The Synthesis of Regulator Logic Using State-Variable Concepts

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Invited Paper

Abstract—This is a tutorial paper. It was written for technically knowledgeable people who are *not* specialists in automatic control. It might also be of some use in university courses in automatic control.

Linear feedback of the state variables for stationary linear plants permits choice of the gains to place poles (eigenvalues) of the controlled system in desired locations. Continuous estimates of the state variables can often be made from one or more measurements of the output using a filter of the same (or lower) order as the plant called "an observer": the poles of the observer can be placed arbitrarily through choice of the observer gains. Combining an observer with state-variable feedback yields a "compensator" design; the poles of the controlled system are the poles of the observer plus the poles of plant with state-variable feedback, which greatly simplifies the design problem.

I. Introduction

A REGULATOR is a control system that keeps the outputs of a machine, a vehicle, or a process near the desired outputs in the presence of disturbances. It consists of 1) *sensors* that measure the outputs, 2) a *logic unit* that receives transduced signals from the sensors and computes control signals, and 3) *effectors* that receive the control signals and produce corresponding control forces, torques, displacements, etc. (See Fig. 1.) We limit our discussion here to the design of the *logic* for the logic unit, and assume that the system to be controlled can be modeled as an nth order stationary linear dynamic system.

The design techniques currently in use are based mainly on *frequency response* and *root-locus* concepts. (See e.g., [1]–[3], where the logic unit is referred to as a "compensator".)

II. Classical Regulator Synthesis

For single-input, single-output plants (i.e., systems to be controlled), the plant dynamics can be expressed in terms of a *transfer function* $G(s)$, which is defined as the ratio $z(s)/u(s)$, where $z(s)=$Laplace transform of output, $u(s)=$Laplace transform of input. For an nth order plant it is the ratio of two polynomials in s where the denominator polynomial is of order n, and the numerator polynomial is of order n or less. The classical regulator synthesis problem (somewhat over-simplified) may be stated by the following. Given $G(s)$, synthesize an $H(s)$ to keep $z(t)$ near zero in the presence of all likely disturbances. (See Fig. 2.)

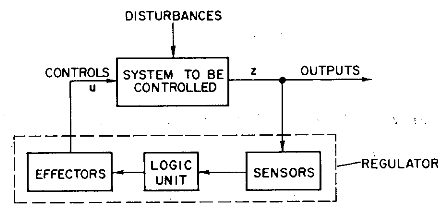

Fig. 1.

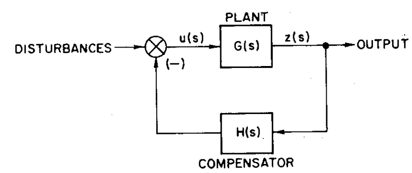

Fig. 2.

The classical methods for choosing a compensator (or compensating network) $H(s)$ are cut-and-try methods based on experience and intuition using lead networks, lag networks, lead–lag networks, double lead–lag networks, etc.

Example

Suppose

$$G(s) = \frac{1}{s^2 + \omega_0^2}, \tag{1}$$

i.e., the plant is an undamped harmonic oscillator; then a typical $H(s)$ is

$$H(s) = C\frac{s + \alpha k}{s + k}, \qquad 0 < \alpha < 1, \qquad C > 0, \tag{2}$$

which is a *lead network*. By choice of C, k, and α, the controlled response $x(t)$ can be made small for a large variety of disturbances.

III. State Variables

If the plant model is written as a set of coupled first-order ordinary differential equations, the differentiated variables are called *state variables*, and the other variables are called *control variables*. If the differential equations are linear and

Manuscript received May 7, 1970; revised August 10, 1970. This research was supported in part by NASA Grants NGL 05-020-007 and NGR 05-020-431. *This invited paper is one of a series planned on topics of general interest—The Editor.*

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there are n independent equations and n state variables, it is possible to put them into the form

$$\dot{x} = Fx + Gu \tag{3}$$

where

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

is the *state vector* where x_i, $i=1,\cdots,n$ are the state variables,

$$u = \begin{bmatrix} u_1 \\ \vdots \\ u_p \end{bmatrix}$$

is the control vector where u_i, $i=1,\cdots,p$ are the control variables,

$$F = \begin{bmatrix} F_{11} & \cdots & F_{1n} \\ \vdots & & \vdots \\ F_{n1} & \cdots & F_{nn} \end{bmatrix}$$

is the *plant matrix*, and

$$G = \begin{bmatrix} G_{11} & \cdots & G_{1p} \\ \vdots & & \vdots \\ G_{n1} & \cdots & G_{np} \end{bmatrix}$$

is the *control matrix.*

Example A

For the undamped harmonic oscillator,

$$\ddot{y} + \omega_0^2 y = u. \tag{4}$$

Let $\dot{y}=v$. Then y and v are state variables of the plant and the plant model in state-variable form is

$$\begin{aligned} \dot{y} &= v \\ \dot{v} &= -\omega_0^2 y + u. \end{aligned} \tag{5}$$

Example B

(See Fig. 3.) A model for a cart with a stick hinged on top of it and acted upon by a force u, is[1]

$$\begin{aligned} \ddot{\theta} &= \theta + u \\ \ddot{y} &= \beta\theta - u. \end{aligned} \tag{6}$$

If we let $\dot{y}=v$, $q=\dot{\theta}$, then θ, q, y, v are state variables of the plant and the plant model in state-variable form is:

$$\begin{aligned} \dot{\theta} &= q \\ \dot{q} &= \theta + u \\ \dot{y} &= v \\ \dot{v} &= \beta\theta - u. \end{aligned} \tag{7}$$

[1] In this model, time t is measured in units of $[6(m+M)g/(m+4M)l]^{-\frac{1}{2}}$; displacement y is measured in units of $2l/3$, force u is measured in units of $(m+M)g$; $\beta=[(3/4)(m/m+M)]$. This example was suggested to us by Prof. Robert Cannon of Stanford University.

IV. Single-Input Regulator Synthesis with State-Variable Feedback

For the single-input case, u is a scalar ($p=1$). If u is constructed as a linear combination of all the state variables, i.e.,

$$u = -Cx, \tag{8}$$

where

$$C = [C_1, \cdots, C_n],$$

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix},$$

then, by choice of the n gains $C_1,\cdots,C_n$, the n eigenvalues (poles) of the controlled system can be selected *arbitrarily.* Of course, this eigenvalue selection must be done with some judgement so that the required values of u are indeed available (i.e., the speed of response is limited by the maximum available values of $|u|$).

Example A

For the undamped harmonic oscillator, the state-variable feedback law involves two gains, C_y and C_v:

$$u = -C_y y - C_v v. \tag{9}$$

A block diagram of the controlled system is shown in Fig. 4. The model of the controlled system is

$$\begin{aligned} \dot{y} &= v \\ \dot{v} &= -(\omega_0^2 + C_y)y - C_v v. \end{aligned} \tag{10}$$

Assuming a solution of (10) in the form $y = y_0 e^{\lambda t}$, $v=v_0 e^{\lambda t}$, yields

$$\begin{bmatrix} \lambda, & -1 \\ \omega_0^2 + C_y, & \lambda + C_v \end{bmatrix} \begin{bmatrix} y_0 \\ v_0 \end{bmatrix} = 0 \tag{11}$$

and for nontrivial solutions of (11), the determinant of the coefficient matrix must vanish, yielding the following second-order characteristic equation for the controlled system:

$$\lambda^2 + C_v\lambda + \omega_0^2 + C_y = 0. \tag{12}$$

If we wish to have controlled system eigenvalues at $\lambda=-\sigma_1$ and $\lambda=-\sigma_2$, then the *desired* characteristic equation is

$$(\lambda + \sigma_1)(\lambda + \sigma_2) = \lambda^2 + (\sigma_1 + \sigma_2)\lambda + \sigma_1\sigma_2 = 0. \tag{13}$$

Comparing coefficients of like powers of λ in (12) and (13) yields

$$\begin{aligned} C_v &= \sigma_1 + \sigma_2 \\ C_y &= \sigma_1\sigma_2 - \omega_0^2. \end{aligned} \tag{14}$$

Note that if σ_1 and σ_2 are complex conjugate numbers, C_v and C_y are real numbers.

Example B

(See Fig. 3.) If we wish a regulator logic for this system that balances the stick (i.e., keeps $\theta=0$) *and* keeps the cart

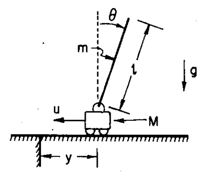

Fig. 3.

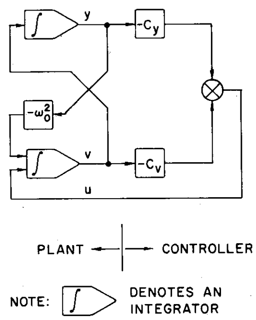

Fig. 4.

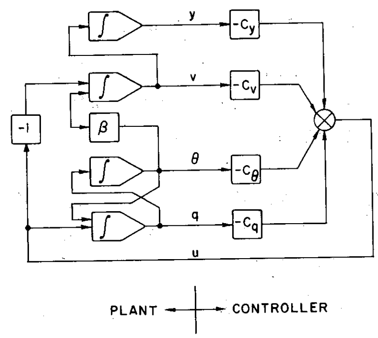

Fig. 5.

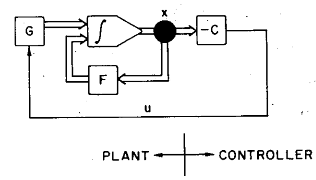

Fig. 6.

centered near $y=0$, then we can use the following state-variable feedback law:

$$u = -C_\theta \theta - C_q q - C_y y - C_v v. \tag{15}$$

A block diagram of the controlled system is shown in Fig. 5. The model of the controlled system is

$$\begin{aligned} \ddot{\theta} &= (1 - C_\theta)\theta - C_q\dot{\theta} - C_y y - C_v \dot{y} \\ \ddot{y} &= (\beta + C_\theta)\theta + C_q\dot{\theta} + C_y y + C_v \dot{y}. \end{aligned} \tag{16}$$

Assuming $\theta = \theta_0 e^{\lambda t}$, $y = y_0 e^{\lambda t}$ yields

$$\begin{bmatrix} \lambda^2 + C_q\lambda + C_\theta - 1, & C_v\lambda + C_y \\ -C_q\lambda - (C_\theta + \beta), & \lambda^2 - C_v\lambda - C_y \end{bmatrix} \begin{bmatrix} \theta_0 \\ y_0 \end{bmatrix} = 0. \tag{17}$$

From (17), the characteristic equation of the controlled system is

$$\lambda^4 + (C_q - C_v)\lambda^3 + (C_\theta - C_y - 1)\lambda^2 + C_v(1 + \beta)\lambda + C_y(1 + \beta) = 0. \tag{18}$$

Now the eigenvalues of the uncontrolled system are $(1, -1, 0, 0)$.

Suppose we wished to make the eigenvalues of the controlled system $(-1, -1, -1+i, -1-i)$; the desired characteristic equation would be

$$(\lambda + 1)^2(\lambda^2 + 2\lambda + 2) = \lambda^4 + 4\lambda^3 + 7\lambda^2 + 6\lambda + 2 = 0. \tag{19}$$

Equating the coefficients of like powers of λ in (18) and (19) yields

$$\begin{aligned} C_y(1+\beta) &= 2 \\ C_v(1+\beta) &= 6 \\ C_\theta - C_y - 1 &= 7 \\ C_q - C_v &= 4 \end{aligned} \Rightarrow \begin{cases} C_y = \dfrac{2}{1+\beta} \\ C_v = \dfrac{6}{1+\beta} \\ C_\theta = 8 + \dfrac{2}{1+\beta} \\ C_q = 4 + \dfrac{6}{1+\beta}. \end{cases} \tag{20}$$

General Case

(See Fig. 6.) Using $u = -Cx$, the controlled system is described by

$$\dot{x} = (F - GC)x. \tag{21}$$

Assuming $x = x_0 e^{\lambda t}$ yields

$$[\lambda I - F + GC]x_0 = 0 \tag{22}$$

where $I = n \times n$ identity matrix. From (22) the characteristic equation of the controlled system is

$$|\lambda I - F + GC| = 0 \tag{23}$$

where $|\ |$ means "determinant of."

Equation (23) is an nth order polynominal equation with the n gains $C_1, \cdots, C_n$ appearing in the coefficients. *If* the system is controllable, then we can choose $C_1, \cdots, C_n$ so that the eigenvalues have any desired values (so long as complex eigenvalues occur in conjugate pairs).

V. Estimation of State Variables

It is *not* necessary to measure all of the state variables to use the feedback scheme of Section IV. From one or more measurements it is often possible to *estimate* the state variables using an auxiliary dynamic system (usually electronic) called a *filter*. The combination of this filter with the state-variable feedback constitutes a compensator.

We consider here a special filter design called an observer [4]–[6] whose structure can be motivated as follows. Suppose we are given a plant model

$$\dot{x} = Fx + Gu, \tag{24}$$

a single continuous measurement of the outputs of the plant $z(t)$, where

$$z(t) = Hx(t), \qquad H = [H_1, \cdots, H_n], \tag{25}$$

and a measurement of the input $u(t)$. We are going to try to estimate $x(t)$ using only $z(t)$, $u(t)$, *and* our knowledge of F and G in (24). If we let

$$\hat{x} = \text{estimate of } x, \tag{26}$$

then the estimated value of the measurement is $H\hat{x}$. We will regard our estimate $\hat{x}$ to be in error when $H\hat{x}$ differs from the actual measurement z, i.e., we wish to keep $z - H\hat{x}$ close to zero.

With our knowledge of F, G, $u(t)$, and initial conditions $x(t_0)$, we could "estimate" $x(t)$ by feeding $u(t)$ into an electronic model of the plant:

$$\dot{\hat{x}} = F\hat{x} + Gu; \qquad x(t_0) \text{ given.} \tag{27}$$

The obvious trouble with this *open-loop* estimator is that F, G, $u(t)$, and $x(t_0)$ are *not* known perfectly and, after a short time, the estimate becomes quite inaccurate. To remedy this difficulty, and yet keep the estimator *linear*, it has been suggested that the error, $z - H\hat{x}$, be fed back into every equation of (27), i.e.,

$$\dot{\hat{x}} = F\hat{x} + Gu + K(z - H\hat{x}) \tag{28}$$

where

$$K = \begin{bmatrix} K_1 \\ \vdots \\ K_n \end{bmatrix}.$$

The estimator represented by (28) is a *closed-loop* estimator called an *observer*. If we define

$$\tilde{x} = \hat{x} - x \tag{29}$$

as the error in the estimate, then $\tilde{x}$ can be determined by subtracting (24) from (28):

$$\dot{\tilde{x}} = (F - KH)\tilde{x} \tag{30}$$

where we used $z - H\hat{x} = H(x - \hat{x}) = -H\tilde{x}$. If the n gains $K_1, \cdots, K_n$ can be chosen to make (30) a stable system, then

$$\tilde{x}(t) \to 0 \quad \text{as } t \text{ increases,} \tag{31}$$

or, in other words

$$\hat{x}(t) \to x(t) \quad \text{as } t \text{ increases,} \tag{32}$$

and (28) is a reasonable estimator. In fact, if the plant is *observable* with the measurement z, then we can choose $K_1, \cdots, K_n$ so that the eigenvalues of (30) have *any desired values* (so long as the complex eigenvalues occur in conjugate pairs).

Example A

For the undamped harmonic oscillator with measurement $z = y$, an observer design is (see also Fig. 7)

$$\begin{aligned} \dot{\hat{y}} &= \hat{v} + K_y(z - \hat{y}); & \hat{y}(0) &= 0 \\ \dot{\hat{v}} &= -\omega_0^2\hat{y} + K_v(z - \hat{y}) + u; & \hat{v}(0) &= 0. \end{aligned} \tag{33}$$

The initial estimates are taken as zero in (33) in the absence of any information about the actual values. Subtracting (5) from (33) gives the error equations

$$\begin{aligned} \dot{\tilde{y}} &= \tilde{v} - K_y\tilde{y}; & \tilde{y}(0) &= -y(0) \\ \dot{\tilde{v}} &= -(\omega_0^2 + K_v)\tilde{y}; & \tilde{v}(0) &= -v(0). \end{aligned} \tag{34}$$

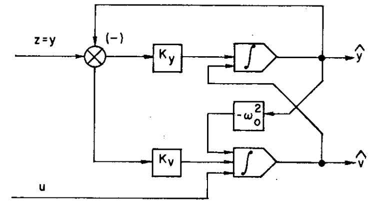

Fig. 7.

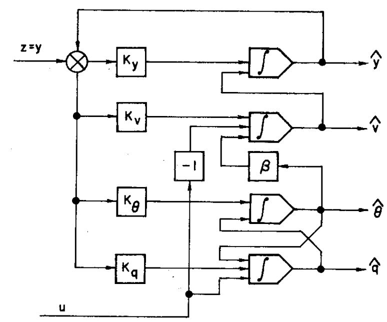

Fig. 8.

The characteristic equation of (34) is

$$\lambda^2 + K_y\lambda + \omega_0^2 + K_v = 0.$$

Clearly, as in (12), K_y and K_v can be chosen to place the two eigenvalues in any position; in particular if eigenvalues with negative real parts are chosen, $\tilde{y}(t) \to 0$ and $\tilde{v}(t) \to 0$ as t increases.

Example B

For the cart with a hinged stick, using only the measurement $z = y$, an observer design is (see also Fig. 8)

$$\begin{aligned} \dot{\hat{\theta}} &= \hat{q} + K_\theta(z - \hat{y}); & \hat{\theta}(0) &= 0 \\ \dot{\hat{q}} &= \hat{\theta} + K_q(z - \hat{y}) + u; & \hat{q}(0) &= 0 \\ \dot{\hat{y}} &= \hat{v} + K_y(z - \hat{y}); & \hat{y}(0) &= 0 \\ \dot{\hat{v}} &= \beta\hat{\theta} + K_v(z - \hat{y}) - u; & \hat{v}(0) &= 0. \end{aligned} \tag{35}$$

Subtracting (7) from (35) gives the error equations

$$\begin{aligned} \dot{\tilde{\theta}} &= \tilde{q} - K_\theta\tilde{y} \\ \dot{\tilde{q}} &= \tilde{\theta} - K_q\tilde{y} \\ \dot{\tilde{y}} &= \tilde{v} - K_y\tilde{y} \\ \dot{\tilde{v}} &= \beta\tilde{\theta} - K_v\tilde{y}. \end{aligned} \tag{36}$$

The characteristic equation of (36) is

$$\lambda^4 + K_y\lambda^3 + (K_v - 1)\lambda^2 + (\beta K_\theta - K_y)\lambda + \beta K_q - K_v = 0. \tag{37}$$

Suppose we wished to put a double pole of the observer at $\lambda = -3$ and two poles at $\lambda = -3 \pm 3i$. The desired characteristic equation would then be

$$(\lambda + 3)^2(\lambda + 3 + 3i)(\lambda + 3 - 3i)$$
$$= \lambda^4 + 12\lambda^3 + 63\lambda^2 + 162\lambda + 162 = 0. \tag{38}$$

Comparing coefficients of like powers of s in (37) and (38) yields

$$K_y = 12$$
$$K_v = 64$$
$$K_\theta = 174/\beta$$
$$K_q = 266/\beta. \quad (39)$$

VI. A Limitation on Observers

It is *not* always possible to estimate all of the state variables from one measurement of the output. For example, in Example B (just considered), suppose the measurement was $z=\theta$ instead of $z=y$; clearly θ can *not* tell us where the cart is relative to an arbitrary reference point! The error equations for an observer using $z=\theta$ are

$$\dot{\tilde{\theta}} = -K_\theta\tilde{\theta} - \tilde{q}$$
$$\dot{\tilde{q}} = -K_q\tilde{\theta}$$
$$\dot{\tilde{y}} = -K_y\tilde{\theta} + \tilde{v}$$
$$\dot{\tilde{v}} = (\beta - K_v)\tilde{\theta}. \quad (40)$$

The system (40) has a characteristic equation

$$\lambda^2(\lambda^2 + K_\theta\lambda + K_q - 1) = 0. \quad (41)$$

Hence, two eigenvalues are zero, i.e., two linear combinations of the errors do *not* approach 0 as t increases. In fact it is easy to see that $\tilde{y}$ and $\tilde{v}$ do *not* attenuate, so this observer only estimates θ and q.

VII. Use of Observers to Estimate Bias Errors

Suppose the measurement contains a constant but unknown bias error b, i.e.,

$$z = Hx + b \quad (42)$$

where

$$\dot{b} = 0; \qquad b(0) = ?. \quad (43)$$

We can estimate b and x by considering b another state variable, so that the augmented plant model is

$$\dot{x} = Fx + Gu$$
$$\dot{b} = 0. \quad (44)$$

Example

For the undamped harmonic oscillator the following applies.

Measurement:

$$z = y + b. \quad (45)$$

Augmented plant model:

$$\dot{y} = v$$
$$\dot{v} = -\omega_0^2 y + u$$
$$\dot{b} = 0. \quad (46)$$

Observer:

$$\dot{\hat{y}} = \hat{v} + K_y(z - \hat{y} - \hat{b}); \qquad \hat{y}(0) = 0$$
$$\dot{\hat{v}} = -\omega_0^2\hat{y} + u + K_v(z - \hat{y} - \hat{b}); \qquad \hat{v}(0) = 0$$
$$\dot{\hat{b}} = K_b(z - \hat{y} - \hat{b}); \qquad \hat{b}(0) = 0. \quad (47)$$

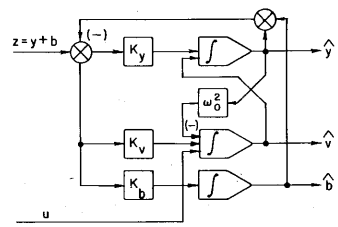

Fig. 9.

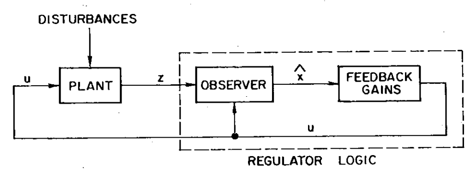

Fig. 10.

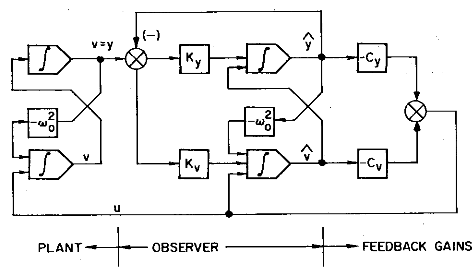

Fig. 11.

A block diagram of (47) is shown in Fig. 9. The error equations are obtained by subtracting (46) from (47), yielding

$$\dot{\tilde{y}} = \tilde{v} - K_y(\tilde{y} + \tilde{b})$$
$$\dot{\tilde{v}} = -\omega_0^2\tilde{y} - K_v(\tilde{y} + \tilde{b})$$
$$\dot{\tilde{b}} = -K_b(\tilde{y} + \tilde{b}). \quad (48)$$

The characteristic equation of the system (48) is

$$\lambda^3 + K_y\lambda^2 + (\omega_0^2 + K_v)\lambda + K_b\omega_0^2 = 0. \quad (49)$$

Clearly we can place the three eigenvalues of the observer wherever we wish by a proper choice of K_y, K_v, and K_b. If we choose eigenvalues with negative real parts, $\tilde{b}(t)$, $\tilde{v}(t)$, and $\tilde{y}(t) \to 0$ as t increases.

VIII. Regulator Synthesis Using Observers

Regulator logic can be synthesized by combining state-variable feedback with an observer that gives estimates of the state variables. We use

$$u = -C\hat{x} \quad (50)$$

where C is determined, as in Section IV, assuming that $\hat{x} \equiv x$, and we obtain $\hat{x}$ from an observer, using the measurement $z=Hx$, as in Section V. This is shown symbolically in Fig. 10.

Example A

A regulator design for the oscillator combining the feedback gains of Fig. 4 with the observer of Fig. 7 is shown in Fig. 11.

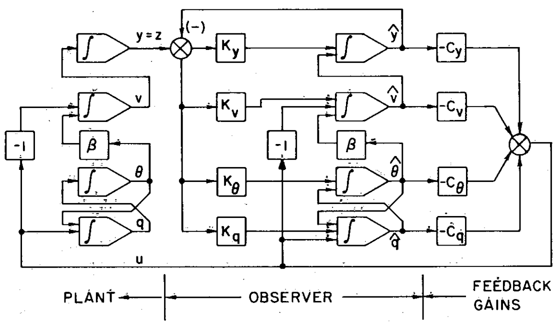

Fig. 12.

Example B

A regulator design that balances the stick and centers the cart (i.e., makes $y \cong 0$) of Fig. 3, combines the feedback gains of Fig. 5 with the observer of Fig. 8; it is shown in Fig. 12.

IX. The Separation of Eigenvalues

A remarkable and very useful fact about the controlled system of Fig. 10 is that its eigenvalues are the eigenvalues of the observer (i.e., of $F-KH$), plus the eigenvalues of the plant with direct state-variable feedback control (i.e., of $F-GC$). Thus, choice of the observer gains K determines n of the eigenvalues of the controlled system and the choice of the feedback gains C determines the other n.

This fact is easily seen if we use $(\tilde{x}, \hat{x})$ as the state variables of the controlled system instead of $(x, \hat{x})$:

$$\dot{\tilde{x}} = (F - KH)\tilde{x} \qquad (51)$$

$$\dot{\hat{x}} = (F - GC)\hat{x} - KH\tilde{x} \qquad (52)$$

where we have used $u = -C\hat{x}$, $\tilde{x} = \hat{x} - x$, and $z = Hx$. Clearly $\hat{x}$ depends on $\tilde{x}$ but not vice-versa since (51) does *not* contain $\hat{x}$. Thus the eigenvalues of the $(2n)$th order controlled system are the n eigenvalues of $F-KH$ and the n eigenvalues of $F-GC$, allowing completely separate design of the observer and the feedback gains C.

Example A

The fourth-order characteristic equation of the controlled system of Fig. 11 factors into two second-order terms:

$$(\lambda^2 + C_v\lambda + C_y + \omega_0^2)(\lambda^2 + K_y\lambda + K_v + \omega_0^2) = 0. \qquad (53)$$

Example B

The eighth-order characteristic equation of the controlled system of Fig. 12 factors into two fourth-order terms:

$$[\lambda^4 + (C_q - C_v)\lambda^3 + (C_\theta - C_y - 1)\lambda^2 + C_v(1+\beta)\lambda + C_y(1+\beta)]$$
$$\cdot [\lambda^4 + K_y\lambda^3 + (K_v - 1)\lambda^2 + (\beta K_\theta - K_y)\lambda + \quad - K_v] = 0. \qquad (54)$$

X. Supplemental Observers

An obvious question to ask about the observers for Examples A and B (Figs. 7 and 8) is, "Why estimate $\hat{y}$ when the measurement itself is already y?" Let us consider the following examples.

Example A

Why not just estimate $\hat{v}$ in Fig. 7 and use $\hat{y} = y$? This is a good idea, but the following rather obvious approach to it fails: from (5) and the fact that $z = y$, we might try

$$\dot{\hat{v}} = -\omega_0^2 z + u \qquad (55)$$

where the measurement z and the input u are known. However, if we subtract the second equation of (5) from (55), we get

$$\dot{\tilde{v}} = 0, \qquad (56)$$

that is, the error in the estimate of v does *not* tend to zero as time increases. Thus (55) does *not* produce a satisfactory estimate of v.

In [4] and [5], a way around this difficulty was suggested.[2] Estimate a linear combination of v and y, say

$$w = v - ky \qquad (57)$$

where the constant k is selected by writing the differential equation for w

$$\begin{aligned} \dot{w} &= \dot{v} - k\dot{y} \\ &= -\omega_0^2 y + u - kv \\ &= -k(v - ky) - (\omega_0^2 + k^2)y + u \end{aligned}$$

or

$$\dot{w} + kw = -(\omega_0^2 + k^2)y + u. \qquad (58)$$

From (58) and the fact that $z = y$, an observer for w can be designed as follows:

$$\dot{\hat{w}} + k\hat{w} = -(\omega_0^2 + k^2)z + u; \qquad \hat{w}(0) = 0. \qquad (59)$$

Subtracting (58) from (59) yields an equation for the error in the estimate of w, $\tilde{w} = \hat{w} - w$:

$$\dot{\tilde{w}} + k\tilde{w} = 0, \qquad \tilde{w}(0) = -w(0).$$

Clearly, if we choose $k > 0$, $\tilde{w}(t) \to 0$ as t increases. We can then estimate v from (57),

$$\hat{v} = \hat{w} + kz, \quad \text{since } z \equiv y. \qquad (60)$$

A block diagram of this *supplemental observer* is shown in Fig. 13. The *feedforward loop* is an essential part of any supplemental observer; it has the obvious disadvantage that noise in the measurement signals is passed on directly to the control signals.

Example B

For this case (see Fig. 8), we might try three linear combinations of v, θ, and q with the measured variable y:

$$\begin{aligned} w_1 &= v - k_1 y \\ w_2 &= \theta - k_2 y \\ w_3 &= q - k_3 y. \end{aligned} \qquad (61)$$

Differentiating (61) with respect to time and using (7) and

[2] In [6], a different but nearly equivalent approach was suggested.

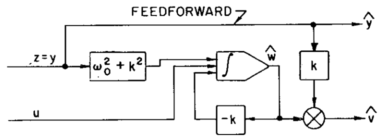

Fig. 13.

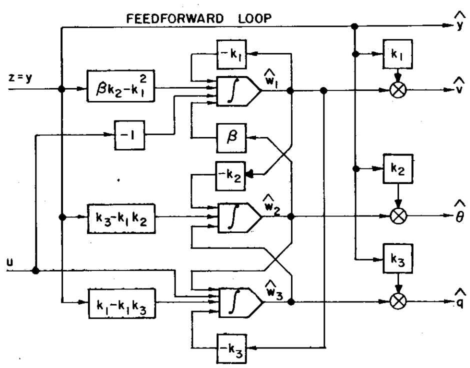

Fig. 14.

(61) again, it is easy to show that

$$\begin{aligned} \dot{w}_1 + k_1 w_1 - \beta w_2 &= (\beta k_2 - k_1^2)y - u \\ \dot{w}_2 - w_3 + k_2 w_1 &= (k_3 - k_1 k_2)y \\ \dot{w}_3 - w_2 + k_3 w_1 &= (k_1 - k_1 k_3)y + u. \end{aligned} \tag{62}$$

From (62) and the fact that $z=y$, an observer for w_1, w_2, and w_3 can be designed as follows:

$$\begin{aligned} \dot{\hat{w}}_1 + k_1 \hat{w}_1 - \beta \hat{w}_2 &= (\beta k_2 - k_1^2)z - u; \quad \hat{w}_1(0) = 0 \\ \dot{\hat{w}}_2 - \hat{w}_3 + k_2 \hat{w}_1 &= (k_3 - k_1 k_2)z; \quad \hat{w}_2(0) = 0 \\ \dot{\hat{w}}_3 - \hat{w}_2 + k_3 \hat{w}_1 &= (k_1 - k_1 k_3)z + u; \quad \hat{w}_3(0) = 0. \end{aligned} \tag{63}$$

From (61) clearly $\hat{v} = \hat{w}_1 + k_1 z$, $\hat{\theta} = \hat{w}_2 + k_2 z$, and $\hat{q} = \hat{w}_3 + k_3 z$. A block diagram of (63) is shown in Fig. 14. Subtracting (62) from (63) yields

$$\begin{aligned} \dot{\tilde{w}}_1 + k_1 \tilde{w}_1 - \beta \tilde{w}_2 &= 0 \\ \dot{\tilde{w}}_2 - \tilde{w}_3 + k_2 \tilde{w}_1 &= 0 \\ \dot{\tilde{w}}_3 - \tilde{w}_2 + k_3 \tilde{w}_1 &= 0. \end{aligned} \tag{64}$$

The characteristic equation of the system (64) is

$$\lambda^3 + k_1 \lambda^2 + (\beta k_2 - 1)\lambda + \beta k_3 - k_1 = 0. \tag{65}$$

If, for example, we wished to place the three eigenvalues of this supplemental observer at $\lambda = -3, -3+3i, -3-3i$, the desired characteristic equation is

$$(\lambda+3)(\lambda+3-3i)(\lambda+3+3i) = \lambda^3 + 9\lambda^2 + 36\lambda + 54 = 0. \tag{66}$$

Comparing coefficients of (65) and (66), we can find the required k_i's:

$$k_1 = 9, \qquad k_2 = 37/\beta, \qquad k_3 = 63/\beta. \tag{67}$$

General Case

For the case with plant model

$$\dot{x} = Fx + Gu \qquad (x \text{ an } n\text{-vector}) \tag{68}$$

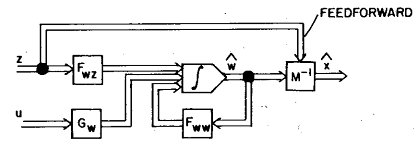

Fig. 15.

and measurement vector

$$z = Hx \qquad (z \text{ an } m\text{-vector}, m < n) \tag{69}$$

a supplemental observer of order $n-m$ can be designed as follows.

Change state variables from x to z and w (an $n-m$ vector):

$$\begin{bmatrix} w \\ \hline z \end{bmatrix} = \begin{bmatrix} K \\ \hline H \end{bmatrix} x \tag{70}$$

where K is *chosen* so that the n by n matrix

$$M \triangleq \begin{bmatrix} K \\ \hline H \end{bmatrix} \tag{71}$$

is *nonsingular*.

Multiply (68) by M and use

$$x = M^{-1} \begin{bmatrix} w \\ \hline z \end{bmatrix} \tag{72}$$

to yield

$$\begin{bmatrix} \dot{w} \\ \hline \dot{z} \end{bmatrix} = MFM^{-1} \begin{bmatrix} w \\ \hline z \end{bmatrix} + MGu. \tag{73}$$

Now partition the matrices MFM^{-1} and MG so that

$$\begin{bmatrix} \dot{w} \\ \hline \dot{z} \end{bmatrix} = \left[\begin{array}{c|c} F_{ww} & F_{wz} \\ \hline F_{zw} & F_{zz} \end{array}\right] \begin{bmatrix} w \\ \hline z \end{bmatrix} + \begin{bmatrix} G_w \\ \hline G_z \end{bmatrix} u. \tag{74}$$

From (74), an observer for w can be synthesized:

$$\dot{\hat{w}} = F_{ww}\hat{w} + F_{wz}z + G_w u; \qquad \hat{w}(0) = 0. \tag{75}$$

Then, from (72) and (75)

$$\hat{x} = M^{-1} \begin{bmatrix} \hat{w} \\ \hline z \end{bmatrix}. \tag{76}$$

A block diagram of this supplemental observer is shown in Fig. 15.

Subtracting the $\dot{w}$ equation of (74) from (75) gives the error equation

$$\dot{\tilde{w}} = F_{ww}\tilde{w}; \qquad \tilde{w}(0) = -w(0). \tag{77}$$

If the K matrix of (70) can be chosen so that the eigenvalues of F_{ww} have negative real parts, then $\tilde{w}(t) \to 0$ as t increases, and

$$\tilde{x}(t) = M^{-1} \begin{bmatrix} \tilde{w}(t) \\ \hline 0 \end{bmatrix} \to 0 \quad \text{as } t \text{ increases}. \tag{78}$$

XI. Regulator Synthesis Using Supplemental Observers

Regulator logic can be synthesized by combining state-variable feedback with a supplemental observer, i.e., a filter of lower order than the plant.[3] This is shown symbolically in Fig. 16. Regulators designed in this way have lead network or lead–lag network properties, which are illustrated in the following example.

Example

A first-order regulator for the second-order oscillator, which combines the feedback gains of Fig. 4 with the supplemental observer of Fig. 13, is shown in Fig. 17. The transfer function of this regulator is easily found to be

$$u(s) = -C\frac{s+\alpha}{s+k}y(s) \tag{79}$$

where

$$C = C_y + kC_v > 0, \qquad \alpha = \frac{C_y - \omega_0^2 C_v/k}{C_y + kC_v} < 1.$$

This is a *lead compensator*, exactly the type of regulator an experienced designer would have prescribed for this system with a measurement of displacement [compare with (2)]. Moreover, the gains C_y, C_v, and k can be selected to give any desired eigenvalues of the third-order controlled system; in fact, we again have *separability* of the eigenvalues, i.e., the characteristic equation of the controlled system (Fig. 17) is easily shown to be factorable as follows:

$$(\lambda + k)(\lambda^2 + C_v\lambda + \omega_0^2 + C_y) = 0. \tag{80}$$

Thus the eigenvalue of the observer ($\lambda = -k$) is also an eigenvalue of the controlled system, and the other two eigenvalues are determined by the state-variable feedback gains C_v and C_y.

XII. The Separation of Eigenvalues

The property noted for the Example in Section XI can be shown to exist for *all* systems with regulators made up of supplemental observers and state-variable feedback gains. The eigenvalues of the controlled system of Fig. 16 are the eigenvalues of the supplemental observer (i.e., of F_{ww}) plus the eigenvalues of the plant with state-variable feedback control (i.e., of $F-GC$). Thus the choice of the matrix K in $w=Kx$ [see (70)] determines $n-m$ of the eigenvalues of the controlled system, and the choice of the feedback gains C determines the other n.

As in Section XI, this fact is easily demonstrated if we use $(\tilde{w}, \hat{x})$ as the state variables of the controlled system instead of $(x, \hat{w})$:

$$\dot{\tilde{w}} = F_{ww}\tilde{w} \tag{81}$$

$$\dot{\hat{x}} = (F - GC)\hat{x} - M^{-1}\left[\begin{array}{c} 0 \\ \hline F_{zw} \end{array}\right]\tilde{w}. \tag{82}$$

[3] For a single feedback control, only one linear combination of the state variables is required. In [5] it is shown that an observer for this linear combination can be constructed with order even lower than $n-m$.

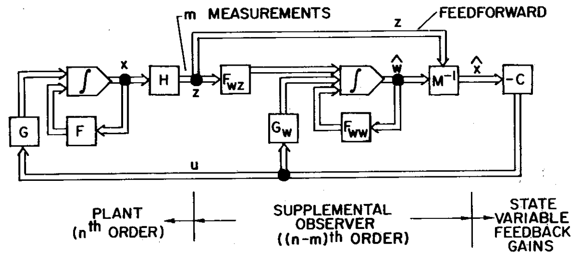

Fig. 16.

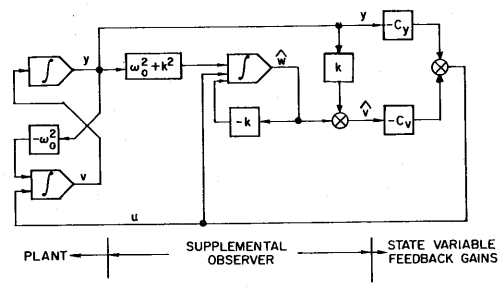

Fig. 17.

Clearly $\hat{x}$ depends on $\tilde{w}$ but not vice-versa since (81) does not contain $\hat{x}$. Thus the eigenvalues of the $(2n-m)$th order controlled system are the $n-m$ eigenvalues of F_{ww} and the n eigenvalues of $F-GC$, allowing completely separate design of the observer and the feedback gains C.

To derive (82), start with

$$\dot{x} = Fx - GC\hat{x}, \tag{83}$$

and use $x = \hat{x} - \tilde{x}$:

$$\dot{\hat{x}} = (F - GC)\hat{x} + \dot{\tilde{x}} - F\tilde{x}. \tag{84}$$

Next, use

$$\tilde{x} = M^{-1}\left[\begin{array}{c} \tilde{w} \\ \hline 0 \end{array}\right],$$

so that

$$\dot{\tilde{x}} - F\tilde{x} = M^{-1}\left[\begin{array}{c} \dot{\tilde{w}} \\ \hline 0 \end{array}\right] - FM^{-1}\left[\begin{array}{c} \tilde{w} \\ \hline 0 \end{array}\right]. \tag{85}$$

From (73) and (74), we have

$$MFM^{-1} = \left[\begin{array}{c|c} F_{ww} & F_{wz} \\ \hline F_{zw} & F_{zz} \end{array}\right] \Rightarrow FM^{-1} = M^{-1}\left[\begin{array}{c|c} F_{ww} & F_{wz} \\ \hline F_{zw} & F_{zz} \end{array}\right]. \tag{86}$$

Substituting (86) into (85) gives

$$\dot{\tilde{x}} - F\tilde{x} = M^{-1}\left[\begin{array}{c} \dot{\tilde{w}} - F_{ww}\tilde{w} \\ \hline -F_{zw}\tilde{w} \end{array}\right]. \tag{87}$$

Using (77) in (87) and substituting the result into (84) yields (82).

Example

Consider regulator design for an unstable first-order plant using a measurement of the state containing a bias error (i.e., an unknown additive constant).

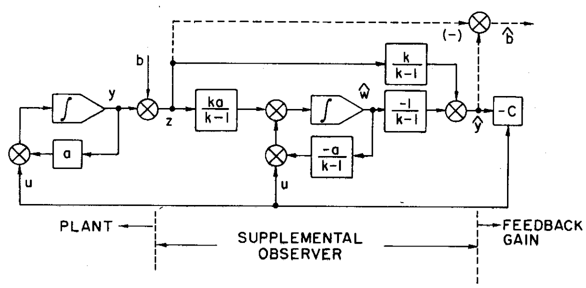

Fig. 18.

Augmented plant:

$$\dot{y} = ay + u; \qquad a > 0 \tag{88}$$

$$\dot{b} = 0. \tag{89}$$

Measurement:

$$z = y + b. \tag{90}$$

To design a supplemental observer to estimate b, let

$$w = y + kb, \tag{91}$$

where $k \neq 1$. Using the method outlined the regulator design is

$$\dot{\hat{w}} + \frac{a}{k-1}\hat{w} = \frac{ka}{k-1}z + u; \qquad \hat{w}(0) = 0 \tag{92}$$

$$\hat{y} = \frac{-1}{k-1}\hat{w} + \frac{k}{k-1}z \tag{93}$$

$$\hat{b} = \frac{1}{k-1}\hat{w} - \frac{1}{k-1}z \tag{94}$$

$$u = -C\hat{y}. \tag{95}$$

A block diagram of the controlled system is shown in Fig. 18. The loop for estimating $\hat{b}$ is shown dashed since it would usually *not* be implemented (no feedback on $\hat{b}$).

The characteristic equation of the controlled system is

$$(\lambda + C - a)\left(\lambda + \frac{a}{k-1}\right) = 0. \tag{96}$$

Hence the eigenvalues separate, as predicted: $\lambda = -(C-a)$ and $\lambda = -a/(k-1)$. Thus, the system is stable if $C > a$, $k > 1$. Note the transfer function of the regulator is

$$u(s) = -\frac{ck}{k-1}\,\frac{s}{s - \dfrac{c-a}{k-1}}\,z(s) \tag{97}$$

which is *not* a conventional compensator!

XIII. Observer Design when Control Inputs Appear Directly in the Measurements

If control inputs u appear directly in the measurements z, in the form

$$z = Hx + Ju \tag{98}$$

where H, J are known; the obvious minor modification of the observer design of Section V is

$$\dot{\hat{x}} = F\hat{x} + Gu + k(z - H\hat{x} - Ju); \qquad \hat{x}(0) = 0. \tag{99}$$

Subtracting $\dot{x} = Fx + Gu$ from (99) and using (98) gives, as in Section V,

$$\dot{\tilde{x}} = (F - KH)\tilde{x}; \qquad \tilde{x}(0) = -x(0) \tag{100}$$

where

$$\tilde{x} = \hat{x} - x = \text{error in the estimate}.$$

XIV. Extensions

Section IV can be extended to multiple inputs. Also, the gain matrix C may be determined by minimizing an integral quadratic form in the state x and the controls u (see e.g., [8]). This latter concept can be extended to time-varying linear systems and terminal control problems.

The Kalman–Bucy filter [9] has the same form as the observer in Section V. However additive purely random Gaussian noise is assumed to occur in the measurements and as forcing functions in the plant; the gains K are determined to minimize the mean square error of the estimates. This approach applies to time-varying linear systems.

XV. Summary

1) Classical regulator synthesis uses compensating networks and frequency response techniques.
2) State-variable feedback design for $\dot{x} = Fx + Gu$ uses $u = -Cx$. C can be chosen to give the controlled system arbitrary dynamics.
3) An "observer" is a closed-loop state-variable estimator of the form $\dot{\hat{x}} = F\hat{x} + Gu + K(z - H\hat{x})$, where $z = Hx$ are output measurements. K can be chosen to give the observer arbitrary dynamics.
4) Observers can often estimate bias errors in the output measurements.
5) A regulator can be designed by combining state-variable feedback with an observer using $u = -C\hat{x}$.
6) The eigenvalues of the controlled system separate into the eigenvalues of the observer, and the eigenvalues of the plant with direct state-variable feedback.
7) Supplemental observers have lower order than the plant, and involve feedforward.
8) A regulator can be designed by combining state-variable feedback with a supplemental observer and is essentially a "compensating network." The eigenvalues of the controlled system separate just as in item 6.

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Reprinted from the PROCEEDINGS OF THE IEEE
VOL. 58, NO. 11, NOVEMBER, 1970
pp. 1803–1811