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QUASI-CONVEX PROGRAMMING*

DAVID G. LUENBERGER†

1. Introduction. In differential form, the Kuhn-Tucker theorem and the duality theorem for convex programming problems have been extended to problems satisfying various weakened convexity criteria [1]–[3]. There have not, however, appeared any global programming results analogous to and yet extending the classical results for convex programming [4]–[6]. In this paper a global theory for minimization of quasi-convex functions is presented that parallels the results for convex programming.

The main result is that, although quasi-convex programs cannot be reduced to saddle-value problems of a Lagrangian function, a Lagrange multiplier does exist in a modified sense. This leads to a complete global theory requiring no differentiability hypotheses.

2. Quasi-convex functions. Let Ω be a convex subset of R^n and let f be a real-valued function on Ω. The function f is said to be *quasi-convex* if, for every real c,

$$\{x: x \in \Omega, f(x) < c\} \tag{1}$$

s convex. Any convex function is quasi-convex. An alternative definition s that f is quasi-convex if, for every c,

$$\{x: x \in \Omega, f(x) \leqq c\} \tag{2}$$

is convex. Condition (1) implies (2) since

$$\{x: x \in \Omega, f(x) \leqq c\} = \bigcap_{d>c} \{x: x \in \Omega, f(x) < d\}.$$

Conversely (2) implies (1) since if (2) holds $f(x_1) \leqq f(x_2) < c$ implies $f(\alpha x_1 + \alpha^* x_2) \leqq f(x_2) < c$ for $\alpha \geqq 0, \alpha^* \geqq 0, \alpha + \alpha^* = 1$. Quasi-concave functions are, of course, defined by reversing the inequalities in (1) and (2).

It is natural to consider global minimization of quasi-convex functions on a convex set since although a relative minimum is not necessarily a global minimum for these functions the following results do apply.

LEMMA 1. *Let $x_0 \in \Omega$ be a strong relative minimum of the quasi-convex function f (i.e., there is a neighborhood S of x_0 such that $f(x) > f(x_0)$ for all $x \in \Omega \cap S, x \neq x_0$). Then x_0 is the global minimum of f over Ω.*

Proof. Suppose there were $x_1 \in \Omega$ with $f(x_1) \leqq f(x_0)$. Then $f(\alpha x_0 + \alpha^* x_1)$

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† Department of Engineering-Economic Systems, Stanford, California 94305. This research was supported by the National Science Foundation under Grant NSF GK 1683.

$\leqq f(x_0)$ for $0 < \alpha$, $0 < \alpha^*$, $\alpha + \alpha^* = 1$; but for small α^*, $\alpha x_0 + \alpha^* x_1 \in S \cap \Omega$, which is a contradiction.

LEMMA 2. *The set of points in Ω at which f achieves its minimum is convex.*

3. The multiplier theorem. Throughout the remainder of this paper we consider problems of the form

$$\text{(3)} \qquad \text{minimize} \quad f(x) \quad \text{subject to} \quad G(x) \leqq 0, \qquad x \in \Omega,$$

where Ω is a convex subset of R^n, f is quasi-convex on Ω and G is a convex[1] mapping from Ω into R^m. In addition we assume satisfaction of the following two regularity conditions:

(i) The function f is upper semicontinuous along lines;[2] i.e., for every $x_1, x_2 \in \Omega$, $f(\alpha x_1 + (1 - \alpha)x_2)$ is an upper semicontinuous function of α for $\alpha \in [0, 1]$.

(ii) There is an $x_1 \in \Omega$ such that $G(x_1) < 0$.

THEOREM 1. *There is a $\lambda_0 \in R^m$, $\lambda_0 \geqq 0$, and $\lambda_0 \neq 0$, such that if x_0 is a point at which $f(x)$ achieves a minimum subject to the constraints $x \in \Omega$, $G(x) \leqq 0$, that is, x_0 solves problem (3), then x_0 solves the following problem:*

$$\text{(4)} \qquad \textit{minimize} \quad f(x) \quad \textit{subject to} \quad \lambda_0' G(x) \leqq 0.$$

Proof. Let x_0 solve (3) and let $\mu_0 = f(x_0)$. Denote the closed negative orthant of R^m by N and define the sets Γ, $A \subset R^m$ by

$$\text{(5)} \qquad \begin{aligned} \Gamma &= \{z : \exists x \in \Omega \quad \text{such that} \quad G(x) \leqq z\}, \\ A &= \{z : \exists x \in \Omega \quad \text{such that} \quad G(x) \leqq z, f(x) < \mu_0\}. \end{aligned}$$

It follows from the nature of f and G that the sets Γ and A are convex. The set Γ is nonempty, and the set A can be assumed to be nonempty, for otherwise the theorem is true for any positive λ_0. Furthermore, by definition of μ_0 it follows that $A \cap N$ is empty. Thus, by the separating hyperplane theorem, there is a $\lambda_0 \in R^m$ such that $\lambda_0' z \geqq 0$ for all $z \in A$, $\lambda_0' z \leqq 0$ for all $z \in N$. It follows from the last inequality that $\lambda_0 \geqq 0$.

Suppose there is a $z_2 \in \Gamma$ satisfying $\lambda_0' z_2 = 0$. Let $x_2 \in \Omega$ be any vector such that $G(x_2) \leqq z_2$. Let x_1 be selected according to assumption (ii) so that $G(x_1) = z_1 < 0$. Since $z_1, z_2 \in \Gamma$ and Γ is convex, $z_\alpha = \alpha z_1 + (1 - \alpha)z_2 \in \Gamma$ for $\alpha \in [0, 1]$. Also by choice of z_1 and z_2, $z_\alpha \notin A$ for $\alpha \in (0, 1]$. Since G is convex, $G(\alpha x_1 + (1 - \alpha)x_2) \leqq z_\alpha$, and it follows that $f(\alpha x_1 + (1 - \alpha)x_2) \geqq \mu_0$ for $\alpha \in (0, 1]$. Then, by the upper semicontinuity of f along

[1] One might consider problems with G quasi-convex rather than convex. Since, however, G is used only to specify the constraint set, quasi-convex constraints are usually replaceable by equivalent convex constraints.

[2] Any convex function, for example, is upper semicontinuous along lines.

lines, $f(x_2) \geqq \mu_0$. Since $x_2 \in \Omega$ was arbitrary with $G(x_2) \leqq z_2$, it follows that $z_2 \notin A$. Thus, since $\lambda_0' z \geqq 0$ for $z \in A$ by definition of λ_0, and since we have shown that $\lambda_0' z = 0$ implies $z \notin A$, we conclude that $\lambda_0' z \leqq 0$ implies $z \notin A$. Therefore, $x \in \Omega$, $\lambda_0' G(x) \leqq 0$ imply $f(x) \geqq \mu_0$. Since x_0 is feasible to (4), it follows immediately that x_0 solves (4).

The results of Theorem 1 can be rephrased as: x_0 solves (3) if and only if there exists a vector λ_0 such that (x_0, λ_0) is a saddlepoint of $K(x, \lambda)$ on $\Omega \times S$, where S is the simplex in R^m, $S = \{\lambda: \lambda \geqq 0, \sum_{i=1}^m \lambda_i = 1\}$, and

$$K(x, \lambda) = \begin{cases} f(x) & \text{if} \quad \lambda' G(x) \leqq 0, \\ +\infty & \text{if} \quad \lambda' G(x) > 0. \end{cases} \tag{6}$$

Here K is quasi-convex in x and quasi-concave in λ but does not satisfy the semicontinuity requirements of the familiar minimax theorem of Sion [7].

The classical analogue to Theorem 1 is the Lagrangian theorem for convex programs, which essentially reduces a constrained minimization problem to an unconstrained problem. In the present case we are able to reduce a problem with m convex constraints to a problem having a single convex constraint. Just as in the classical theorem, the correspondence is in one direction only since if x_0 solves (4) it does not necessarily solve (3). Some additional requirement, such as the saddle condition, must be imposed to guarantee $G(x_0) \leqq 0$.

In the classical version the m components of the Lagrange multiplier must be determined, while in the present case the vector λ_0 need be determined only to within a positive multiple; hence, the trade-off of constraints for unknowns is almost identical for the two theories.

In general we do not have $\lambda_0' G(x_0) = 0$ for the quasi-convex programming problem as we do for convex programming. For example, if

$$f(x_1, x_2) = \begin{cases} 0, & x_1 + x_2 > 0, \\ 1, & x_1 + x_2 \leqq 0, \end{cases}$$

the problem of minimizing $f(x_1, x_2)$ subject to $x_1 \leqq 0$, $x_2 \leqq 0$ is equivalent to minimizing $f(x_1, x_2)$ subject to $x_1 + x_2 \leqq 0$, so in this case $\lambda_0' = (1, 1)$. However, $x_0' = (-1, -1)$ solves the original problem and $\lambda_0' G(x_0) = -2$. This phenomenon is essentially a consequence of the fact that relative minima need not be global minima.

THEOREM 2. *Let x_0 solve problem (3) and define $\mu_0 = f(x_0)$. Then either there is an $\bar{x}_0 \in \Omega$, $G(\bar{x}_0) < 0$ and $f(\bar{x}_0) = \mu_0$, or there is a $\lambda_0 \geqq 0$ such that x_0 solves problem (4) and $\lambda_0' G(x_0) = 0$.*

Proof. Define the convex set $B \subset R^m$ as

$$B = \{z: \exists x \in \Omega \text{ such that } G(x) \leqq z, f(x) \leqq \mu_0\}.$$

If there is no $\bar{x}_0$ satisfying the first hypothesis of the theorem, then B contains no interior points of the negative orthant N. Thus, by the separating hyperplane theorem, there is a $\lambda_0 \geqq 0$ such that $\lambda_0'z \geqq 0$ for all $z \in B$. The point $z_0 = G(x_0)$ is in both B and N, and hence $\lambda_0'G(x_0) = 0$.

4. Duality. As before let $\Gamma \subset R^m$ be defined as the convex set

$$\Gamma = \{z: \exists x \in \Omega \text{ such that } G(x) \leqq z\}. \tag{7}$$

On the set Γ we define the primal function

$$\omega(z) = \inf \{f(x): G(x) \leqq z, x \in \Omega\}. \tag{8}$$

LEMMA 3. *The primal function ω is quasi-convex on Γ and upper semicontinuous along lines.*

Proof. Given a real constant c, let $z_1, z_2 \in \Gamma$ satisfy $\omega(z_1) < c$, $\omega(z_2) < c$. Then there are $x_1, x_2 \in \Omega$ with $G(x_1) \leqq z_1$, $G(x_2) \leqq z_2$, $f(x_1) < c$ and $f(x_2) < c$. For any $\alpha > 0$, $\alpha^* > 0$, $\alpha + \alpha^* = 1$ we have

$$\omega(\alpha z_1 + \alpha^* z_2) = \inf \{f(x): x \in \Omega, G(x) \leqq \alpha z_1 + \alpha^* z_2\} \leqq f(\alpha x_1 + \alpha^* x_2) < c.$$

Hence ω is quasi-convex.

To prove the upper semicontinuity, it is sufficient to show that ω is upper semicontinuous at the right end of any line. Given $z_1, z_2 \in \Gamma$ and $\epsilon > 0$, select $x_1, x_2 \in \Omega$ such that $G(x_1) \leqq z_1$, $G(x_2) \leqq z_2$, $f(x_2) < \omega(z_2) + \epsilon$. Then

$$\lim_{\alpha \to 0} \omega(\alpha z_1 + \alpha^* z_2) \leqq \lim_{\alpha \to 0} f(\alpha x_1 + \alpha^* x_2) \leqq f(x_2) < \omega(z_2) + \epsilon.$$

We introduce the dual function on the positive orthant in R^m:

$$\mu(\lambda) = \inf \{f(x): x \in \Omega, \lambda' G(x) \leqq 0\}. \tag{9}$$

In general $\mu(\lambda)$ may not be finite. Let D be the subset of the positive orthant on which μ is finite.

LEMMA 4. *The dual function can be expressed as*

$$\mu(\lambda) = \inf \{\omega(z): \lambda' z \leqq 0, z \in \Gamma\}. \tag{10}$$

Proof. This follows from the identity

$$\inf \{f(x): x \in \Omega, \lambda' G(x) \leqq 0\} = \inf_{\substack{\lambda' z < 0 \\ z \in \Gamma}} \inf \{f(x): x \in \Omega, G(x) \leqq z\},$$

which holds for $\lambda \geqq 0$.

LEMMA 5. *The dual function μ is quasi-concave and upper semicontinuous throughout D.*

Proof. Given a real constant c define the (convex) set

$$K = \{z: \omega(z) < c, z \in \Gamma\}.$$

Let

$$K^+ = \{\lambda : \lambda' z \geqq 0 \text{ for all } z \in K\}.$$

K^+ is clearly a closed convex cone and we have $K^+ \supset \{\lambda : \mu(\lambda) \geqq c\}$. An argument similar to that contained in the proof of Theorem 1 shows that $\lambda \in K^+$, $\lambda' z = 0$ imply $z \notin K$. Hence $K^+ = \{\lambda : \mu(\lambda) \geqq c\}$. Since K^+ is convex and closed, μ is quasi-concave and upper semicontinuous.

We also have the following duality theorem.

THEOREM 3. *Assume that* $\mu_0 = \inf\{f(x) : x \in \Omega, G(x) \leqq 0\}$ *is finite. Then,* $\mu_0 = \max_{\lambda \geqq 0} \mu(\lambda)$, *where the maximum is achieved for some* $\lambda_0 \geqq 0$.

Proof. From (9) it follows immediately that $\mu(\lambda) \leqq \mu_0$ for all $\lambda \geqq 0$. To conclude the proof it is only necessary to exhibit a $\lambda_0 \geqq 0$ for which equality holds. This follows from the construction of a hyperplane separating N and $A = \{z : \exists x \in \Omega \text{ such that } G(x) \leqq z, f(x) < \mu_0\}$.

We note that the problem of determining $\mu_0 = \inf\{f(x) : G(x) \leqq 0, x \in \Omega\}$ is by definition of the primal function ω equivalent to the minimization problem

$$\mu_0 = \min\{\omega(z) : z \leqq 0, z \in \Gamma\}, \tag{11}$$

which has a minimum achieved at $z = 0$. Problem (11) provides a symmetric companion to the problem

$$\mu_0 = \max\{\mu(\lambda) : \lambda \geqq 0, \lambda \in D\}, \tag{12}$$

which achieves a maximum at λ_0. The manner in which we have defined μ in terms of homogeneous constraints does not actually lead to perfect duality since the dual of μ is not ω.

5. Relation to convex programming. The standard sensitivity interpretation of Lagrange multipliers carries over in a weakened form. It follows directly from (10) that $\lambda_0' z \leqq 0$ implies that $\omega(z) \geqq \omega(0) = \mu_0$. Therefore, if ω is continuously differentiable, λ_0 points in the direction of the negative gradient of $\omega(z)$ at $z = 0$. The magnitude of λ_0 is not necessarily related to the magnitude of the gradient, however, since in fact λ_0 is determined only to within a positive scale factor.

If f were convex rather than merely quasi-convex, the problem

$$\text{minimize} \quad f(x) \quad \text{subject to} \quad G(x) \leqq 0, \qquad x \in \Omega, \tag{3}$$

could be reduced to

$$\text{minimize} \quad f(x) \quad \text{subject to} \quad \lambda_0' G(x) \leqq 0, \qquad x \in \Omega, \tag{4}$$

by Theorem 1. This would then be a convex programming problem having a single constraint and could be converted to

$$\text{minimize} \quad f(x) + \beta\lambda_0' G(x) \quad \text{over} \quad x \in \Omega, \tag{13}$$

where $\beta \geqq 0$ is the Lagrange multiplier for the convex problem (4). In this case the undetermined scale factor in λ_0 is thus determined.

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