OBSERVERS FOR MULTIVARIABLE SYSTEMS

BY

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Observers for Multivariable Systems

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Abstract—Often in control design it is necessary to construct estimates of state variables which are not available by direct measurement. If a system is linear, its state vector can be approximately reconstructed by building an observer which is itself a linear system driven by the available outputs and inputs of the original system. The state vector of an nth order system with m independent outputs can be reconstructed with an observer of order $n-m$.

In this paper it is shown that the design of an observer for a system with M outputs can be reduced to the design of m separate observers for single-output subsystems. This result is a consequence of a special canonical form developed in the paper for multiple-output systems.

In the special case of reconstruction of a single linear functional of the unknown state vector, it is shown that a great reduction in observer complexity is often possible.

Finally, the application of observers to control design is investigated. It is shown that an observer's estimate of the system state vector can be used in place of the actual state vector in linear or nonlinear feedback designs without loss of stability.

I. Introduction

MANY CONTROL system designs are based on state vector feedback, where the input to the system is a function only of the current state vector. For example, in the case of a continuous, linear, time-invariant system of the form

$$\dot{x} = Ax + Du \tag{1}$$

where

- x is an $n \times 1$ state vector
- u is an $r \times 1$ input vector
- A is an $n \times n$ transition matrix
- D is an $n \times r$ distribution matrix,

such a design would express $u(t)$ as a function of x and t: $u(t) = F[x(t), t]$. The function F is determined by the particular design scheme employed. It may be the control function which in some sense optimizes system performance or it may be determined by some other design technique—possibly trial and error. Such state vector feedback designs offer many advantages with respect to both system performance and analysis. There is, however, one major disadvantage. In many control situations the system state vector is not available for direct measurement. In these situations, it is not possible to evaluate the control function $F[x(t), t]$, and hence, either the control scheme must be abandoned, or a reasonable substitute for the state vector must be found.

Typically there will be associated with the system (1) an output vector y of dimension $m < n$ of the form

$$y = H^* x \tag{2}$$

where H^* is an $m \times n$ output matrix.[1] It will be assumed that the outputs represented by the components of y are independent—or equivalently, that the matrix H^* has rank m.

The problem discussed in this paper is that of reconstructing the state vector from the available outputs. The system which performs the reconstruction is called an observer.

One possible method for obtaining the state vector is to build a model of the given system, drive the model with the same inputs as the original system, and use the state vector of the model as an approximation to the unknown state vector. In this method the dynamic behavior of the observer is identical with that of the system it observes. If initial conditions were not set properly or if there were slight disturbances, the model generally would not recover fast enough to provide an estimate suitable for control.

Another approach is to differentiate the available outputs a number of times and then combine these derivatives appropriately to obtain the state vector. In this case, the estimate responds instantaneously to disturbances, but it is severely degraded by a small quantity of additive noise in the measurements.

It is important to design an observer which is a compromise between these two simple procedures. It is desirable that the dynamic elements of the observer be faster than those of the system it observes, but not so fast as to possess the undesirable characteristics of differentiators (which correspond to poles at $-\infty$).

It has been shown [1] that an observer for the system (1) can be constructed which is itself a linear, time-invariant system driven by the system it observes. The observer need only contain $n-m$ poles and these may be chosen arbitrarily.

In Section II of this paper the general theory of observers is reviewed with particular emphasis on single-output systems.

Section III contains the main contribution of this paper. A new procedure is developed for designing observers for systems which have several outputs. Essentially, the problem is reduced to a series of observer designs for single-output systems—these individual designs being relatively straightforward.

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[1] The "star" notation represents the transpose of a matrix. Thus the $m \times n$ matrix H^* is the transpose of an $n \times m$ matrix H.

The new procedure, based on a special canonical form for multiple-output systems developed in the Appendix, not only leads to simpler observer designs but also to stronger theoretical results than obtained previously.

In Section IV the problem of reconstructing a single linear functional of the state rather than the entire state vector is considered. It is shown that considerable reduction in observer complexity is then possible. This procedure is similar to one given by Bass and Gura [2] for the design of single-input, linear control systems.

Finally, the stability implications of using the reconstructed state vector rather than the actual state vector is discussed. It is shown that observers may be used to realize both linear and nonlinear control laws without loss of stability. Thus, it is concluded that observers can effectively surmount the difficulties associated with control design when the state is not measurable.

II. Observers with Arbitrary Dynamics

The basic observer configuration is illustrated in Fig. 1. The system S_1 is assumed to be a linear time-invariant system. It is assumed initially that the system is free, i.e., unforced. The available outputs of this system drive a second system S_2 which is the observer. Theorem 1 (proved in [1]) shows that under these conditions, the observer's state vector will nearly always tend toward a linear transformation of the first system's state vector.

Theorem 1 (Observation of a free system): Let S_1 be a free system: $\dot{x}=Ax$ which drives S_2: $\dot{z}=Bz+Cx$. Suppose there is a transformation T satisfying $TA-BT=C$. If $z(0)=Tx(0)$, then $z(t)=Tx(t)$ for all $t\geq 0$. Or more generally,

$$z(t) = Tx(t) + e^{Bt}[z(0) - Tx(0)]. \tag{3}$$

Notice that in Theorem 1 it is not necessary for A and B to be the same size; they only have to be square. If A and B have no common eigenvalues, there is always a unique T satisfying $TA-BT=C$ [1].

Assume now that the system S_1 is governed by

$$\dot{x} = Ax + Du \tag{4}$$

where u is an input vector. An observer driven by both the plant input and outputs governed by

$$\dot{z} = Bz + Cx + TDu \tag{5}$$

will behave according to (3). Therefore, observers designed for a free system can be applied to the forced system if the input is suitably connected to the observer.

The construction of an observer rests on the solution of the matrix equation $TA-BT=C$. The solution T must have rank great enough to guarantee the recovery of the unmeasurable state variables. As illustrated below, observer design consists in choosing B and the part of C which is unspecified in such a way that T has that property.

In this section attention is focused on the problem of constructing observers for single-output systems. In Section III it is shown that the multiple-output case can be reduced to this form.

Two possible constructions for an observer are considered here; both based on a well-known [3] canonical form for an observable single-output system. An approach more suitable for computation, which does not employ canonical forms, may be found in [1].

Suppose an nth order system is governed by

$$\dot{x} = Ax \tag{6}$$

and has a single output[2] $y=h'x$. It is assumed that the system is completely observable [3], i.e., the vectors h, $A^*h, \cdots, (A^*)^{n-1}h$ are linearly independent. For such a system there exists a coordinate system in which the system is represented in the special canonical form [3]

$$\dot{x} = \begin{bmatrix} -\alpha_1 & 1 & 0 & \cdots & 0 \\ -\alpha_2 & 0 & 1 & & \vdots \\ \vdots & & & & \vdots \\ -\vdots & & & & 1 \\ -\alpha_n & 0 & & \cdots & 0 \end{bmatrix} x \tag{7}$$

$$y = x_1.$$

The system in this form is represented schematically in Fig. 2. Here the vector x is represented in the new coordinate system so that the new state variables $x_1, x_2, \cdots, x_n$ do not necessarily correspond to the original state variables—the original variables are suppressed from consideration for the present purposes.

It is readily verified that the characteristic polynomial of the matrix in (7) is $s^n+\alpha_n s^{n-1}+\alpha_{n-1}s^{n-2}+\cdots+\alpha_1$.

Consider an nth order observer for this system governed by

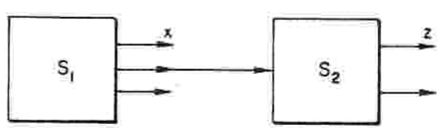

Fig. 1. A simple observer.

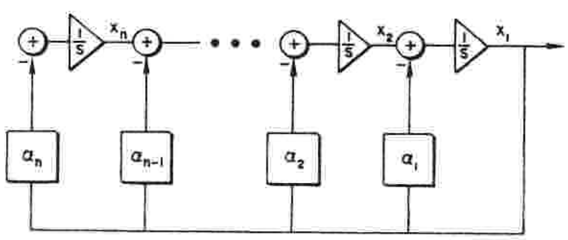

Fig. 2. Canonical form of observable system.

[2] As a matter of convention the transpose of a vector h is represented by h' throughout this paper rather than by h^*.

$$\dot{z} = \begin{bmatrix} -\beta_1 & 1 & 0 \cdots 0 \\ -\beta_2 & 0 & 1 \\ \vdots & & & \\ \vdots & & & 1 \\ -\beta_n & & \cdots & 0 \end{bmatrix} z + \begin{bmatrix} (\beta_1 - \alpha_1) \\ (\beta_2 - \alpha_2) \\ \vdots \\ \vdots \\ (\beta_n - \alpha_n) \end{bmatrix} x_1$$
$$= Bz + Cx. \tag{8}$$

In this case it is readily verified that $TA-BT=C$ is satisfied by $T=I$ so that state variables of z are in direct correspondence with those of x. Since the characteristic polynomial of this system is $s^n+\beta_n s^{n-1}+ \cdots +\beta_1$ and the coefficients β_i are arbitrary, it is clear that the poles of an observer of this form are arbitrary.

Consider now the problem of constructing an $(n-1)$th order observer for (7). If the observer is taken as

$$\dot{z} = \begin{bmatrix} -\beta_1 & 1 & 0 & & 0 \\ -\beta_2 & 0 & 1 & 0 \cdots & 0 \\ \vdots & & & & \\ \vdots & & & & 1 \\ -\beta_{n-1} & & & 0 & 0 \end{bmatrix} z + \begin{bmatrix} -\beta_1(\alpha_1 - \beta_1) + (\beta_2 - \alpha_2) \\ -\beta_2(\alpha_1 - \beta_1) + (\beta_3 - \alpha_3) \\ \vdots \\ -\beta_{n-1}(\alpha_1 - \beta_1) - \alpha_n \end{bmatrix} x_1 \tag{9}$$

$TA-BT=C$ is satisfied by

$$\mathrm{T} = \begin{bmatrix} -\beta_1 & 1 & 0 \cdots & 0 \\ -\beta_2 & 0 & 1 \cdots & 0 \\ \vdots & & & \\ \vdots & & & \\ -\beta_{n-1} & & & 1 \end{bmatrix} \tag{10}$$

which is an $(n-1)\times n$ matrix. The x variables can be recovered from the z variables according to

$$x_k = z_{k-1} + \beta_{k-1}x_1, \qquad 1 < k \leq n. \tag{11}$$

Again, since the β_i are arbitrary, the poles of this $(n-1)$th order observer are also arbitrary. The result can be formulated as

Theorem 2: Let S_1: $\dot{x}=Ax$, $y=h'x$ be an nth order, completely observable system. An observer of order $n-1$ can be constructed for S_1. Furthermore, the $n-1$ pole locations of the observer can be chosen arbitrarily.

III. Multiple-Output Systems

The general theory developed in the early part of Section II applies to multiple-output systems as well as single-output systems. The state of a system driven by the outputs will tend to follow a linear transformation T of the unknown state vector. In general, however, it is not obvious how to design the observer so that the state can be completely reconstructed while at the same time using the theoretically minimum number of dynamic elements: $n-m$.

As with the single-output situation, an essential assumption is that of complete observability with respect to the outputs. A system

$$\dot{x} = Ax$$
$$y = H^*x \tag{12}$$

is completely observable [3] if the $n\times(nm)$ matrix $[H, A^*H, \cdots, (A^*)^{n-1}H]$ has rank n. Often a rank of n will be achieved with a smaller number of powers of A^* times H. The observability index ν of the system (12) is defined as the least positive integer for which the matrix $[H, A^*H, \cdots, (A^*)^{\nu-1}H]$ has rank n. The observability index plays a key role in the theory of observers for multivariable systems.

For some systems the extension to the multiple-output case is elementary. For example, consider the fourth-order system shown in Fig. 3. It is assumed that the two variables x_1 and x_3 are available for direct measurement.

The fourth-order system may be regarded as two coupled second-order subsystems as indicated by the dashed-line boxes in the figure. The output of the first box is the measurable variable x_1 and the input is the measurable variable x_3. Therefore, since it is possible to measure the input and output of this second-order subsystem, a first-order observer may be constructed for this subsystem. Similar considerations apply to the second box, so it is seen that an observer for the total system can be built up from two separate observers, each observing a single-output subsystem.

The seemingly fortuitous situation in the above example is actually commonplace. In fact, all multiple-output observing problems can be reduced to the observation of single-output subsystems. This is a result of the following theorem (proved in the Appendix) guaranteeing the existence of a special canonical form.

Theorem 3 (Canonical Representation of Multiple-Output Systems): Suppose that the nth order system $\dot{x}=Ax$ with associated output vector $y=H^*x$ is completely observable with observability index ν. Suppose further that y consists of m independent components. Then there is a nonsingular linear coordinate change such that in terms of the new coordinates the system has the representation shown in Fig. 4. In this form the system consists of m component subsystems, each with one observable output which is a linear combination of the components of y. The orders of the subsystems satisfy $n_1+n_2+ \cdots +n_m=n$, and the largest subsystem is of order ν. The subsystems are coupled to each other only through their outputs.

From the previous developments it is obvious that an observer can be constructed for each of the single-output subsystems, since the inputs to the subsystems are available for measurement. Furthermore, the kth observer can be designed with n_k-1 arbitrary poles. Thus, by employing Theorems 2 and 3, it is easy to deduce

Theorem 4: An observer for the system (12) can be constructed of order $n-m$. (m is the rank of H^*.)

Furthermore, the $n-m$ poles of the observer are arbitrary.

This result is slightly stronger than the corresponding conclusion of [1]. Furthermore, in the form developed here there is established a simple algorithm for computing the observer in terms of the single-output subsystems, whereas the method of [1] is not so straightforward.

Example: The system illustrated in Fig. 3 is already in canonical form appropriate for design of a second-order observer. The poles of the observer are arbitrary and will both be chosen to be -3. The design is carried out separately for each subsystem.

System S_1 is governed by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x_3. \tag{13}$$

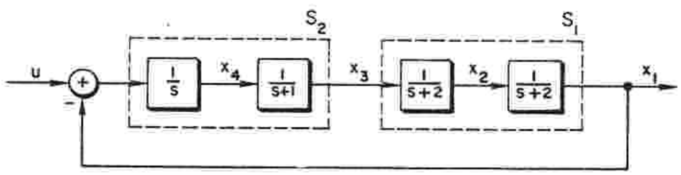

Fig. 3. Fourth-order system of example.

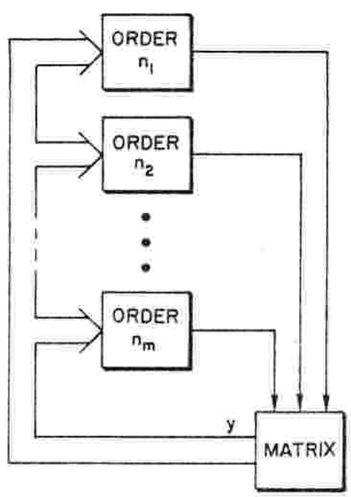

Fig. 4. Canonical form of multiple-output system.

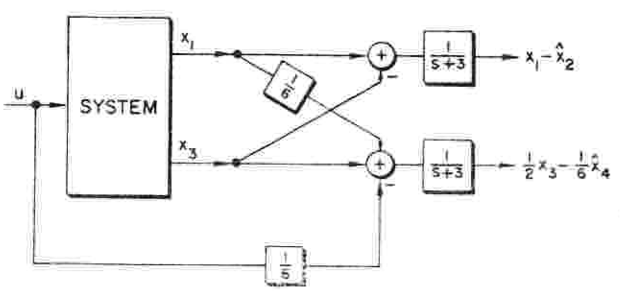

Fig. 5. Observer for fourth-order system.

According to the results of Section II, an observer with a pole at -3 driven by $x_1 = [1 \;\; 0]x$ will produce Tx where

$$T \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix} + 3T = [1 \;\; 0] \tag{14}$$

or $T = [1 \;\; -1]$. The observer will be governed according to (5) which in this case is

$$\dot{z} = -3z + x_1 - x_3. \tag{15}$$

The estimate $\hat{x}_2$ of the variable x_2 is constructed from the measurement x_1 and z according to

$$\hat{x}_2 = x_1 - z. \tag{16}$$

A similar procedure applied to the subsystem S_2 leads to the observer

$$\dot{w} = -3w + x_3 - \tfrac{1}{6}(u - x_1) \tag{17}$$

$$\hat{x}_4 = 3x_3 - 6w. \tag{18}$$

The complete observer for the system is shown in Fig. 5.

IV. Observing a Single Linear Functional

Sometimes it is only necessary to estimate a single (but prespecified) linear functional of a sytem's state vector. This is the situation, for example, in the design of linear, time-invariant, state feedback for a single-input system. In these instances, an observer of considerably reduced complexity can often be constructed which will produce this single quantity.

Observation of a single linear functional is similar in concept to a feedback design procedure developed by Bass and Gura [2]. The method in [2] is not an observer method in the sense of this paper and does not enjoy the closed-loop stability properties developed in Section V. The conclusions in this section concerning the required dynamic order of such an observer, however, coincide with the corresponding conclusions of [2].

Imagine an observer constructed for a multiple-output system according to to the scheme of Section III. The output of the observer is an estimate of the system state vector x. In order to obtain an estimate of a linear functional of x, say $a'x$, the same linear functional of the observer output is taken. The result is shown in Fig. 6.

The largest block in the observer has exactly $\nu-1$ poles which may be chosen arbitrarily. Suppose these poles are chosen first. Then the poles of each of the other blocks of the observer can be chosen to be a subset of the poles of this largest block. Now, corresponding to each output y_k there is a transfer function of the form $\Delta_k(s)/\Delta(s)$ from y_k through the observer to $a'\hat{x}$. The polynomial $\Delta(s)$ is the characteristic polynomial of the largest block in the observer, and $\Delta_k(s)$ is a polynomial of degree no greater than that of $\Delta(s)$.

Thus, it may be concluded that the observer of Fig. 6 is equivalent to the form shown in Fig. 7 when the individual blocks of the original observer have common poles. An observer of this form can be realized by a system of order $\nu-1$; therefore the following theorem is established.

Theorem 5: A single linear functional of the state of a linear system can be observed by a system with $\nu-1$ arbitrary poles. (ν is the observability index of the system.)

As pointed out in Bass and Gura [2], $\nu-1$ is often considerably less than $n-m$, the order of a complete observer. In fact, $(n/m)-1 \leq \nu-1 \leq n-m$. A twenty-fifth order system with five outputs, for example, may require as few as four arbitrary poles to construct an estimate of a single linear functional of the state vector.

Example: Consider again the fourth-order system in Fig. 5. Suppose it is desired to reconstruct the single linear functional x_2+x_4. According to Theorem 5 an observer with a single arbitrary pole is sufficient. If the pole is chosen to be at -3, the observer constructed in Section III for this system can be used as a first step in the design procedure.

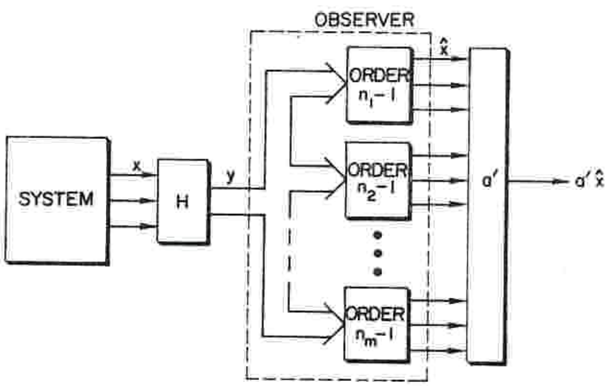

Fig. 6. Observing a single linear functional.

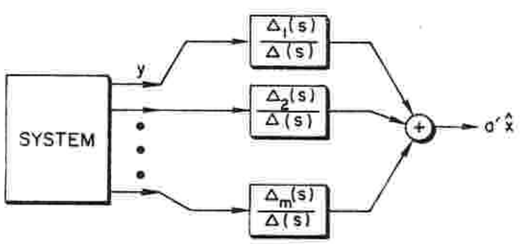

Fig. 7. Reduced observer for single linear functional.

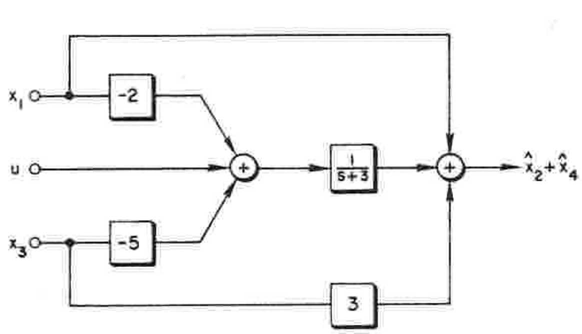

Fig. 8. First-order observer for example.

Using the results obtained in the previous example

$$\hat{x}_2 + \hat{x}_4 = x_1 + 3x_3 - z - 6w. \tag{19}$$

By carrying out some straightforward manipulations, the observer shown in Fig. 8 is obtained.

In general, it is possible to shortcut the two-step design procedure outlined in this section. One may boldly hypothesize the required form of the observer (according to Theorem 5) and solve for the unknown numerator polynomials directly—without first constructing the canonical form. It turns out that this shortcut procedure saves only a small percentage of the labor involved and does not offer much insight into the observer process. This shortcut procedure follows the spirit of Bass and Gura [2], however, and is certainly recommended for actual computations of observers of this type.

V. Closed-Loop Stability Properties

As stated in the Introduction, the investigation of observers is aimed at circumventing the difficulty of realizing state vector feedback control when the entire state vector is unavailable for measurement. Consider a system governed by

$$\dot{x} = Ax + Du \tag{20}$$

with outputs $y = H^*x$. Suppose that a (possibly nonlinear) control law of the form $u = F(x)$ has been derived for this system by some design scheme. An appropriate observer for (20) is

$$\dot{z} = Bz + Cx + TDu \tag{21}$$

where $TA - BT = C$. Cx must be derivable from the output vector y; hence, $C = GH^*$ for some appropriate matrix G. The estimated state is a linear combination of the system outputs and the state vector of the observer

$$\hat{x} = Lx + Kz \tag{22}$$

where $L + KT = I$ (the identity). The control law $u = F(x)$ can be approximated by a control law $\hat{u} = F(\hat{x})$ based on the estimated state vector. The complete system is then governed by the equations

$$\begin{aligned} \dot{x} &= Ax + DF(\hat{x}) \\ \dot{z} &= Bz + Cx + TDF(\hat{x}) \\ \hat{x} &= Lx + Kz. \end{aligned} \tag{23}$$

It is the purpose of this section to investigate the stability properties of the control system governed by (23).

The system equations (23) can be rearranged so that many of their stability properties become clearly apparent. Defining $\tilde{z} = z - Tx$, $\tilde{x} = \hat{x} - x$ and then subtracting T times the first equation in (23) from the second leads to the equivalent system

$$\dot{x} = Ax + DF(\hat{x})$$
$$\dot{\bar{z}} = B\bar{z}$$
$$\hat{x} = K\bar{z}. \quad (24)$$

If, initially, the estimated state vector is equal to the actual state vector, i.e., $\bar{z}(0)=0$, equality will be maintained for all future time. This important fact is due to what might be described as the complete *uncontrollability* of the observer from u. It implies that if there is initial equality between the state and its estimate the closed-loop system using the estimate behaves exactly like the closed-loop system using the actual state to obtain the control.

Generally the initial equality between the state and its estimate will not obtain, and stability properties of the complete system, including the observer, must be investigated. A first result in this regard applies to linear control laws. If $F(x)=Fx$ is linear, the closed-loop system using the actual state is $\dot{x}=(A+DF)x$. It has been shown [1], [4] that if an observer with transition matrix B is used to supply an estimate of the state vector, the closed-loop poles of the overall system (23) are the eigenvalues of $A+DF$ and of B. In other words, the observer does not disturb the poles of the original system but merely adds its own poles.

In a similar fashion, it is possible to investigate the effect of an observer in realizing a nonlinear control law. Suppose that the closed-loop system

$$\dot{x} = Ax + DF(x) \quad (25)$$

is asymptotically stable in the large [5]. It is assumed here that the asymptotic stability of (25) is established by the construction of a continuously differentiable Liapunov function $V(x)$ for the system which satisfies the following conditions

1) $V(x)>0$ for $x\neq\theta$, $V(\theta)=0$
2) $V(x)\to\infty$ as $\|x\|\to\infty$
3) $U(x)\equiv\dot{V}(x)\equiv\nabla_x V(x)[Ax+DF(x)]<0$ for $x\neq\theta$
4) $\lim_{\|x\|\to\infty} -U(x)/\|\nabla_x V(x)\| = +\infty$.

The first three assumptions are sufficient to guarantee asymptotic stability of (25), while the fourth is an additional assumption which is often satisfied in practice.[3]

Theorem 6 shows that under relatively mild conditions the observer scheme outlined above leads to an asymptotically stable system.

Theorem 6: Assume that there is available a Liapunov function for the system $\dot{x}=Ax+DF(x)$ which satisfies the conditions 1)–4) listed above. If $F(x)$ satisfies a uniform Lipschitz condition and the observer is asymptotically stable in the large, i.e., B has its eigenvalues in the left half plane. The complete system (24) is asymptotically stable in the large.

[3] For example, if V is a pd quadratic form, there is a $c>0$ such that $-U/\|\nabla_x V\|\geq c\|x\|\to\infty$.

Proof: As a first step in the proof a quadratic Liapunov function is constructed for the observer $\dot{\bar{z}}=B\bar{z}$ by the standard procedure for stable, linear, time-invariant systems [5], [6]. For this purpose define P as the unique solution to the matrix equation $PB+B^*P=-I$. It is well known that the matrix P so defined is positive definite and that $\bar{z}'P\bar{z}\equiv\|\bar{z}\|^2_P$ is a Liapunov function for the system $\dot{\bar{z}}=B\bar{z}$ with derivative $(d/dt)\|\bar{z}\|^2_P=-\|\bar{z}\|^2$. For the overall system (24) define $W(x,\bar{z})=V(x)+\|\bar{z}\|^2_P$. $W(x,\bar{z})$ is clearly positive definite. Also

$$\begin{aligned}\dot{W}(x,\bar{z}) &= \nabla_x V(x)[Ax+DF(\hat{x})]-\|\bar{z}\|^2\\ &= U(x)+\nabla_x V(x)D[F(\hat{x})-F(x)]-\|\bar{z}\|^2\\ &\leq U(x)+c_1\|\nabla_x V(x)\|\cdot\|\hat{x}\|-\|\bar{z}\|^2 \quad (26)\end{aligned}$$

where the positive constant c_1 is determined by the Lipschitz condition.

Using $\hat{x}=K\bar{z}$ from (24), the above inequality can be converted to

$$\dot{W}(x,\bar{z})\leq U(x)+c_2\|\nabla_x V(x)\|\cdot\|\bar{z}\|-\|\bar{z}\|^2. \quad (27)$$

Using the function W it will now be shown that any trajectory of the system (24) is bounded. Obviously $\bar{z}$ is bounded on any trajectory. Condition 4) on the Liapunov function V implies that for sufficiently large x and bounded $\bar{z}$ the function $\dot{W}(x,\bar{z})$ is negative definite. Therefore, since $W(x,\bar{z})\to\infty$ as $\|x\|\to\infty$ it is impossible that $\|x\|$ increase without bound. Thus there is an $R>0$ such that for all $t>0$, $\|x(t)\|<R$.

Now, since $\|\bar{z}\|\to 0$ as $t\to\infty$, given $\epsilon>0$ there is a finite time T such that for $t>T$, $U(x)+c_2\|\nabla_x V(x)\|\cdot\|\bar{z}\|<0$ throughout the annulus $\epsilon<\|x\|<R$. Thus for $t>T$ the function $\dot{W}$ is negative definite in this annulus and x must tend toward the circle $\|x\|\leq\epsilon$. Since ϵ was arbitrary, x tends to θ. This establishes asymptotic stability in the large.

VI. Conclusions

The observer theory developed in this paper can be compared with other methods of state estimation. In the case of noisy measurements and unknown noisy disturbance inputs, an optimal least-mean-square estimate of the state may be generated by a linear estimator [7]. If the estimator operates on the infinite past and the noise statistics are stationary, the estimator is a linear time-invariant system driven by the measurements. Such an estimator will act as an observer in the sense of this paper if the noise disturbances are suddenly disconnected so that the system becomes free and noiseless. Therefore, optimal estimators can be regarded as observers with their pole locations determined by the statistical properties of the noise. In many practical situations (namely those in which the noise level is significant), statistically optimal estimators offer excellent advantages over other estimation schemes. As the noise level decreases, however, the optimal pole locations move toward $-\infty$ and in the limiting case of perfect (noise-free) measurements, the statistically optimal estimator consists of a number of differentiators [8]. When noise level is low, noise is not a critical design factor and observer pole locations should be based on other criteria (e.g., parameter variation effects, reliability, ease of synthesis, etc.). Generally, there seems to be little reason to choose observer poles much faster than the other poles of the closed-loop system. So far, however, other than in the statistical case, there is little theory devoted to the problem of choosing observer pole locations.

The procedure outlined in this paper decomposes control design into state reconstruction and control of a completely measurable system. The effectiveness and practicality of the method lies in the fact that stable observer poles do not affect overall system stability. Theorem 2 of [1] and Theorem 6 of this paper are but two results concerned with the stability of combined observer and control systems. Other questions concerning speed of response, time-varying systems, and various types of stability under various assumptions offer fruitful areas for research.

One of the most important results described in this paper is the canonical form given in the Appendix. This canonical form can be used to extend many well-known properties of single output or single input systems to multivariable systems.

Appendix

Theorem 3 (Canonical Representation of Multiple Output Systems): Suppose that the nth order system $\dot{x}=Ax$ with associated output vector $y=H^*x$ is completely observable with observability index ν. Suppose further that y consists of m independent components. Then there is a nonsingular linear coordinate change such that in terms of the new coordinates the system has the representation shown in Fig. 4. In this form the system consists of m component subsystems, each with one observable output which is a linear combination of the components of y. The orders of the subsystems satisfy $n_1+n_2+\cdots+n_m=n$ and the largest subsystem is of order ν. The subsystems are coupled to each other only through their outputs.

Proof: The first step in the proof is the generation of a certain set of n linear independent vectors. The procedure used here is identical with that employed by Bass and Gura [2] for another purpose.

Since the matrix $[H, A^*H, \cdots, A^{*\nu-1}H]$ has rank n, n independent vectors can be taken as a certain n columns of this matrix. To define these vectors precisely

a) Start with the columns $h_1, h_2, \cdots, h_m$ of the matrix H.
b) Adjoin to these the columns $A^*h_1, A^*h_2, \cdots, A^*h_m$ one by one, checking that each new column is linearly independent of the previous ones. (Use the Gram-Schmidt orthogonalization procedure.)
c) If any of the new columns is found to be dependent, omit it from the matrix and go on to the next.
d) After A^*h_m has been tested, continue with $(A^*)^2h_1$, $(A^*)^2h_2, \cdots, (A^*)^2h_m$, etc., until n linearly independent columns have been found.
e) If a column $(A^*)^ih_j$ has been skipped because of linear dependence, all columns of the form $(A^*)^kh_j$ where $k>i$ can be skipped, since they also must be dependent on the previous columns.

As a result of this procedure, there is defined an array of n independent vectors

$$\begin{array}{l} h_1,\ A^*h_1, \cdots, (A^*)^{\nu_1-1}h_1 \\ h_2,\ A^*h_2, \cdots, (A^*)^{\nu_2-1}h_2 \\ \vdots \\ h_m,\ A^*h_m, \cdots, (A^*)^{\nu_m-1}h_m \end{array}$$

where for each k, $\nu_k \leq \nu$. Furthermore, by construction there are coefficients $\alpha_{ij}(k)$ such that

$$(A^*)^{\nu_k}h_k = \sum_{j=1}^{m} \sum_{i=0}^{\nu-1} \alpha_{ij}(k)(A^*)^ih_j \tag{28}$$

where $\alpha_{ij}(k)=0$ for $i>\nu_k$ and $\alpha_{ij}(k)=0$ for $i=\nu_k$ if $j\geq k$.

The desired canonical form of the system will have a structure similar to the structure of the above array in that the kth subsystem will be of order ν_k. However, the state variables of the kth subsystem will be defined in terms of vectors from the complete array rather than just the kth row. Because of the complexity of notation due to the several indexes required, the explicit transformation defining the appropriate new state variables is not particularly illuminating and therefore will be suppressed. The new coordinates are instead defined implicitly in terms of a schematic diagram. The kth subsystem takes the form shown in Fig. 9.

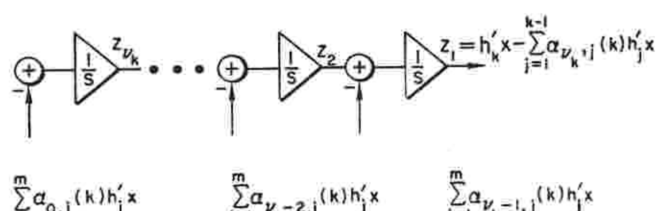

Fig. 9. kth subsystem of canonical form

The outputs of the m subsystems are each linear combinations of the original outputs and hence are themselves measurable quantities. Conversely, the new outputs are linearly independent so the old outputs can be recovered from the new. (The independence of the new outputs follows from the fact that the transformation matrix relating the old and new outputs is triangular with 1's along the diagonal.)

In order to establish that the proposed canonical form is in fact a linear coordinate change of the original system, it is only necessary to verify that all variables of the form $z=k'x$ in the canonical form satisfy $\dot{z}=(A^*k)'x$. That this requirement is satisfied by the kth subsystem shown in Fig. 9, follows directly from (28).

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