - T_{n-1})^+ \le t\} = \{s - T_{n-1} \le t\} &= \{s \le T_{n-1} + t\} \in \mathcal{G}^{n-1}_{T_{n-1}+t} \\
\{H_n(s) \le t\} &= \{T_n - T_{n-1} \le t\} \cup \{s - T_{n-1} \le t\} \\
&= \{T_n \le T_{n-1} + t\} \cup \{s \le T_{n-1} + t\} \in \mathcal{G}^{n}_{T_{n-1}+t}.
\end{aligned}
$$

□

Lemma A.5. *For all $n \ge 1$, $G_n((s - T_{n-1})^+)$ is $\mathcal{G}^{n-1}_s$-measurable.*

Proof. Fix $s \ge 0$. We know that $(s - T_{n-1})^+$ is a stopping time for $\mathbb{G}^{n-1}$ by Remark A.4, and that G_n is $\mathbb{G}^{n-1}$-adapted. Thus $G_n((s-T_{n-1})^+)$ is measurable with respect to $\mathcal{G}^{n-1}_{T_{n-1}+(s-T_{n-1})^+}$. But

$$T_{n-1} + (s - T_{n-1})^+ \le s \vee T_{n-1}.$$

So $G_n(H_n(s))$ is $\mathcal{G}^{n-1}_{s\vee T_{n-1}}$-measurable, and hence for any $t \ge 0$,

$$\{G_n((s - T_{n-1})^+) \le t\} \in \mathcal{G}^{n-1}_{s\vee T_{n-1}}.$$

By definition of this σ-algebra,

$$\{G_n((s - T_{n-1})^+) \le t\} \cap \{s \vee T_{n-1} \le s\} \in \mathcal{G}^{n-1}_s.$$

But $\{s \vee T_{n-1} \le s\} = \{T_{n-1} \le s\}$ and $\{T_{n-1} \le s\} \in \mathcal{G}_s^{n-1}$, so we have that

$$\begin{aligned}
&\{G_n((s - T_{n-1})^+) \le t\} \\
&\qquad = (\{G_n((s - T_{n-1})^+) \le t\} \cap \{T_{n-1} \le s\}) \cup (\{G_n(0) \le t\} \cap \{s < T_{n-1}\}) \\
&\qquad = (\{G_n((s - T_{n-1})^+) \le t\} \cap \{s \vee T_{n-1} \le s\}) \cup \{s < T_{n-1}\} \in \mathcal{G}_s^{n-1}
\end{aligned}$$

and we are done. □

Lemma A.6. *For all $n \ge 1$, $G_n(H_n(s))$ is $\mathcal{G}_s^n$-measurable.*

Proof. Fix $s \ge 0$. We know that $H_n(s)$ is a stopping time for $\mathbb{G}^n$ by Remark A.4, and that G_n is $\mathbb{G}^{n-1}$-adapted. Thus $G_n(H_n(s))$ is $\mathcal{G}^n_{T_{n-1}+H_n(s)}$-measurable. But

$$T_{n-1} + H_n(s) \le T_{n-1} + (s - T_{n-1})^+ \le s \vee T_{n-1}.$$

The remainder of the proof follows that of the preceding Lemma, with $(s - T_{n-1})^+$ replaced by $H_n(s)$ and $\mathcal{G}^{n-1}$ replaced by $\mathcal{G}^n$. □

Remark A.7. Lemma A.6 and (24) imply that G is adapted to $\mathbb{G}$.

Lemma A.8. *The filtration $(\mathcal{G}_{G^{-1}(t)})_{t \ge 0}$ satisfies the usual hypotheses and for $t \ge 0$, $\mathcal{G}_{G^{-1}(t)} \subset \mathcal{H}_t$.*

Proof. Since G is a continuous, strictly increasing, $\mathbb{G}$ adapted process, it is well known that the time changed filtration $(\mathcal{G}_{G^{-1}(t)})_{t \ge 0}$ satisfies the usual hypotheses. The fact that $\mathcal{G}_{G^{-1}(t)} \subset \mathcal{H}_t$ follows from the Monotone Class Theorem and the fact that $1_{\{S_n \le t\}} = 1_{\{T_n \le G^{-1}(t)\}}$. □

Proposition A.9. *For any $t_0 \ge 0$, $G(t_0)$ is a stopping time for $\mathbb{H}$.*

Proof. Fix $t_0 \ge 0$. Since G is continuous and strictly increasing, Lemma A.8 implies

$$\{G(t_0) \le t\} = \{t_0 \le G^{-1}(t)\} \in \mathcal{G}_{G^{-1}(t)} \subset \mathcal{H}_t.$$

Thus $G(t_0)$ is an $\mathbb{H}$ stopping time. □

Proposition A.10. *If $G(t)$ is strictly increasing, $\mathcal{G}_t \subset \mathcal{H}_{G(t)}$ for all t.*

Proof. Fix $t \ge 0$. For any n, the process $(1_{\{S_n \le t\}})$ is $\mathbb{H}$-adapted, and càdlàg. Since $G(t)$ is a finite stopping time, Theorem 6 in Protter (2004, Chapter I) yields that $1_{\{S_n \le G(t)\}}$ is measurable with respect to $\mathcal{H}_{G(t)}$ for all t. But, since G is strictly increasing, $1_{\{S_n \le G(t)\}} = 1_{\{T_n \le t\}}$ and hence $1_{\{T_n \le t\}}$ and $Z_k 1_{\{T_n \le t\}}$ are $\mathcal{H}_{G(t)}$-measurable for all t.

Thus, since n was arbitrary,

$$\sigma(1_{\{T_n \le s\}}, Z_n 1_{\{T_n \le t\}}, s \le t, n \ge 1) \subset \mathcal{H}_{G(t)}.$$

Furthermore, since $\mathcal{F}_t \subset \mathcal{F}_\infty \subset \mathcal{H}_0$, and $0 \le G(t)$, we get $\mathcal{F}_t \subset \mathcal{H}_{G(t)}$. This implies

$$\mathcal{F}_t \vee \sigma(1_{\{T_n \le s\}}, Z_n 1_{\{T_n \le s\}}, s \le t, n \ge 1) \subset \mathcal{H}_{G(t)}$$

since $\mathcal{H}_{G(t)}$ is a σ-algebra containing each of the two components, and the left-hand side is the smallest σ-algebra containing them. Now, since $\mathcal{G}_t$ is the right continuous version of the left-hand side, and $\mathcal{H}_{G(t)}$ satisfies the usual hypotheses, $\mathcal{G}_t \subset \mathcal{H}_{G(t)}$ by Remark A.3. □

Proof of Theorem 3.3. By definition, $(1_{\{T_n \le t\}})$ is $\mathbb{G}$-adapted, hence so is N. By Remarks A.2 and A.7, G is $\mathbb{G}$-predictable, and hence it remains to prove that $N - G$ is a $\mathbb{G}$-martingale.

Since N^0 is a Poisson process with respect to $\mathbb{H}$, the process M^0 given by $M_t^0 = N_t^0 - t$ is an $\mathbb{H}$-local martingale with localizing sequence $(S_n)_{n \ge 0}$. The process G is continuous and increases to infinity almost surely. Given $T_n \to \infty$ almost surely, Remark A.2 and Proposition A.9 imply $G(t)$ is a finite $\mathbb{H}$-stopping time for all t. Applying Proposition A.1, it follows that the time changed process

$$M_t = M_{G(t)}^0 = N_{G(t)}^0 - G(t) = N_t - G(t)$$

is a local martingale with respect to $(\mathcal{H}_{G(t)})_{t \ge 0}$ with localizing sequence $T_n = G^{-1}(S_n)$. By Proposition A.10, $\mathbb{G}$ is a subfiltration of $(\mathcal{H}_{G(t)})_{t \ge 0}$ and so $N - G$ is also a $\mathbb{G}$ local martingale. Thus G is the compensator of N. □

Proof of Proposition 3.1. Is is clear that N does not explode if and only if $\lim_{n \to \infty} T_n = \infty$ almost surely.

Suppose we are given the subsequence n_k, $\epsilon > 0$, and positive $(M_k)_{k \ge 0}$ satisfying the given properties. Since $(S_n - S_{n-1})_{n \ge 0}$ are i.i.d. exponential random variables, the Borel Cantelli lemma implies that

$$\mathbb{P}(S_{n_k} - S_{n_k - 1} > M_k \text{ i.o.}) = 1.$$

But since G_n^{-1} is increasing, and $G_{n_k}^{-1}(M_k) > \epsilon$ almost surely for all k, we have that with probability one, $G_{n_k}^{-1}(S_{n_k} - S_{n_k - 1}) > \epsilon$ infinitely often. Hence

$$\lim_{n \to \infty} T_n = \sum_{n=1}^{\infty} G_n^{-1}(S_n - S_{n-1}) \ge \sum_{k=1}^{\infty} G_{n_k}^{-1}(S_{n_k} - S_{n_k - 1}) = \infty$$

almost surely. □

Proof of Proposition 3.2. Let the process $\hat{N}$, with arrivals $(\hat{T}_n)$ and time change functions $(\hat{G}_n)$ be given. Suppose that for all $n \geq n_0$, $G_n(t) \leq \hat{G}_n(t)$ for all t. Thus, for all $n \geq n_0$ and all s, $G_n^{-1}(s) \geq \hat{G}_n^{-1}(s)$. Hence,

$$\lim_{n\to\infty} T_n = \sum_{n=1}^{\infty} G_n^{-1}(S_n - S_{n-1}) \geq \sum_{n=n_0}^{\infty} \hat{G}_n^{-1}(S_n - S_{n-1}) = \infty$$

since $T_n^1 \to \infty$. $\square$

Lemma A.11. *The Poisson inter-arrival time $S_n - S_{n-1}$ is independent of $\mathcal{G}^{n-1}_{T_{n-1}+t}$.*

Proof. Since N^0 is a Poisson process, $S_n - S_{n-1}$ is independent of $\mathcal{H}_{S_{n-1}}$. Note that $(\mathcal{G}_t^{n-1})_{t\geq 0}$ only contains information about $\mathbb{F}$ and the first $n-1$ arrivals. Since all these arrivals have happened by T_{n-1}, the monotone class theorem, and Proposition A.10 imply

$$\mathcal{G}^{n-1}_{T_{n-1}+t} \subset \mathcal{G}^{n-1}_{T_{n-1}} \vee \mathcal{F}_\infty \subset \mathcal{G}_{T_{n-1}} \vee \mathcal{F}_\infty \subset \mathcal{H}_{G(T_{n-1})} = \mathcal{H}_{S_{n-1}}.$$

Thus $S_n - S_{n-1}$ is also independent of $\mathcal{G}^{n-1}_{T_{n-1}+t}$. $\square$

Proof of Theorem 3.7. (1) Fix $t \geq 0$. Lemma A.11 implies that $S_n - S_{n-1}$ has an exponential distribution given $\mathcal{G}^{n-1}_{T_{n-1}+t}$, and since $G_n(t)$ is $\mathcal{G}^{n-1}_{T_{n-1}+t}$-measurable, it follows that

$$\mathbb{P}(T_n - T_{n-1} > t | \mathcal{G}^{n-1}_{T_{n-1}+t}) = \mathbb{P}(S_n - S_{n-1} > G_n(t) | \mathcal{G}^{n-1}_{T_{n-1}+t}) = e^{-G_n(t)}.$$

Hence

$$\begin{aligned}\mathbb{P}(T_n - T_{n-1} > t | \mathcal{G}^{n-1}_{T_{n-1}+s}) &= \mathbb{E}[\mathbb{P}(T_n - T_{n-1} > t | \mathcal{G}^{n-1}_{T_{n-1}+t}) | \mathcal{G}^{n-1}_{T_{n-1}+s}] \\ &= \mathbb{E}[e^{-G_n(t)} | \mathcal{G}^{n-1}_{T_{n-1}+s}].\end{aligned}$$

(2) Since the S_n's are arrivals of the standard Poisson process N^0, we have that for any $\mathbb{H}$-stopping time $\sigma \geq S_{n-1}$, and fixed $t > 0$

$$\mathbb{P}(S_n > \sigma + t | \mathcal{H}_\sigma) = e^{-t} 1_{\{S_n > \sigma\}}.$$

By Lemma A.11, we know that for any $s < t$, $G_n(s)$ and $G_n(t)$ are $\mathcal{H}_{S_{n-1}}$-measurable, and hence also $\mathcal{H}_\sigma$-measurable. Thus, the variable $G_n(t) - G_n(s)$ is $\mathcal{H}_\sigma$-measurable, implying that

$$\mathbb{P}(S_n > \sigma + G_n(t) - G_n(s) | \mathcal{H}_\sigma) = e^{-(G_n(t) - G_n(s))} 1_{\{S_n > \sigma\}}.$$

Now, note that since $G_n(s) > 0$ is $\mathcal{H}_{S_{n-1}}$-measurable, $\sigma = S_{n-1} + G_n(s)$ is an $\mathbb{H}$-stopping time by definition of $\mathcal{H}_{S_{n-1}}$. Thus,

$$\mathbb{P}(S_n > S_{n-1} + G_n(t)|\mathcal{H}_{S_{n-1}+G_n(s)}) = e^{-(G_n(t)-G_n(s))}1_{\{S_n > S_{n-1}+G_n(s)\}}.$$

Since $T_n - T_{n-1} = G_n^{-1}(S_n - S_{n-1})$,

$$\mathbb{P}(T_n - T_{n-1} > t|\mathcal{H}_{S_{n-1}+G_n(s)}) = e^{-(G_n(t)-G_n(s))}1_{\{T_n - T_{n-1} > s\}}.$$

But, by construction, $S_{n-1} + G_n(s) = G(T_{n-1} + s)$ for $T_{n-1} + s < T_n$, and hence, since $G_n(s)$ is increasing, $S_{n-1} + G_n(s) \geq G((T_{n-1} + s) \wedge T_n)$. Thus, by Proposition A.10,

$$\mathcal{H}_{S_{n-1}+G_n(s)} \supset \mathcal{H}_{G((T_{n-1}+s)\wedge T_n)} \supset \mathcal{G}_{(T_{n-1}+s)\wedge T_n}.$$

It follows that

$$\mathbb{P}(T_n - T_{n-1} > t|\mathcal{G}_{(T_{n-1}+s)\wedge T_n}) = \mathbb{E}[e^{-(G_n(t)-G_n(s))}1_{\{T_n - T_{n-1} > s\}}|\mathcal{G}_{(T_{n-1}+s)\wedge T_n}]$$

which implies the result on the event $\{T_n - T_{n-1} > s\} = \{T_{n-1} + s < T_n\}$. □

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