Tracking of Goal Seeking Vehicles

DAVID G. LUENBERGER, MEMBER, IEEE

Abstract—In many competitive situations it is necessary to track an enemy vehicle on the basis of noisy measurements received at a tracking station. In such situations it is often not appropriate to model the enemy vehicle as a dynamic system subject to white noise inputs since the enemy may be moving toward some specific objective which is unknown or partially unknown to the tracker. The tracker should be able to incorporate this fact in computing his opti-

---

Manuscript received May 19, 1967; revised October 11, 1967. This work was performed under Contract N123-(953) 54486A for the U. S. Naval Electronics Labs., San Diego, Calif. This work was performed while the author was a consultant with the Stanford Research Institute, Menlo Park, Calif.

The author is with the Stanford Electronics Laboratories, Stanford University, Stanford, Calif.

mal estimate of the vehicle's position. A simple model of these competitive situations is proposed in which the enemy is assumed to be governed by a system of linear differential equations controlled so as to move optimally toward a goal unknown to the tracker. The optimal estimation equations for the tracker are derived, and it is shown that the tracker can use a linear dynamic system to form the best estimate of the enemy position and his goal.

INTRODUCTION

Usual estimation and tracking procedures often assume that the object being tracked behaves as a dynamic system subject to noisy disturbances and unknown initial conditions.[1] The tracking station obtains noisy measurements of the object and from these computes an estimate of the object's position and velocity coordinates. In the standard formulation, no account is taken of intelligence within the object which may be guiding it toward a specific goal which is unknown or only partially known to the observer at the tracking station.

Problems of this type arise in competitive or combat situations when one force is tracking an adversary who is initiating an attack at an unknown position. For instance, a destroyer may be tracking a submarine that threatens to attack a convoy. In this case presumably the submarine is likely to move toward the convoy to a point convenient for attack. A tracking station on the destroyer, although uncertain of the submarine's precise destination, should be able to utilize its current estimate of the destination to reduce its tracking error and to continuously improve its estimate of the submarine's destination.

Any major analysis of tactical situations such as those mentioned quickly leads to a complex formulation in the realm of dynamic game theory for which no general techniques of solution are yet available. The analysis contained in this paper presents a preliminary attempt to study problems of this type.

PROBLEM STATEMENT

The tactical situation described above can be viewed from two points of view: that of the attacker and that of the tracker. The attacker is assumed to have perfect information about his own position. Initially he decides on an appropriate destination and then moves toward that point. His method for determining the optimal destination is not considered here since that depends strongly on the specific nature of the tactical problem. It is assumed, however, that such a choice can be reasonably made at the outset and that the choice, once made, is not changed en route.

Specifically, the attacker is assumed to have control over his vehicle which is governed by the dynamic system

$$\dot{x}(t) = F(t)x(t) + D(t)u(t) \tag{1}$$

where

$x(t)$ is an $n \times 1$ state vector,
$F(t)$ is an $n \times n$ system matrix,
$D(t)$ is an $n \times r$ distribution matrix, and
$u(t)$ is an $r \times 1$ input (control) vector.

It is assumed that initially the system is at state $x(0) = x_0$.

The prescribed final destination of the vehicle is denoted x_F and it is assumed that the final time T is also prescribed.

The problem of the attacker is that of determining a control action that will take him to x_F at time T. It is assumed here that this control action is chosen so as to minimize the quadratic cost functional

$$J = \frac{1}{2}\int_0^T [x'(t)R_1(t)x(t) + \{x(t) - x_F\}'R_2(t)\{x(t) - x_F\} + u'(t)Q(t)u(t)]dt \tag{2}$$

subject to the constraint that $x(T) = x_F$. The $r \times r$ symmetric matrix $Q(t)$ is assumed to be positive definite; R_1 and R_2 are positive semidefinite.

Presumably the cost functional J given by (2) represents a combination of operating costs: fuel, wear, etc.; and tactical costs: noise of engines, water depth of submarine, etc. The choice of parameters of this functional is not considered here, but it is assumed that they are uniquely defined and known both to the attacker and the tracker.

The problem of the attacker may be summarized as consisting only of solving the deterministic control problem of minimizing (2) while satisfying $x(T) = x_F$.

This paper is primarily concerned with the problem from the tracker's point of view. It is assumed that he has an estimate of the attacker's initial position and an estimate, based on tactics, of the attacker's destination. These estimates, denoted by $\hat{x}_0$ and $\hat{x}_F$, have associated covariances

$$P_1(0) = E[(x_0 - \hat{x}_0)(x_0 - \hat{x}_0)']$$

and

$$P_3(0) = E[(x_F - \hat{x}_F)(x_F - \hat{x}_F)']$$

respectively.

As time progresses the tracker obtains noisy measurements of the attacker's position in the form

$$y(t) = M(t)x(t) + w(t) \tag{3}$$

where

$y(t)$ is an $m \times 1$ observation vector,
$M(t)$ is an $m \times n$ matrix, and
$w(t)$ is an $m \times 1$ vector of white noise with mean zero and covariance $E[w(t)w'(s)] = W(t)\delta(t-s)$.

From the original estimates, the observed data, and a knowledge of the attacker's capabilities, i.e., his system (1) and his objective (2), the tracker is required to form optimal estimates of both the attacker's current position and his destination. Since the tracker is assumed to know the exact quadratic criterion being used by the attacker and the final time T, the tracker knows the attacker's motion to within $2n$ variables: x_0 and x_F. Thus the estimation problem can be viewed as a special kind of linear regression problem.

SOLUTION OF THE CONTROL PROBLEM

The control problem stated in the previous section is now solved. Given:

$$\begin{aligned} \dot{x}(t) &= F(t)x(t) + D(t)u(t) \\ x(0) &= x_0 \\ x(T) &= x_F. \end{aligned} \tag{4}$$

Minimize

$$J = \frac{1}{2}\int_0^T [x'R_1x + (x - x_F)'R_2(x - x_F) + u'Qu]dt. \tag{5}$$

The standard optimal control methodology[2] can be employed and the adjoint variable $\lambda(t)$ introduced, defined in this case by

$$-\dot{\lambda}(t) = F'(t)\lambda(t) + R_1(t)x(t) + R_2(t)[x(t) - x_F]. \tag{6}$$

The optimal control can then be expressed in terms of λ according to

$$u(t) = -Q^{-1}D'\lambda(t). \tag{7}$$

Together, (4), (6), and (7) define the optimal trajectory and the optimal control. This set of equations, however, can be transformed into an explicit feedback control law for $u(t)$.

Following Bullock,[3] a solution is sought of the form

$$x(t) - b(t) = S(t)\lambda(t) \tag{8}$$

where $b(t)$ and $S(t)$ are yet to be determined.

Differentiating (8) with respect to t we obtain

$$\dot{x}(t) - b(t) = \dot{S}(t)\lambda(t) + S(t)\dot{\lambda}(t) \tag{9}$$

which upon substitution of (4) and (6) becomes

$$Fx + Du - b = \dot{S}\lambda - SF'\lambda - SR_1x - SR_2x + SR_2x_F. \tag{10}$$

Substitution of (7) and (8) in (10) yields

$$FS\lambda + Fb - DQ^{-1}D'\lambda - \dot{b} = \dot{S}\lambda - SF'\lambda - S(R_1 + R_2)S\lambda - S(R_1 + R_2)b + SR_2x_F. \tag{11}$$

Now (11) can be identically satisfied by choosing b and S to satisfy

$$-\dot{b} + Fb = -S(R_1 + R_2)b + SR_2x_F \tag{12}$$

and

$$\dot{S} - FS - SF' - S(R_1 + R_2)S + DQ^{-1}D' = 0. \tag{13}$$

The boundary conditions for (12) and (13) will be chosen so that the boundary condition $x(t) = x_F$ is satisfied. Inspection of (8) shows that if

$$b(T) = x_F \tag{14}$$

$$S(T) = 0 \tag{15}$$

the boundary condition is satisfied.

Collecting the results of these manipulations, the solution is obtained in feedback form. Equations (7) and (8) give

$$u(t) = -Q^{-1}(t)D'(t)S^{-1}(t)[x(t) - b(t)] \tag{16}$$

where $b(t)$ and $S(t)$ are found by integrating (12) and (13) backwards from $t = T$ using the boundary conditions (14) and (15).

There are several interesting features of this solution that will play an important role in the following section in the derivation of the tracking solution.

The first point to note is that the matrix Riccati equation (13) does not depend on x_F. Thus presumably both the attacker and the tracker can integrate this equation to obtain $S(t)$ even before any decision about x_F is made.

The second point to notice is that $b(t)$ can be computed after $S(t)$ is computed and after x_F is chosen. The *initial* state x_0 does not enter into the computation of $b(t)$. Thus the tracker's uncertainty of the attacker's position at any time does not directly affect his estimate of $b(t)$—only the uncertainty about the final state affects the estimate of $b(t)$.

SOLUTION TO THE TRACKING PROBLEM

At the outset the tracker prepares to accept measurements by first integrating the Riccati equation (13) just as he knows the attacker must do.

$$\begin{aligned} \dot{S} &= FS + SF' + S(R_1 + R_2)S \\ S(T) &= 0. \end{aligned} \tag{17}$$

Next, not knowing x_F precisely, the tracker must integrate the equation determining $b(t)$ backward [see (12)] for all possible x_F which, since the equation is linear, can be done by integrating a single matrix equation.

$$\begin{aligned} -\dot{B} + FB &= -S(R_1 + R_2)B + SR_2 \\ B(T) &= I. \end{aligned} \tag{18}$$

Here $B(t)$ is a square matrix. If x_F were known by the tracker he could obtain $b(t)$ by

$$b(t) = B(t)x_F. \tag{19}$$

At this point the tracker can describe the motion of the attacker

$$\dot{x}(t) = F(t)x(t) + D(t)u(t) \tag{20}$$

by the homogeneous set of equations

$$\begin{aligned} \dot{x} &= Fx - DQ^{-1}D'S^{-1}[x - Bx_F] \\ \dot{x}_F &= 0 \end{aligned} \tag{21}$$

where the explicit expression (16) for the control and (19) have been substituted in (20).

In matrix form (21) becomes

$$\begin{bmatrix} \dot{x} \\ \dot{x}_F \end{bmatrix} = \begin{bmatrix} F - DQ^{-1}DS^{-1} & DQ^{-1}DS^{-1}B \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ x_F \end{bmatrix} \tag{22}$$

or, by introducing a $2n$-dimensional vector,

$$\bar{x} = \begin{bmatrix} x \\ x_F \end{bmatrix}.$$

Equations (22) can be written

$$\dot{\bar{x}} = \bar{A}\bar{x}. \tag{23}$$

The tracker can now regard his problem as one of estimating the single $2n$-dimensional vector $\bar{x}(t)$ from the noisy measurements

$$y = M(t)x(t) + w(t)$$

or equivalently

$$y = \bar{M}\bar{x} + w \tag{24}$$

where

$$M = \underbrace{M \quad 0}$$

and from his initial estimate of $\bar{x}(0)$. Thus the tracker jointly estimates both the attacker's current state and his destination. In this form the problem can be resolved by conventional techniques.

The standard approach to estimation problems of this type is to solve an appropriate Riccati equation for the covariance matrix of the optimal estimation error and then to construct the estimate from it.[1] Since the system (23) is of order $2n$, the appropriate Riccati equation is also $2n$-dimensional.

The appropriate Riccati equation for the error covariance is

$$\begin{aligned} \dot{P} &= AP + PA + P\bar{M}'W^{-1}\bar{M}P \\ P(0) &= \begin{bmatrix} P_1(0) & 0 \\ 0 & P_3(0) \end{bmatrix}. \end{aligned} \tag{25}$$

From $P(t)$ the optimal estimate can be derived from the measurements according to

$$\dot{\hat{\bar{x}}}(t) = A\hat{\bar{x}} + P\bar{M}'W^{-1}[y(t) - \bar{M}\hat{\bar{x}}(t)]. \tag{26}$$

Equations (25) and (26) are both $2n$-dimensional and, in particular, (25) is a $2n \times 2n$ matrix differential equation.

By partitioning the matrices in (25) some insight is obtained. Let

$$P = \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix}$$

$$A = \begin{bmatrix} \bar{F} & \bar{D} \\ 0 & 0 \end{bmatrix}$$

$$\bar{M} = [M \quad 0].$$

Then separate equations can be written for P_1, P_2, and P_3 that are equivalent to (25):

$$\dot{P}_1 = \bar{F}P_1 + P_1\bar{F}' + \bar{D}P_2 + P_2\bar{D}' - P_1M'W^{-1}P_1 \tag{27}$$

$$\dot{P}_2 = FP_2 + DP_3 - P_1M'W^{-1}MP_2 \tag{28}$$

$$\dot{P}_3 = P_2M'W^{-1}MP_2. \tag{29}$$

These equations are coupled and apparently cannot be uncoupled in a simple way; thus the decomposition does not lead to any significant reduction in the computation required. The corresponding estimation equation becomes

$$\dot{\hat{x}}(t) = \bar{F}\hat{x} + \bar{D}\hat{x}_F + P_1M'W^{-1}[y(t) - M\hat{x}(t)] \tag{30}$$

$$\dot{\hat{x}}_F = P_2M'W^{-1}[y(t) - M\hat{x}(t)]. \tag{31}$$

Although these equations are fairly complex, due to the coupling between them, it is possible to obtain some feeling for their meaning by examining certain limiting cases. Of particular interest is the case where the tracker knows x_F precisely. In that case $P_3(0) = P_2(0) = 0$. Inspection of (28) and (29) reveals that their P_2 and P_3 remain 0 for all time, i.e., if the final state x_F is originally known, it remains known.

CONCLUSIONS

The problem discussed in this paper represents only an initial attempt at developing analytical approaches to tactical problems such as outlined in the Introduction. The method has several limitations, the primary one being the assumption that the enemy moves in an optimal fashion determined by a fixed deterministic criterion known to both the attacker and the tracker.

On the other hand, the method seems to be the first estimation technique to account for the goal seeking nature of the target. Furthermore, the method, based on a fairly complex problem formulation, nevertheless results in a straightforward computational procedure not much more complex than standard Kalman filter techniques.

No evaluation of the method proposed here has been made in the sense of computer simulations or comparison with other possible methods. Since the method, by its construction, is optimal for the given problem formulation, the utility of the method rests primarily on the validity of the proposed problem formulation as a model of the given tactical situation.

It is hoped that this approach to the tactical estimation problems will stimulate further research in this area.

ACKNOWLEDGMENT

The author would like to acknowledge Dr. J. Peschon who first brought problems of this type to his attention.

REFERENCES

[1] R. E. Kalman and R. Bucy, "New results in linear filtering and prediction theory," *Trans. ASME, J. Basic Engrg.* ser. D, vol. 83, pp. 95–108, 1961.

[2] L. S. Pontryagin *et al.*, *The Mathematical Theory of Optimal Processes.* New York: Interscience, 1962.

[3] T. E. Bullock, "Computation of optimal controls by a method based on second variations," Dept. of Aeronautics and Astronautics, Stanford University, Stanford, Calif., SUDAAR 297, pp. 60–70, December 1966.