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Products of random mappings†

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Abstract. Suppose that $X^1, X^2, \ldots$ is a stationary stochastic process of positive $k \times k$ matrices, and let ${}^nY^1 = X^n X^{n-1} \ldots X^1$ be the corresponding product matrices. For a special case, Bellman showed that the elements $[{}^nY^1]_{ij}$ converge in the sense that $n^{-1}\mathrm{E}\{\log[{}^nY^1]_{ij}\} \to a$ as $n \to \infty$. The constant a is independent of i and j. Bellman also conjectured that, asymptotically, the $n^{-1/2}\{\log[{}^nY^1]_{ij} - na\}$ terms are distributed according to a normal distribution with a common variance, independent of ij. Later Furstenberg and Kesten generalized and strengthened Bellman's result and established the validity of his conjecture.

This paper extends these results to the case of nonlinear mappings that are monotonic and homogeneous of degree one on R^k_+. Specifically, given a stationary process $H^1, H^2, \ldots$ of such mappings, we define the composite mappings ${}^nF^1(\cdot) = H^n(H^{n-1}(\ldots(H^1(\cdot))\ldots))$. Under appropriate conditions, the components $[{}^nF^1(x^0)]_i$ have the property that, almost surely, $n^{-1}\log[{}^nF^1(x^0)]_i \to a$ independent of x^0 and i. Furthermore the components $n^{-1/2}\{\log[{}^nF^1(x^0)]_i - na\}$ are asymptotically distributed according to a normal distribution with a common variance.

1. *Introduction*

Let $X^1, X^2, \ldots, X^m, \ldots$ be a stationary stochastic process of $k \times k$ matrices. Bellman [**1**] and Furstenberg and Kesten [**2**] initiated an investigation of the limiting behaviour of the corresponding products

$${}^nY^1 = X^n X^{n-1} \ldots X^1.$$

In [**2**], with the norm of a $k \times k$ matrix A having real or complex entries defined by $\|A\| = \max_i |A_{ij}|$, it is shown that the sequence of terms

$$n^{-1}\{\log \|{}^nY^1\|\} \tag{1}$$

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converges in mean and in probability under various conditions. Also, in the case where the entries of each X^i are positive, it is shown that under fairly weak assumptions the individual entries of the product converge together in the sense that

$$\lim_{n\to\infty} n^{-1} \log({}^nY^1)_{ij} = a \tag{2}$$

almost surely, where a is a (finite) positive constant independent of ij. (This extends [**1**] where convergence in mean is shown.) Furthermore, as conjectured in [**1**], the ijth terms

$$n^{-1/2}\{\log({}^nY^1)_{ij} - na\} \tag{3}$$

are asymptotically distributed according to a normal distribution with a common variance.

This paper extends these results from the case of matrices to a stochastic process $H^1, H^2, \ldots$ of general (nonlinear) homogeneous-of-degree-one mappings from R^k to R^k, and the corresponding product mappings ${}^nF^1(\cdot) = H^n(H^{n-1}(\ldots H^1(\cdot))\ldots)$. We show that the norms of the product mappings obey the same limit properties as established earlier for the norms of products of random matrices.

Additional results are obtained in the case where the mappings H^i are monotone on R^k_+. As an analogue to the results concerning the asymptotic behaviour of the entries of the products of positive matrices, we investigate the behaviour of the terms $[{}^nF^1(x^0)]_i$, the ith component of ${}^nF^1$ applied to x^0. We show that under suitable assumptions there are constants a and b such that

$$(nb)^{-1/2}\{\log[{}^nF^1(x^0)]_i - na\}$$

is asymptotically normally distributed, according to the standard unit normal distribution. The constants a and b are independent of i and x^0.

For the case of positive matrices, the result (2) can be viewed as a generalization of both the strong law of large numbers and the Frobenius–Perron theorem. Indeed, in the case where the X^i matrices are all equal to a positive matrix X, a defined by (2) is exactly the logarithm of the Frobenius–Perron eigenvalue of X. In a similar way, the results of this paper can be regarded as generalizations of the classic probability limit theorems and nonlinear versions of the Frobenius–Perron theorem. The results of this paper therefore have application to optimal investment with commissions, nonlinear economic growth theory, nonlinear population dynamics, and other areas described by nonlinear dynamics and uncertainty.

2. *Basic ergodic results*

We consider a stationary random process $H^1, H^2, H^3, \ldots$ of mappings from R^k to R^k. Compositions of the form $H^2\{H^1(\cdot)\}$ are written simply in product form H^2H^1. Also, we often write ${}^nF^m = H^nH^{n-1}\ldots H^m$ for $n \geq m$.

Any H that is an element of the process is assumed to satisfy various assumptions. The most elementary and important of these is the following.

ASSUMPTION A. (Homogeneity) *For any $x \in R^k$ and any $\alpha \geq 0$, $H(\alpha x) = \alpha H(x)$ (that is, H is (positively) homogeneous of degree 1).*

For $x \in R^k$ we define $\|x\| = \sum_i |x_i|$. Also for H satisfying assumption A, we define the *norm*

$$\|H\| = \sup_x \left\{ \sum_i |H(x)_i| : \|x\| = 1, x \in R^k \right\}.$$

This definition satisfies:

(1) $\|H(x)\| \leq \|H\| \cdot \|x\|$;

(2) $\|H_1 + H_2\| \leq \|H_1\| + \|H_2\|$;

(3) $\|H_1 H_2\| \leq \|H_1\| \cdot \|H_2\|$.

Although continuity of H would imply that H has a finite norm, we instead introduce the following assumption explicitly.

ASSUMPTION B. *H has finite norm.*

With just these simple preliminaries, it is possible to obtain a strong result concerning the asymptotic behaviour of the stochastic process. (Essentially we extend Kingman's approach for products of random matrices [3] to the more general setting.)

THEOREM 1. *Let $H^1, H^2, \ldots$ be a stationary stochastic process with elements satisfying Assumptions A and B. Suppose that*

$$\mathrm{E}\{(\log \|H^1\|)^+\} < \infty.$$

Then

$$\xi = \lim_{n \to \infty} n^{-1} \log \|H^n H^{n-1} \ldots H^1\|$$

exists in $-\infty \leq \xi < \infty$ with probability one, and

$$\mathrm{E}(\xi) = \lim_{n \to \infty} n^{-1} \mathrm{E}\{\log \|H^n H^{n-1} \ldots H^1\|\}.$$

Proof. For s, t positive integers with $s < t$, let

$$d_{st} = \log \|H^t H^{t-1} \ldots H^{s+1}\|.$$

Then:

(S_1) for $s < t < u$, there holds $d_{su} \leq d_{st} + d_{tu}$;

(S_2) the joint distributions of the process $\{d_{s+1,t+1}\}$ are the same as those of $\{d_{st}\}$;

(S_3') $\mathrm{E}(d_{01}^+) < \infty$.

Therefore, the result follows immediately from subadditive ergodic theory; see [3, Theorems 1, 2 and 6]. □

Theorem 1 above states a result concerning the asymptotic convergence of the norm of the product. A similar result concerning the components of the resultant vector may be obtained by introducing an additional assumption.

ASSUMPTION C. (Positivity and Monotonicity) *H is a mapping from $R_+^k = \{x : x \in R^k, x_i \geq 0, i = 1, 2, \ldots, k\}$ into R_+^k. If $x \in R_+^k$, $y \in R_+^k$, with $y \geq x$, then $H(y) \geq H(x)$.*

Let $v \in R_+^k$ be a fixed non-zero vector, and let x be arbitrary in R_+^k. We define x_v, the vth generalized component of x as $\overline{\alpha}$, where $\overline{\alpha}$ is the maximum α such that $x = \alpha v + y$ with $y \in R_+^k$. Note that if $v = e_i$, the ith unit vector, then $x_v = x_i$, the ordinary ith component of x; and we continue to use the notation x_i for this case. In general $0 \leq x_v = \min_i \{x_i / v_i : v_i > 0\}$. Hence $x_v v \leq x$.

THEOREM 2. *Let* $H^1, H^2, \ldots$ *be a stationary process with elements satisfying Assumptions A–C. Suppose that*

$$\mathrm{E}\{\log \|H^1\|\} < \infty.$$

Let $v \in R^k_+$, $v \neq 0$. *Then*

$$a_v = \lim_{n\to\infty} n^{-1} \log[H^n H^{n-1} \ldots H^1(v)]_v$$

exists with probability one and in the mean.

Proof. The proof is a modification of [**3**, Theorem 5], for the case of positive matrices. Without loss of generality assume $\|v\| = 1$. Let

$$z_{st} = [H^t H^{t-1} \ldots H^{s+1}(v)]_v.$$

Then for $s < t < u$ we have

$$z_{su} = [H^u H^{u-1} \ldots H^{t+1}(H^t \ldots H^{s+1}(v))]_v.$$

Note that (from the general rule that $x_v v \leq x$) we have $z_{st} v \leq H^t H^{t-1} \ldots H^{s+1}(v)$. Hence by Assumption C

$$z_{su} \geq [H^u H^{u-1} \ldots H^{t+1}(z_{st} v)]_v = z_{tu} z_{st}$$

where the last equality follows from homogeneity. Therefore

$$d_{st} = -\log z_{st}$$

satisfies (S_1) in the proof of Theorem 1. By stationarity (S_2) is also satisfied.

Let $g_n = \mathrm{E}\{d_{0n}\} = -\mathrm{E}\{\log z_{0n}\}$. Then

$$\begin{aligned} -g_n &= \mathrm{E}\{\log[H^n H^{n-1} \ldots H^1(v)]_v\} \\ &\leq \mathrm{E}\{\log \|H^n H^{n-1} \ldots H^1\|\} \\ &\leq \mathrm{E}\left\{\sum_{j=1}^{n} \log \|H^j\|\right\} \\ &\leq n\{\mathrm{E}(\log \|H^1\|)\} < \infty. \end{aligned}$$

Therefore

$$\inf g_n/n \geq -\mathrm{E}\{\log \|H^1\|\} > -\infty$$

and condition (S_3) of [**3**] is satisfied. The result then follows from [**3**, Theorem 1]. □

The above theorem shows, roughly, that the growth in any direction v, as a result of repeated transformation by the mappings of the stochastic process, is exponential with a given (possibly stochastic) coefficient a_v. With an additional assumption, it is possible to show that these coefficients are all equal.

ASSUMPTION D. (Primitivity) *There is a positive integer* p *such that for any* $v \in R^k_+$, *and any* i, $i = 1, 2, \ldots, k$

$$\mathrm{E}\{\log[H^p H^{p-1} \ldots H^1(v)]_i\}$$

is finite.

This assumption guarantees that the elements of ${}^pF^1(v)_i$ are sufficiently positive so that growth in any one component eventually causes similar growth in other components. Since, in general, $x_w = \min\{x_i/w_i : w_i > 0\}$, the above assumption can, without loss of generality, be extended to arbitrary generalized components.

THEOREM 3. *Assume* $H^1, H^2, \ldots$ *satisfy Assumptions A–D and* $\mathrm{E}\{\log \|H^1\|\} < \infty$. *Then for any* $v \in R^k_+$, $v \neq 0$ *and any* $i = 1, 2, \ldots, k$,

$$a = \lim_{n\to\infty} n^{-1} \log[H^n H^{n-1} \ldots H^1(v)]_i$$

exists with probability one and in the mean and is independent of v *and* i.

Proof. Let a be defined by Theorem 2 when $v = e_1$ and let z_{st} be defined as in the proof of Theorem 2 with $v = e_1$. To obtain the result for other components and values of v, we write (for $n > 2p$)

$$[{}^nF^1(v)]_i \geq [{}^nF^{n-p+1}(e_1)]_i z_{p,n-p}[{}^pF^1(v)]_1.$$

Hence

$$\liminf_{n\to\infty} n^{-1} \log[{}^pF^1(v)]_i \geq \liminf_{n\to\infty} n^{-1} \log\{[{}^nF^{n-p+1}(e_1)]_i\} + a + \liminf_{n\to\infty} n^{-1} \log\{[{}^pF^1(v)]_1\}.$$

The finiteness of $\mathrm{E}[\log\{[{}^pF^1(v)]_1\}]$ shows (as in [**3**, p 892]) that the final term is zero. Hence

$$P\left\{\liminf_{n\to\infty} n^{-1} \log[{}^nF^1(v)]_i \geq a\right\} = 1. \tag{4}$$

In a similar way we write for $n > 2p$

$$z_{0,n} \geq [{}^nF^{n-p+1}(e_i)]_1[{}^{n-p}F^{p+1}(v)]_i[{}^pF^1(e_1)]_v.$$

This shows that

$$P\left\{\liminf_{n\to\infty} n^{-1} \log[{}^{n-p}F^{p+1}(v)]_i \leq a\right\} = 1.$$

Stationarity then implies

$$P\left\{\liminf_{n\to\infty} n^{-1} \log[{}^nF^1(v)]_i \leq a\right\} = 1. \tag{5}$$

The two conditions, (4) and (5), establish convergence with probability one. Convergence in mean is shown in a similar fashion. □

There are additional results along the lines above. In particular, the underlying space can be much more general in Theorem 1; and if the H process is metrically transitive, the ξ, a_v and a of Theorems 1–3, respectively, are constant almost surely.

3. *Invariant measure*

In order to make further progress and consider the central limit problem for random mappings, we introduce additional structure. Specifically we assume that the random mappings are characterized by a finite number of random parameters.

Let $q^1, q^2, \ldots$ form a stationary stochastic process with values in R^r, for some positive integer r. Let $H(\cdot, \cdot)$ be a mapping from $R^k \times R^r$ into R^k. By fixing $\overline{q} \in R^r$, $H(\cdot, \overline{q})$ is a mapping from R^k into R^k. We write

$$H^i(\cdot) = H(\cdot, q^i) \tag{6}$$

and then $H^1, H^2, \ldots$ forms a stochastic process of random mappings as in §2. We assume that each of these mappings satisfies Assumptions A and B.

Given an initial point $x^0 \in R^k$, the process $\{q^i\}$ generates a random process $x^1, x^2, \ldots$ through the recursions

$$x^{i+1} = H^{i+1}(x^i) = H^{i+1}(x^i, q^i). \tag{7}$$

Paralleling [**2**, p. 459], we introduce an additional associated process.

Let Ω be the set of all sequences $(\overline{q}_1, \overline{z}_1), (\overline{q}_2, \overline{z}_2), \ldots$ where $(\overline{q}_n, \overline{z}_n)$ are all points in $R^r \times R^k$ with $\overline{q}_n$ in the range of q_n and the $\overline{z}_n$ satisfy

$$\overline{z}_{n+1}\|H(\overline{z}_n, \overline{q}_{n+1})\| = H(\overline{z}_n, \overline{q}_{n+1}), \quad \|\overline{z}_n\| = 0 \text{ or } 1. \tag{8}$$

The variables (q^n, z^n) are then defined on Ω as the coordinate functions on Ω. On the subset of Ω where $\overline{z}_1 = H^1(x^0, \overline{q}_1)/\|H^1(x^0, \overline{q}_1)\|$ (with $\overline{z}_1 = 0$ if $H^1(x^0, \overline{q}_1) = 0$ and $\overline{z}_{n+1} = 0$ if $H^{n+1}(\overline{z}_n, \overline{q}_{n+1}) = 0$), the $\overline{z}_n$ are functions of $\overline{q}_n$ and hence this subset can be taken as the sample space for the q-process. Consequently we can define a probability measure μ_1 on Ω by carrying over to this subset the probability measure of the q-process. (It can be noted that on the support of μ_1 the z-process is related to the x-process of (7) by $z^n = x^n/\|x^n\|$, or $z^n = 0$ if $x^n = 0$.) Let T be the shift operator on Ω defined by $T\{(\overline{q}_n, \overline{z}_n)\} = \{(\overline{q}_{n+1}, \overline{z}_{n+1})\}$, and define the measures μ_k on Ω by

$$\mu_k(\Omega') = \mu_1(T^{-k+1}\Omega') \tag{9}$$

for $\Omega' \subset \Omega$. Then exactly as in [**2**] we have the following lemma.

LEMMA 1. *Let $\nu_n = n^{-1}\sum_{k=1}^{n} \mu_k$. There exists a subsequence ν_{n_i} converging weakly to a probability measure μ on Ω in the sense that the finite dimensional joint distribution functions of the variables q^n and z^n with respect to the ν_{n_i} converge to the corresponding distribution functions of q^n and z^n with respect to μ at each continuity point of the latter. The measure μ is stationary; i.e. invariant under T, and on subsets of Ω defined by the q^n alone, μ agrees with the given probability measure.*

It should be noted that in general the invariant measure will depend on x^0.

4. *Asymptotic distribution*

In this section we show that under a strict monotonicity assumption the components $[^nF(x^0)]_i$ satisfy a central limit theorem. The basic idea of the proof is that, under strict monotonicity, each H mapping has a contraction property in the sense that for any two vectors $x, y \in R^k_+$ the angle between $H(x)$ and $H(y)$ is less than that between x and y. (We assume throughout that the dimension k has $k \geq 2$.) Hence, if all these vectors are projected onto the unit simplex (where $\|\cdot\| = 1$), H appears to act like a strict contraction. Under suitable assumptions on the q-process, the contraction property implies that the z-process is nearly independent of its distant past, since all starting points eventually lead to essentially the same future sequence. This near independence in turn leads to the central limit result.

We require the following additional assumption on the structure of the mappings.

ASSUMPTION E. *For each H there holds $H(0) = 0$ and there are constants $d > 0$, $D > 0$ such that for any $x \in R^k_+$, any $\delta > 0$, and every $i, j = 1, 2, \ldots, k$*

$$d\delta \leq [H(x_1, x_2, \ldots, x_{i-1}, x_i + \delta, x_{i+1}, \ldots, x_n)]_j - [H(x_1, x_2, \ldots, x_n)]_j \leq D\delta. \tag{10}$$

Furthermore, there is a constant $C < \infty$ such that

$$1 \leq \frac{Dk}{d} \leq C \tag{11}$$

for all H in the range of the process $H^1, H^2, \ldots$. (Note in particular that $C \geq 2$.)

Example 1. If each $H = [h_{ij}]$ is a $k \times k$ matrix with $h_{ij} > 0$, for all i, j, then Assumption E is satisfied if there is an $M < \infty$ such that for all H in the range of the H process

$$\left(\max_{i,j} h_{ij}\right) / \left(\min_{i,j} h_{ij}\right) \leq M.$$

Example 2. Suppose $k = 2$ and

$$H(x)_1 = x_1 + h|x_1 - x_2| + x_2$$
$$H(x)_2 = x_1 + x_2$$

where h takes on values from a stationary random process. Then Assumption E is satisfied if there is an M such that $(1 + |h|)/(1 - |h|) \leq M$ for all h in the range of the process (i.e. if $|h|$ is uniformly less than one).

It is clear that Assumption E implies continuity and monotonicity, and because $H(0) = 0$, it implies positivity and primitivity (with $p = 1$). In fact a somewhat stronger result will be used.

LEMMA 2. *Assume H satisfies Assumption E. If x is in the range of H, then*

$$C^{-1} \leq \frac{x_i}{x_j} \leq C$$

for all $i, j = 1, 2, \ldots, k$.

Proof. We have $x = H(y)$ for some $y \in R^k_+$. Suppose, say, $y_n = \max_i y_i$. Then by Assumption E,

$$dy_n \leq x_i \leq Dky_n, \quad Dky_n \geq x_j \geq dy_n.$$

Dividing these two yields the result. □

We now state the contraction property of homogeneous positive monotone mappings.

LEMMA 3. *Let H satisfy Assumptions A and E. Let x, y be non-zero elements of R_+^k in the range of H and let*

$$a = \min_i \frac{x_i}{y_i}, \quad A = \max_i \frac{x_i}{y_i}.$$

Let $x' = H(x)$, $y' = H(y)$, and

$$a' = \min_i \frac{x_i'}{y_i'}, \quad A' = \max_i \frac{x_i'}{y_i'}.$$

Then

$$a' \geq a + \frac{(A-a)}{C^2}, \tag{12}$$

$$A' \leq A - \frac{(A-a)}{C^2}. \tag{13}$$

Proof. We may arbitrarily assume that $x_1/y_1 = A$. We have

$$\begin{aligned} x_i' &= H_i(x_1, x_2, \ldots, x_n) \\ &\geq H_i(x_1, ay_2, ay_3, \ldots, ay_n) \\ &= aH_i(y) + a\left\{H_i\left(\frac{x_1}{a}, y_2, \ldots, y_n\right) - H_i(y)\right\} \\ &= ay_i' + a\left\{H_i\left(\frac{Ay_1}{a}, y_2, \ldots, y_n\right) - H_i(y)\right\} \\ &\geq ay_i' + ad\left(\frac{A}{a} - 1\right) y_1. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{x_i'}{y_i'} &\geq a + \frac{d(A-a)y_1}{H_i(y_1, y_2, \ldots, y_n)}, \\ &\geq a + \frac{d(A-a)y_1}{kD \max_i y_i}. \end{aligned}$$

By Lemma 2, we obtain

$$\frac{x_i'}{y_i'} \geq a + \frac{(A-a)}{C^2}.$$

A similar proof holds for the opposite inequality. □

LEMMA 4. *Assume that Assumptions A and E are satisfied. For any x, $y \in R_+^k$ and any $n > 1$, m, j*

$$\left|\log \frac{{}^{n+m}F^m(x)_j}{{}^{n+m-1}F^m(x)_j} - \log \frac{{}^{n+m}F^m(y)_j}{{}^{n+m-1}F^m(y)_j}\right| = O((1 - 2C^{-2})^n).$$

Proof. For any $p, 0 \le p \le n$, let

$$a_p = \min_i \frac{{}^{p+m}F^m(x)_i}{{}^{p+m}F^m(y)_i}, \quad A_p = \max_i \frac{{}^{p+m}F^m(x)_i}{{}^{p+m}F^m(y)_i}.$$

By subtracting the two inequalities (12) and (13) of Lemma 3,

$$A_{p+1} - a_{p+1} \le (A_p - a_p)(1 - 2C^{-2}).$$

Thus

$$A_p - a_p \le (1 - 2C^{-2})^{p-1}(A_1 - a_1).$$

We also have from Lemma 3

$$a_{p+1} \le A_{p+1} \le A_p.$$

Thus

$$\frac{a_{p+1}}{a_p} \le \frac{A_p}{a_p} = \frac{A_p - a_p}{a_p} + 1 \le 1 + \frac{(1-2C^{-2})^{p-1}}{a_1}(A_1 - a_1)$$

and hence

$$\begin{aligned} \frac{A_p}{a_{p-1}} &= \frac{A_p - a_p}{a_{p-1}} + \frac{a_p}{a_{p-1}} \\ &\le (1 - 2C^{-2})^{p-1}\frac{(A_1 - a_1)}{a_1} + 1 + (1 - 2C^{-2})^{p-2}\frac{(A_1 - a_1)}{a_1}. \end{aligned}$$

By the strict positivity of vectors in the range of an H, we have $0 < a_1 \le A_1 < \infty$. Therefore for $p > 1$,

$$\frac{A_p}{a_{p-1}} = 1 + O(1 - 2C^{-2})^p.$$

Now

$$\begin{aligned} &\log \frac{{}^{n+m}F^m(x)_j}{{}^{n+m-1}F^m(x)_j} - \log \frac{{}^{n+m}F^m(y)_j}{{}^{n+m-1}F^m(y)_j} \\ &\qquad = \log \left(\frac{{}^{n+m}F^m(x)_j}{{}^{n+m}F^m(y)_j} \cdot \frac{{}^{n+m-1}F^m(y)_j}{{}^{n+m-1}F^m(x)_j} \right) \\ &\qquad \le \log \left(\frac{A_n}{a_{n-1}} \right) = O(1 - 2C^{-2})^n. \end{aligned}$$

The opposite inequality is proved in a similar fashion. □

We consider again the (q, z) process and the invariant measure μ of the preceding section, defined on Ω. We select $x^0 \in R^k_+$ arbitrarily to define μ and the (z, q)-process, and then define

$$a = \int \log \frac{[H^2(z^1)]_1}{z^1_1}\, d\mu = \int \log \frac{[H^{k+2}(z^{k+1})]_1}{z^{k+1}_1}\, d\mu \tag{14}$$

$$\begin{aligned} c_r &= \int \left(\log \frac{[H^{r+1}(z^r)]_1}{z^r_1} - a \right) \left(\log \frac{[H^2(z^1)]_1}{z^1_1} - a \right) d\mu \\ &= \int \left(\log \frac{[H^{r+p+1}(z^{r+p})]_1}{z^{r+p}_1} - a \right) \left(\log \frac{[H^{p+2}(z^{p+1})]_1}{z^{p+1}_1} - a \right) d\mu \end{aligned} \tag{15}$$

and

$$b = c_1 + 2\sum_{r=2}^{\infty} c_r. \tag{16}$$

We use the following 'near independence assumption' [**2**].

ASSUMPTION F. *If Ω^1 is a measurable set in the sample space of the q-process defined in terms of $q^{m+n+1}, q^{m+n+2}, \ldots$, only, then*

$$|P\{\Omega^1|q^1, q^2, \ldots, q^m\} - P\{\Omega^1\}| \le D_1\lambda_1^n P\{\Omega^1\}$$

where D_1 and λ_1 are some fixed positive constants with $0 \le \lambda_1 < 1$.

We note that Assumption F is satisfied if the q^i are mutually independent or if the q-process is a Markov process satisfying certain standard assumptions, see [**4**, p. 224]. Also, we note that if Ω^1 is invariant under the shift operator, then $P\{\Omega^1|q^1, q^2, \ldots, q^m\} = P\{\Omega^1\}$ since Ω^1 is measurable on $q^n, q^{n+1}, \ldots$ for any n. It follows from a version of the zero–one law [5] that $P\{\Omega^1\} = 0$ or 1. Hence, Assumption F implies that the q-process is metrically transitive.

THEOREM 4. *Assume that the stationary process q satisfies Assumption F and the corresponding H process satisfies Assumptions A and E. Assume also that there is a $\delta > 0$ such that*

$$\mathrm{E}\{\log|[H^1(e)]_1|^{2+\delta}\} < \infty \tag{17}$$

where $e \in R_+^k$ has all components equal to unity. Then the constants a, c_r, $r = 1, 2, \ldots$, and b defined by (14), (15), and (16) are independent of $x^0 \in R_+^k$, and for every $j = 1, 2, \ldots, k$

$$\lim_{n\to\infty} P\left\{\log\frac{[^nF(x^0)]_j - na}{(nb)^{1/2}} \le u\right\} = (2\pi)^{-1/2}\int_{-\infty}^{u} e^{-t^2/2}\,dt \tag{18}$$

when $b \ne 0$. If $b = 0$, then

$$\log\frac{^n[F(x^0)]_j - na}{n^{1/2}} \to 0 \tag{19}$$

in probability.

Proof. Lemma 2 shows that

$$|\log[^nF^1(x^0)]_i - \log[^nF^1(x^0)]_j| \le \log C,$$

and hence it is sufficient to establish the result for a single index, say $j = 1$. We define†

$$\xi_k = \log([^kF^1(x^0)]_1/[^{k-1}F^1(x^0)]_1), \quad k > 1$$

and

$$\xi_1 = \log[^1F^1(x^0)]_1 = \log[x^0]_1.$$

† We use k as an index here (as well as the dimension of R^k) so that our notation is consistent with that of [**2**].

Then

$$\log[{}^nF^1(x^0)]_1 = \sum_{k=1}^{n} \xi_k.$$

The idea of the proof is to use the fact that the ξ_k random variables are 'almost independent'. The explicit strategy of the proof follows [**2**] which in turn is a modification of the treatment given in Doob [**4**, Chapter V, §7] for the case of Markov chains. This strategy uses the expansion (cf [**4**, p. 38])

$$\begin{aligned}\mathrm{E}\{\exp(it\gamma y)|I\} = 1 &+ it\gamma \mathrm{E}(y|I)\\ &- \frac{t^2\gamma^2}{2}\mathrm{E}(y^2|I)\\ &+ O(|\gamma t|^{2+\delta}\mathrm{E}(|y|^{2+\delta}|I))\end{aligned}$$

applied to the random variable

$$y = \sum_{s=m+n+1}^{m+n+k} (\xi_s - a)$$

and the information $I = (q^1, q^2, \dots, q^m)$. We must show that there exist positive constants D and λ, with $0 < \lambda < 1$ such that

$$\mathrm{E}\{\xi_s - a|q^1, q^2, \dots, q^m\} \le D\lambda^{s-m} \tag{20}$$

$$\left|\mathrm{E}\left\{\left(k^{-1/2}\sum_{s=m+n+1}^{m+n+k}(\xi_s - a)\right)^2 |q^1, q^2, \dots, q^m\right\} - c_1 - 2k^{-1}\sum_{s=1}^{k}\sum_{r=s+1}^{k} c_{r-s+1}\right| \le D\lambda^m \tag{21}$$

and

$$\mathrm{E}\left\{\left|\sum_{s=m+n+1}^{m+n+k}(\xi_s - a)\right|^{2+\delta'} |q^1, q^2, \dots, q^m\right\} \le Dk^{1+(\delta'/2)} \tag{22}$$

for $\delta' = 0$ or δ. These relations will be derived using Lemmas 2–4, which replace Lemmas 2 and 3 in [**2**].

We note first that with $x^{k-1} = {}^{k-1}F^1(x^0)$,

$$\xi_k = \log\frac{[H^k(x^{k-1})]_1}{x_1^{k-1}} = \log[H^k(x^{k-1}/x_1^{k-1})]_1.$$

By Lemma 2 there are fixed constants $\alpha > 0$, $\beta > 0$ such that

$$ae \le \frac{x^{k-1}}{x_1^{k-1}} \le \beta e$$

uniformly over all possible values of x^{k-1} and k. Therefore by the homogeneity and monotonicity of H, $\xi_k - \log[H^k(e)]_1$ is uniformly bounded above and below. Hence from Assumption F and (17)

$$\mathrm{E}\{|\xi_{m+n+1}|\} \tag{23}$$

is uniformly bounded.

Setting $y = {}^{m+[n/2]}F^1(x^0)$, and using Lemma 4, we may write

$$\xi_{m+n+1} = \log \frac{[{}^{m+n+1}F^{m+[n/2]}(y)]_1}{[{}^{m+n}F^{m+[n/2]}(y)]_1}$$
$$= \log \frac{[{}^{m+n+1}F^{m+[n/2]}(x^0)]_1}{[{}^{m+n}F^{m+[n/2]}(x^0)]_1} + O(1-2C^{-2})^{n/2} \tag{24}$$

uniformly in $q^1, q^2, \ldots, q^m$. Therefore

$$\mathrm{E}\{\xi_{m+n+1}|q^1, q^2, \ldots, q^m\}$$
$$= \mathrm{E}\left\{\log \frac{[{}^{m+n+1}F^{m+[n/2]}(x^0)]_1}{[{}^{m+n}F^{m+[n/2]}(x^0)]_1}|q^1, q^2, \ldots, q^m\right\} + O(1-2C^{-2})^{(n/2)} \tag{25}$$
$$= \mathrm{E}\left\{\log \frac{[{}^{m+n+1}F^{m+[n/2]}(x^0)]_1}{[{}^{m+n}F^{m+[n/2]}(x^0)]_1}\right\} + O(1-2C^{-2})^{n/2} + O(\lambda_1^{n/2}) \tag{26}$$

uniformly in $q^1, q^2, \ldots, q^m$. The last equality follows because the first term on the right-hand side of (25) is the expectation of an expression that depends only on $q^{m+[n/2]}, q^{m+[n/2]+1}, \ldots$ and because the expectation of the absolute value of that same expression is uniformly bounded according to (23). Using the stationarity of the q-process and setting $\lambda_2 = \max(1-2C^{-2}, \lambda_1)^{1/2} > 0$, we may write (24) as

$$\mathrm{E}\{\xi_{m+n+1}|q^1, q^2, \ldots, q^m\} = \mathrm{E}\{\xi_{n-[n/2]+2}\} + O(\lambda_2^n). \tag{27}$$

We have by Lemma 4

$$\mathrm{E}\{\xi_{k+1}\} = \int \log \frac{[{}^{k+1}F^1(x^0)]_1}{[{}^{k}F^1(x^0)]_1}\, d\mu$$
$$= \int \log \frac{[{}^{k+1}F^2(z^1)]_1}{[{}^{k}F^2(z^1)]_1}\, d\mu + O(1-2C^{-2})^k$$
$$= \int \log \frac{[H^{k+1}(z^k)]_1}{z_1^k}\, d\mu = a + O(1-2C^{-2})^k. \tag{28}$$

From (27) and (28) it follows that for $\lambda \le \lambda_2$, there is a D such that (20) is true.

By the same techniques as in (24)–(26), one may show

$$\mathrm{E}\{(\xi_{m+n+r} - a)(\xi_{m+n+1} - a)|q^1, q^2, \ldots, q^m\}$$
$$= \mathrm{E}\{(\xi_{n-[n/2]+r} - a)(\xi_{n-[n/2]+1} - a)\} + O(\lambda_2^n)$$
$$= c_r + O(\lambda_2^n) \tag{29}$$

uniformly in $q^1, q^2, \ldots, q^m$.

From here we may copy the proof of [**2**] to establish (21) and (22). Then, as in [**2**], (22)–(24) replace Lemmas 7.1–7.4 in [**4**, pp. 222–227] and complete the theorem using the techniques of [**4**, pp. 228–230]. □

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