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Stochastic Differential Games with Constrained State Estimators

IAN B. RHODES, MEMBER, IEEE, AND DAVID G. LUENBERGER, MEMBER, IEEE

***Abstract*—Attention is given to stochastic differential games in which the two controllers have available only noise-corrupted output measurements. Consideration is restricted to the case in which the system is linear, the cost functional quadratic, and the noises corrupting the output measurements are independent, white, and Gaussian. A solution to this problem is presented under the constraint that each controller is limited to a linear dynamic system of fixed dimension for the generation of his estimate of the system state. The optimal controls are shown to satisfy a separation theorem, the optimal estimators are shown to be closely related to Kalman filters, and the various terms in the optimal cost are shown to be readily assignable to the appropriate contributing sources.**

I. INTRODUCTION

IN AN EARLIER PAPER [1], attention was given to a number of stochastic differential game problems in which the two opposing controllers had access only to noise-corrupted output measurements. In that paper, as in this one, attention is restricted to the following problem involving a linear system, a quadratic cost functional, and linear output measurements additively corrupted by independent white Gaussian noises.

Problem 1

Consider a linear system described by the vector differential equation

$$\dot{x} = F(t)x + G_1(t)u(t) + G_2(t)v(t) \tag{1}$$

to which Controller 1, controlling u, has available measurements of the form

$$y_1(t) = H_1(t)x(t) + w_1(t) \tag{2a}$$

while Controller 2, controlling v, has available measurements

$$y_2(t) = H_2(t)x(t) + w_2(t). \tag{2b}$$

The vector $x(t) \in E^n$ is the system state, $u(t) \in E^p$ and $v(t) \in E^q$ are the control vectors, $y_1(t) \in E^{m_1}$ and $y_2(t) \in E^{m_2}$ are the measured output vectors, and the matrices $F(t)$, $G_1(t)$, $G_2(t)$, $H_1(t)$, and $H_2(t)$ have the appropriate dimensions. The random disturbances $w_1(t)$ and $w_2(t)$ are assumed independent, white, zero-mean, and Gaussian with covariances

$$\text{cov } [w_1(t),w_1(\tau)] = W_1(t)\delta(t - \tau) \tag{3a}$$

$$\text{cov } [w_2(t),w_2(\tau)] = W_2(t)\delta(t - \tau) \tag{3b}$$

$$\text{cov } [w_1(t),w_2(\tau)] \equiv 0. \tag{3c}$$

It is assumed for simplicity that both controllers consider the initial state $x(t_0)$ to be a Gaussian random vector uncorrelated for all t with $w_1(t)$ and $w_2(t)$ and having mean $\bar{x}_0$ and covariance

$$\text{cov } [x(t_0),x(t_0)] = M_0. \tag{4}$$

Consider also the quadratic cost functional

$$J_\tau[u,v] = \int_\tau^T (u'Q_1u + v'Q_2v)\, dt + x'(T)C'Cx(T) \tag{5}$$

where the matrices $Q_1(t)$ and $Q_2(t)$ are symmetric and, respectively, positive definite and negative definite, the matrix C has dimension $m \times n$ with $m \leqq n$, and the final time T is fixed. Denote by $Y_i(t)$, $i = 1, 2$, the output function measured by Controller i up to time t, i.e.,

$$Y_i(t) = \{(y_i(s),s),\ s \in [t_0,t)\}. \tag{6}$$

It is understood that the measurement set $Y_i(t)$ includes the a priori information available to Controller i so that, in particular, $Y_i(t_0)$ is simply this a priori information. Attention is restricted to feedback control laws $u(\cdot,\cdot)$ and $v(\cdot,\cdot)$ where $u(\cdot,\cdot)$ is a mapping from the pairs $(Y_1(t),t)$, $t \in [t_0,T]$, to E^p and $v(\cdot,\cdot)$ is a mapping from the pairs $(Y_2(t),t)$ to E^q. The objective is to find (if they exist) the control laws $u^0(\cdot,\cdot)$ and $v^0(\cdot,\cdot)$ which are optimal in the sense that for any control laws $u(\cdot,\cdot)$ and $v(\cdot,\cdot)$ there hold for all $\tau \in [t_0,T]$

$$E\{J_\tau[u^0,v^0] | Y_1(\tau)\} \leqq E\{J_\tau[u,v^0] | Y_1(\tau)\} \tag{7a}$$

$$E\{J_\tau[u^0,v] | Y_2(\tau)\} \leqq E\{J_\tau[u^0,v^0] | Y_2(\tau)\}. \tag{7b}$$

By giving separate additional consideration to the special cases in which one (or both) of the controllers has available perfect state information or no output measurements at all, nine problems may be formulated. Of these, eight have been examined in [1]; these are the cases in which at least one of the two controllers has either perfect state information or no measurements at all. The problem in which one controller has perfect state measurements has also been examined in [2], [3]. The remaining problem, in which both controllers have available only noise-

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I. B. Rhodes is with the Department of Electrical Engineering, Massachusetts Institute of Technology, Cambridge, Mass.

D. G. Luenberger is with the Stanford Electronics Laboratories, Stanford University, Stanford, Calif.

corrupted output measurements, is the subject of this paper. It has also been examined in [4].

It is noted that the analysis and discussion that follow may be readily extended to the case where terms of the form

$$\int_{\tau}^{T} [x'R_1x + (x - d)'R_2(x - d)]\, dt + [x(T) - d(T)]'\, R_3[x(T) - d(T)]$$

are added to the cost functional (5), where R_1, R_2, and R_3 are symmetric nonnegative definite matrices and $d(\cdot)$ is a prescribed n-vector valued function defined over $[t_0,T]$. Furthermore, it is formally straightforward to extend this material to the situation where, in addition to the noises corrupting the measurements, Gaussian white additive noise that is uncorrelated with the initial state and the measurement noises is present in the system dynamics (1). As in the corresponding stochastic optimal control situation, noises in the dynamics or measurements that are Markov with rational spectra instead of white may be included in the problem formulation by the standard procedure of defining additional state variables.

II. Formulation of the Problem

The eight problems examined in [1] had one important property in common; conditions were such that each controller required only a finite-dimensional dynamic system (together, possibly, with a singular estimator) to estimate the system state and any other necessary parameters. Alternatively, this common property may be viewed in terms of the fact that in these special cases it was unnecessary for either controller to retain the entire measured output function up to the time under consideration, since the output of a certain finite-dimensional dynamic system was sufficient to represent all the information the controller required from the output function measured up to that time.

It appears, however, that when both controllers only have noise-corrupted measurements available a general solution to the problem will require an infinite-dimensional state estimator for each controller or, equivalently, the preservation and use in the control applied at time t of the entire output function measured up to that time. On the other hand, in the absence of special conditions under which the required estimators turn out to be finite-dimensional dynamic systems, it is of interest to examine the problem in the case where we formally impose the restriction that each controller is limited to an estimator of fixed and finite dimension, the output of which is all that is available to him in the generation of his control at that time. In particular, we examine the case where each controller is allowed an n-dimensional linear dynamic system, the inputs to which at any given time t are restricted to the measured output $y_1(t)$ or $y_2(t)$ and the control input $u(t)$ or $v(t)$ applied by the controller to the main system at that time. Thus Controller 1 is allowed a linear dynamic system, the n-dimensional state vector $z_1(t)$ of which may be considered without loss of generality to satisfy the linear differential equation

$$\dot{z}_1 = A_1(t)z_1 + B_1(t)[y_1(t) - H_1(t)z_1] + D_1(t)u(t) \tag{8a}$$

where the matrices $A_1(t)$, $B_1(t)$, and $D_1(t)$ and the initial condition $z_1(t_0)$ are to be selected by Controller 1. Moreover, Controller 1 is limited at each time t to a control of the form $u(t) = u(z_1(t),t)$. Entirely analogous restrictions apply to Controller 2.

Thus, instead of allowing each controller to preserve and use the entire output function he has measured up to time t, we restrict him to generate and use only a selected n-vector-valued function of it. Furthermore, instead of having available the entire measured output function $Y_1(t)$ or $Y_2(t)$ upon which to condition his expectation of the cost functional or other entities, each controller is restricted to conditioning expectations on knowledge only of $z_1(t)$ or $z_2(t)$ and the a priori information. The combination of $z_1(t)$ and the a priori information will henceforth be denoted $Z_1(t)$, with a similar definition for $Z_2(t)$.

In order to incorporate these considerations, the formulation of Problem 1 is modified to restrict attention to closed-loop control laws $u(\cdot,\cdot)$ and $v(\cdot,\cdot)$ which map, respectively, the pairs $(Z_1(t),t)$ to E^p and the pairs $(Z_2(t),t)$ to E^q. Furthermore, it is convenient to regard the combination of the estimator (8a) and the control law $u(\cdot,\cdot)$, both of which are to be selected by Controller 1, as a single entity, which we call an estimator-controller and denote abstractly by U. Similarly, the combination of estimator and controller for Controller 2 we denote by V. The problem may then be viewed as one of finding the combined estimator-controllers U^0 and V^0 which are optimal in the sense that for all estimator controllers U and V and all τ in $[t_0,T]$ we have

$$E\{J_\tau[U^0,V]|Z(t_0)\} \leq E\{J_\tau[U^0,V^0]|Z(t_0)\} \leq E\{J_\tau[U,V^0]|Z(t_0)\}. \tag{9}$$

This saddle point criterion of optimality is obtained by conditioning the expectations in (7) on $Z_1(t_0)$ and $Z_2(t_0)$ instead of on $Y_1(t)$ and $Y_2(t)$; since $Z_1(t_0)$ and $Z_2(t_0)$ are simply the a priori information sets available to Controllers 1 and 2 and since, by assumption, these information sets are the same, we write $Z_1(t_0) = Z_2(t_0) = Z(t_0)$ and combine the resultant pair of inequalities into the double inequality (9). A formal statement of the problem follows.

Problem 2

Consider the special case of Problem 1 for which, instead of the output measurements (2a), Controller 1 has available only the n-dimensional state vector of the dynamic system

$$\dot{z}_1 = A_1(t)z_1 + B_1(t)[y_1(t) - H_1(t)z_1] + D_1(t)u(t) \tag{8a}$$

where the matrices $A_1(t)$, $B_1(t)$, and $D_1(t)$ and the initial condition $z_1(t_0)$ may be chosen by Controller 1. Similarly, instead of the output measurements (2b), Controller 2 has available only the n-dimensional state vector $z_2(t)$

of the dynamic system

$$\dot{z}_2 = A_2(t)z_2 + B_2(t)[y_2(t) - H_2(t)z_2] + D_2(t)v(t) \tag{8b}$$

where the matrices $A_2(t)$, $B_2(t)$, and $D_2(t)$ and the initial condition $z_2(t_0)$ are to be selected. With the cost functional $J_\tau[u,v]$ defined by (5) we seek, if they exist, the matrices $A_1{}^0(t)$, $B_1{}^0(t)$, $D_1{}^0(t)$, $A_2{}^0(t)$, $B_2{}^0(t)$, and $D_2{}^0(t)$, $t \in [t_0,T]$, the initial conditions $z_1{}^0(t_0)$ and $z_2{}^0(t_0)$, and the control laws

$$u^0(\cdot,\cdot): E^n \times [t_0,T] \rightarrow E^p$$
$$v^0(\cdot,\cdot): E^n \times [t_0,T] \rightarrow E^q$$

which are optimal in the sense that given the matrices $A_2{}^0(t)$, $B_2{}^0(t)$, and $D_2{}^0(t)$, $t \in [t_0,T]$, the initial state $z_2{}^0(t_0)$ and the control law $v^0(\cdot,\cdot)$ [i.e., given the optimal combined estimator-controller V^0 for Controller 2] there holds for all $\tau \in [t_0,T]$

$$E\{J_\tau[u^0,v^0] \| Z_1(t_0)\} \leqq E\{J_\tau[u,v^0] \| Z_1(t_0)\} \tag{10a}$$

and given the matrices $A_1{}^0(t)$, $B_1{}^0(t)$, and $D_1{}^0(t)$, $t \in [t_0,T]$, the initial state $z_1{}^0(t_0)$ and the control law $u^0(\cdot,\cdot)$ [i.e., given the combined estimator-controller U^0 for Controller 1], there holds for all $\tau \in [t_0,T]$

$$E\{J_\tau[u^0,v] \| Z_2(t_0)\} \leqq E\{J_\tau[u^0,v^0] \| Z_2(t_0)\}. \tag{10b}$$

III. Solution of the Problem

Proposition 1: The following control laws $u^0(\cdot,\cdot)$ and $v^0(\cdot,\cdot)$, matrices $A_1{}^0(t)$, $A_2{}^0(t)$, $B_1{}^0(t)$, $B_2{}^0(t)$, $D_1{}^0(t)$, and $D_2{}^0(t)$, and initial conditions $z_1{}^0(t_0)$ and $z_2{}^0(t_0)$ solve Problem 1

$$u^0(t) = -Q_1{}^{-1}(t)G_1{}'(t)P(t)z_1(t) \tag{11a}$$

$$v^0(t) = -Q_2{}^{-1}(t)G_2{}'(t)P(t)z_2(t) \tag{11b}$$

$$A_1{}^0(t) = F(t) - G_2(t)Q_2{}^{-1}(t)G_2{}'(t)P(t) \cdot [I + (M_{20} - M_{21})(M_{00} - M_{01})^{-1}] \tag{12a}$$

$$B_1{}^0(t) = M_{11}(t)H_1{}'(t)W_1{}^{-1}(t) \tag{12b}$$

$$D_1{}^0(t) = G_1(t) \tag{12c}$$

$$A_2{}^0(t) = F(t) - G_1(t)Q_1{}^{-1}(t)G_1{}'(t)P(t) \cdot [I + (M_{10} - M_{12})(M_{00} - M_{02})^{-1}] \tag{13a}$$

$$B_2{}^0(t) = M_{22}(t)H_2{}'(t)W_2{}^{-1}(t) \tag{13b}$$

$$D_2{}^0(t) = G_2(t) \tag{13c}$$

$$z_1{}^0(t_0) = \bar{x}_0 = z_2{}^0(t_0). \tag{14}$$

The $n \times n$ symmetric nonnegative definite matrix $P(t)$ satisfies the Riccati equation

$$\dot{P} + PF(t) + F'(t)P - P[G_1(t)Q_1{}^{-1}(t)G_1{}'(t) + G_2(t)Q_2{}^{-1}(t)G_2{}'(t)]P = 0 \tag{15a}$$

with boundary condition

$$P(T) = C'C. \tag{15b}$$

The $3n \times 3n$ symmetric nonnegative definite matrix $M(t)$, partitioned into $n \times n$ matrices as follows

$$M(t) = \begin{bmatrix} M_{00}(t) & M_{01}(t) & M_{02}(t) \\ M_{10}(t) & M_{11}(t) & M_{12}(t) \\ M_{20}(t) & M_{21}(t) & M_{22}(t) \end{bmatrix} \tag{16}$$

satisfies the differential equation

$$\dot{M} = \bar{F}(t)M + M\bar{F}'(t) + \bar{G}(t)W(t)\bar{G}'(t) \tag{17a}$$

with boundary conditions

$$M_{ij}(t_0) = \begin{cases} x_0x_0{}' + M_0, & i = 0 = j \\ M_0, & \text{otherwise} \end{cases} \tag{17b}$$

where the $3n \times 3n$ matrix $\bar{F}(t)$, the $3n \times (m_1 + m_2)$ matrix $\bar{G}(t)$, and the $(m_1 + m_2) \times (m_1 + m_2)$ matrix $W(t)$ are defined in partitioned form and suppressing the common argument t as follows

$$\bar{F} = \begin{bmatrix} F - G_1Q_1{}^{-1}G_1{}'P - G_2Q_2{}^{-1}G_2{}'P & G_1Q_1{}^{-1}G_1{}'P & G_2Q_2{}^{-1}G_2{}'P \\ F - A_1{}^0 - G_2Q_2{}^{-1}G_2{}'P & A_1{}^0 - B_1{}^0H_1 & G_2Q_2{}^{-1}G_2{}'P \\ F - A_2{}^0 - G_1Q_1{}^{-1}G_1{}'P & G_1Q_1{}^{-1}G_1{}'P & A_2{}^0 - B_2{}^0H_2 \end{bmatrix} \tag{18a}$$

$$\bar{G} = \begin{bmatrix} 0 & 0 \\ B_1 & 0 \\ 0 & B_2 \end{bmatrix} \tag{18b}$$

$$W = \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix}. \tag{18c}$$

As a point of interpretation, we note that the solution to the corresponding deterministic differential game problem is

$$u^0(t) = -Q_1{}^{-1}(t)G_1{}'(t)P(t)x(t) \tag{19a}$$

$$v^0(t) = -Q_2{}^{-1}(t)G_2{}'(t)P(t)x(t) \tag{19b}$$

where $P(t)$ satisfies the Riccati equation (15a) with boundary condition (15b). (See [1], in which conditions for the existence of a solution to this Riccati equation are given.) The optimal controls that solve Problem 2 are thus given by a separation theorem or certainty equivalence principle in the sense that the now unknown state $x(t)$ in the solution (19) to the deterministic problem is replaced by the estimate $z_1(t)$ or $z_2(t)$ of it to give the solution (11) to the stochastic problem.

Before proceeding with the proof of Proposition 1, we derive a preliminary result that is of interest in its own right as well as being important to the subsequent proof: we show that given the optimal estimator-controller V^0 [i.e., given the optimal control law $v^0(\cdot,\cdot)$ and the optimal estimator defined by (8b), (13), and (14)], the estimator defined by (8a) with (12) and (14) in the optimal estimator-controller for Controller 1 is unbiased in the sense that for all t in $[t_0,T]$

$$E\{x(t) \| Z_1(t)\} = z_1(t).$$

A similar argument, which proceeds simultaneously, applies for the optimal estimator of Controller 2. As a byproduct of this analysis, an interpretation is given to the matrices $A_i^0(t)$, $B_i^0(t)$, $D^0_i(t)$, $M(t)$, $\bar{F}(t)$, and $\bar{G}(t)$, $t \in [t_0,T]$ and $i = 1, 2$.

We note that substitution of (11) into the system equation (1) gives on rearrangement of terms

$$\begin{aligned}\dot{x} &= (F - G_1Q_1^{-1}G_1'P - G_2Q_2^{-1}G_2'P)x \\ &\quad + G_1Q_1^{-1}G_1' P(x - z_1) + G_2Q_2^{-1}G_2'P(x - z_2). \qquad (20)\end{aligned}$$

Subtraction of (8a) from (20) yields after some algebra and the use of (12c)

$$\begin{aligned}(\dot{x} - \dot{z}_1) &= (F - A_1^0 - G_2Q_2^{-1}G_2'P)x \\ &\quad + (A_1^0 - B_1^0H_1)(x - z_1) \\ &\quad + G_2Q_2^{-1}G_2'P(x - z_2) - B_1^0w_1. \qquad (21)\end{aligned}$$

Similarly it follows from (8b), (20), and (13c) that

$$\begin{aligned}(\dot{x} - \dot{z}_2) &= (F - A_2^0 - G_1Q_1^{-1}G_1'P)x \\ &\quad + G_1Q_1^{-1}G_1'P(x - z_1) \\ &\quad + (A_2^0 - B_2^0H_2)(x - z_2) - B_2^0w_2. \qquad (22)\end{aligned}$$

Forming the composite vectors $m(t)$ and $w(t)$ defined by

$$m'(t) = [x'(t), x'(t) - z_1'(t), x'(t) - z_2'(t)] \qquad (23)$$

$$w'(t) = [w_1'(t), w_2'(t)] \qquad (24)$$

we have from (20)–(22) with (18)

$$\dot{m} = \bar{F}(t)m - \bar{G}(t)w(t)$$

from which follows (17a) with boundary condition (17b) when $M(t)$ is defined by

$$M(t) = E\{m(t)m'(t)\}. \qquad (25)$$

If we now consider (17a) and, using (18) write out in detail the components $\dot{M}_{01}(t)$ and $\dot{M}_{11}(t)$, we find, after some algebra and the use of (12a) and (12b), that for all $t \in [t_0,T]$,

$$\begin{aligned}\dot{M}_{11}(t) - \dot{M}_{01}(t) &= \Delta(t)[M_{11}(t) - M_{01}(t)] \\ &\quad + [M_{11}(t) - M_{01}(t)]\Lambda(t)\end{aligned}$$

where $\Delta(t)$ and $\Lambda(t)$ are known matrices. Since, by choice of initial condition (17b), $M_{11}(t_0) = M_{01}(t_0)$, we conclude that $M_{11}(t) = M_{01}(t)$ for all $t \in [t_0,T]$. Moreover, since M is a symmetric matrix, we have

$$M_{10}(t) = M_{01}'(t) = M_{11}'(t) = M_{11}(t). \qquad (26)$$

In view of the definition (25) of M we thus have for all $t \in [t_0,T]$

$$\begin{aligned}E\{x(t)[x'(t) - z_1'(t)]\} &= E\{[x(t) - z_1(t)][x'(t) - z_1'(t)]\} \\ &= E\{[x(t) - z_1(t)]x'(t)\}\end{aligned}$$

which implies

$$E\{[x(t) - z_1(t)]z_1'(t)\} = 0.$$

Thus each component of the error $x(t) - z_1(t)$ is orthogonal to each component of $z_1(t)$, and $z_1(t)$ may be regarded as in some sense an estimate of $x(t)$. The sense in which $z_1(t)$ is an estimate of $x(t)$ may be clarified by asking the question: given the matrix $M(t)$ [obtained by integrating forward (17a) with boundary condition (17b)] and given $Z_1(t)$, what is the best estimate of $x(t)$ in the least mean-square error sense? This estimate is also the expected value of $x(t)$ given $Z_1(t)$ and, since all variables are Gaussian, is a linear function of $Z_1(t)$, i.e., we seek the matrix $L(t)$ such that

$$E\{x(t)|Z_1(t)\} = L(t)z_1(t).$$

By standard estimation theory [5], we require that

$$E\{[x(t) - L(t)z_1(t)]z_1'(t)\} = 0$$

which gives, after some algebraic manipulation,

$$L(t)[M_{00}(t) - M_{01}(t)] = [M_{00}(t) - M_{01}(t)].$$

Unless some components of $x(t)$ can be estimated exactly, $[M_{00}(t) - M_{01}(t)]$ is positive definite for all $t \in (t_0,T]$, and thus the unique solution is $L(t) = I$ (identity) for $t \in (t_0,T]$. On the other hand, by choice of the initial condition (14) the same is true at $t = t_0$, so we have, for all $t \in [t_0,T]$, $E\{x(t)|Z_1(t)\} = z_1(t)$, which is (9a). Similarly it can be shown that, for all $t \in [t_0,T]$, $M_{02}(t) = M_{22}(t) = M_{20}(t)$, which implies that $E\{x(t)|Z_2(t)\} = z_2(t)$, which is (9b).

In addition to the above, we ask the question: given the matrix $M(t)$ and given $Z_1(t)$, what is the best estimate of $z_2(t)$? By a procedure analogous to the preceding one, we find after some algebraic manipulation that for all $t \in (t_0,T]$

$$\begin{aligned}E\{z_2(t)|Z_1(t)\} &= [I + (M_{20}(t) - M_{21}(t)) \\ &\quad \cdot (M_{00}(t) - M_{01}(t))^{-1}]z_1(t) \qquad (27)\end{aligned}$$

while, by choice of boundary condition,

$$E\{z_2(t_0)|Z_1(t_0)\} = z_1(t_0). \qquad (28)$$

As a point of interpretation it is noted that, using (11)–(14) and (27), the differential equation (8a) for $z_1(t)$ may be written

$$\begin{aligned}\frac{d}{dt} E\{x(t)|Z_1(t)\} &= F(t)E\{x(t)|Z_1(t)\} + G_1(t)u(t) \\ &\quad + G_2(t)E\{v^0(t)|Z_1(t)\} \\ &\quad + M_{11}(t)H_1'(t)W_1^{-1}(t)[y_1(t) \\ &\quad - H_1(t)E\{x(t)|Z_1(t)\}]\end{aligned}$$

with boundary condition

$$E\{x(t_0)|Z_1(t_0)\} = \bar{x}_0.$$

In view of the system equation (1) and the results of standard filtering theory [6] this is an intuitively reasonable result. Entirely symmetric results apply to $z_2(t)$ and the situation from the viewpoint of Controller 2.

We are now in a position to prove Proposition 1, i.e., to show that the estimator-controllers U^0 and V^0 are indeed optimal in the sense that (9) is satisfied. To do this, use is made of the following sufficiency lemma which is a specialization to the present situation of a more general theorem presented in our earlier paper, [1, Theorem 1]. The proof of this lemma is not given but follows immediately by specializing the proof of the more general result.

Lemma 1

A sufficient condition for two closed-loop control laws $u^0(\cdot,\cdot)$ and $v^0(\cdot,\cdot)$ to satisfy (10*a*) and (10*b*) is that there exists a real-valued function $I(\cdot,\cdot)\colon X \times [t_0,T] \rightarrow R$, differentiable in each variable, which together with $u^0(\cdot,\cdot)$ and $v^0(\cdot,\cdot)$ satisfies for all $t \in [t_0,T]$ the following five conditions. Defining for all $t \in [t_0,T]$ the scalar function $S(\cdot,\cdot,\cdot,\cdot)$ by

$$S(x,u,v,t) = I_t(x,t) + I_x(x,t)[F(t)x + G_1(t)u + G_2(t)v] + u'Q_1(t)u + v'Q_2(t)v \quad (29)$$

(where I_t and I_x denote the partial derivatives of I with respect to t and x) the five conditions are

1) $\min\limits_u E\{S(x(t),u,v^0(Z_2(t),t),t)|Z_1(t)\} = 0$
2) $E\{S(x(t),u^0(Z_1(t),t),v^0(Z_2(t),t),t)|Z_1(t)\} = 0$
3) $\max\limits_v E\{S(x(t),u^0(Z_1(t),t),v,t)|Z_2(t)\} = 0$
4) $E\{S(x(t),u^0(Z_1(t),t),v^0(Z_2(t),t),t)|Z_2(t)\} = 0$
5) $I(x,T) = x'C'Cx$.

Proposition 2: The real valued function $I(\cdot,\cdot)$ defined by

$$I(x,t) = x'P(t)x + b(t) \quad (30)$$

where the $n \times n$ matrix $P(t)$ satisfies (15a) with boundary condition (15b) and

$$b(t) = \operatorname{tr}\left\{\int_t^T P(\tau)G_1(\tau)Q_1^{-1}(\tau)G_1'(\tau)P(\tau)M_{11}(\tau) + P(\tau)G_2(\tau)Q_2^{-1}(\tau)G_2'(\tau)P(\tau)M_{22}(\tau)\,d\tau\right\} \quad (31)$$

satisfies, along with the control laws $u^0(\cdot,\cdot)$ and $v^0(\cdot,\cdot)$ defined by (11) and the optimal estimators defined by (8) and (12)–(14), the five conditions of the Lemma, so that $u^0(\cdot,\cdot)$ and $v^0(\cdot,\cdot)$ satisfy (10).

Proof of Proposition 2: We have in this case

$$S(x,u,v^0,t) = x'\dot{P}x + \dot{b} + u'Q_1u + z_2'PG_2Q_2^{-1}G_2'Pz_2 + 2x'P(Fx + G_1u - G_2Q_2^{-1}G_2'Pz_2) \quad (32)$$

from which it follows that

$$\arg\min_u E\{S(x,u,v^0,t)|Z_1(t)\} = -Q_1^{-1}G_1'PE\{x(t)|Z_1(t)\}. \quad (33)$$

It has been shown above, however, that, under the presently assumed condition that Controller 2 uses his optimum combined estimator-controller, the estimator in the optimum estimator-controller of Controller 1 is unbiased. In other words, use of the optimum estimator defined by (8a), (12), and (14) and the control (33) gives $E\{x(t)|Z_1(t)\} = z_1(t)$ and (33) becomes

$$\arg\min_u E\{S(x,u,v^0,t)|Z_1(t)\} = -Q_1^{-1}G_1'Pz_1(t).$$

which is the optimal control $u^0(z_1(t),t)$. Substitution into (32) then gives after some algebra

$$\begin{aligned}E\{S(x,u^0,v^0,t)|Z_1(t)\} &= E\{x'(\dot{P} + PF + F'P - PG_1Q_1^{-1}G_1'P \\ &\quad - PG_2Q_2^{-1}G_2'P)x + \dot{b} + E[(x \\ &\quad - z_1)'PG_1Q_1^{-1}G_1'P\,(x - z_1) \\ &\quad + (x - z_2)'PG_2Q_2^{-1}G_2'P(x - z_2) \\ &\quad \cdot|Z_1(t)]t)\}|Z_1.\end{aligned}$$

Finally, in view of (15a) and (31) and the choice of a boundary condition (15b) we have

$$E\{S(x,u^0,v^0,t)|Z_1(t)\} = 0$$

which demonstrates the satisfaction of conditions 1), 2), 5) of the Lemma. A symmetrical argument shows the satisfaction of conditions 3), 4), and 5) and so completes the proof.

It has been noted earlier that, in view of (19), the optimum controls for Problem 2 given by (11) satisfy a separation theorem in the sense that the (unknown) state $x(t)$ in the solution to the corresponding deterministic problem is replaced by the best estimate $z_1(t)$ or $z_2(t)$ of it given the available information within the imposed constraints. Furthermore, we note that the function $I(\cdot,\cdot)$ has the property that (see [1])

$$E\{I(x(t),t)|Z_i(t)\} = E\{J_t[u^0,v^0]|Z_i(t)\}, \quad i = 1,2$$

so that from the viewpoint of Controller i, $i = 1, 2$, we have that in this case the expected optimal cost from time t is

$$\begin{aligned}E\{J_t[u^0,v^0]|Z_i(t)\} &= E\{x'(t)P(t)x(t) + b(t)|Z_i(t)\} \\ &= E\{x'(t)P(t)x(t)|Z_i(t)\} + b(t).\end{aligned}$$

Since the optimum cost from time t in the corresponding deterministic problem is $x'(t)P(t)x(t)$ (see [1]), it is seen that from the viewpoint of each controller the expected optimum cost in the nondeterministic case is the sum of two terms: 1) the expected deterministic cost from time t and 2) the term $b(t)$ which, from (31), depends on the future estimation error covariances $M_{11}(\tau)$ and $M_{22}(\tau)$ which in turn are due to the error covariances at time t and the future measurement noise covariances (see (17a)). The first term may itself be written as the sum of two terms

$$\begin{aligned}E\{x'(t)P(t)x(t)|Z_i(t)\} \\ = z_i'(t)P(t)z_i(t) + \operatorname{tr}\{M_{ii}(t)P(t)\}, \quad i = 1,2\end{aligned}$$

i.e., the sum of the deterministic cost with $z_i(t)$ replacing $x(t)$ and a term depending on the estimation error covariance at time t.

To this point, nothing has been said about uniqueness of the solutions to Problem 2. Uniqueness, however, is irrelevant, for suppose that U^* and V^* are also optimal combined estimator-controllers in the sense defined by (9) or, more precisely, by Problem 2. In other words, suppose that for all estimator controllers U and V and all τ in $[t_0,T]$ we have

$$E\{J_\tau[U^*,V]|Z(t_0)\} \leqq E\{J_\tau[U^*,V^*]|Z(t_0)\} \leqq E\{J_\tau[U,V^*]|Z(t_0)\}. \quad (34)$$

In particular, (34) must hold for the estimator-controllers U^0 and V^0 defined in Proposition 1, so that

$$E\{J_\tau[U^*,V^0]|Z(t_0)\} \leqq E\{J_\tau[U^*,V^*]|Z(t_0)\} \leqq E\{J_\tau[U^0,V^*]|Z(t_0)\}. \quad (35)$$

On the other hand, U^0 and V^0 satisfy (9) for all U and V and, in particular, for U^* and V^*, so that we have

$$E\{J_\tau[U^0,V^*]|Z(t_0)\} \leqq E\{J_\tau[U^0,V^0]|Z(t_0)\} \leqq E\{J_\tau[U^*,V^0]|Z(t_0)\}. \quad (36)$$

Noting that the term on the right of (35) coincides with the term on the left of (36), we may combine these two inequalities into the single expression

$$\begin{aligned} E\{J_\tau[U^*,V^0]|Z(t_0)\} &\leqq E\{J_\tau[U^*,V^*]|Z(t_0)\} \\ &\leqq E\{J_\tau[U^0,V^*]|Z(t_0)\} \\ &\leqq E\{J_\tau[U^0,V^0]|Z(t_0)\} \\ &\leqq E\{J_\tau[U^*,V^0]|Z(t_0)\}. \quad (37) \end{aligned}$$

Since the terms on the left and right sides of (37) are identical, equality must hold throughout, and the cost of using any of the four possible combinations (U^0,V^0), (U^0,V^*), (U^*,V^0), and (U^*,V^*) of estimator-controllers is the same.

IV. CONCLUSION

Consideration has been given to a stochastic differential game involving an n-dimensional linear system, a quadratic cost functional, and independent white Gaussian noises additively corrupting the output measurements. A solution to this problem has been presented under the constraint that each controller is limited to a linear dynamic system of dimension n for the generation of his estimate of the system state; this estimate is all that is available to him in the generation of his control vector. Without this additional restriction it appears that any solution to the more general problem will require infinite-dimensional state estimators or, equivalently, the preservation and use in the control at time t of the entire output function measured up to that time. On the other hand, a natural extension of this work is to allow one or both of the controllers a dynamic estimator of dimension larger than n. One such possibility is to allow one controller a linear dynamic estimator of dimension n and the other a linear dynamic estimator of dimension $2n$, with which he could estimate both the state vector and the error in his opponent's state estimate. Alternatively, both controllers could be allowed estimators of dimension $2n$. Continuation of this process leads naturally to the question of whether or not, under suitable conditions, the resulting optimal controls converge toward a limit and, if so, how rapidly. Examination of this problem remains an open area for future investigation.

The optimum control for each controller has been shown to be closely related to that which would result by assuming a separation theorem to hold, and the optimum estimator for each has been shown to be closely related to a Kalman filter. Furthermore, the various terms in the optimal cost are shown to be readily assignable to the appropriate contributing source, including the optimal cost which would result if the problem were instead a deterministic one, the effect of estimation errors, and the effect of measurement noise.

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Ian B. Rhodes (S'64–M'67), for a photograph and biography please see page 38 of the February, 1969, issue of this TRANSACTIONS.

David G. Luenberger (S'57–M'64), for a photograph and biography please see page 38 of the February, 1969, issue of this TRANSACTIONS.