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# NOTES AND COMMENTS

# SINGULAR DYNAMIC LEONTIEF SYSTEMS[1]

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## 1. INTRODUCTION

A DYNAMIC LEONTIEF MODEL of a multisector economy has the form

$$x(k)=Ax(k)+B[x(k+1)-x(k)]+d(k) \tag{1}$$

where $x(k)$ is the vector of output levels, $d(k)$ is the vector of final demands (excluding investment), $A$ is the Leontief input-output matrix, and $B$ is the capital coefficient matrix. The matrices $A$ and $B$ are assumed known and time-invariant.[2] If the dimension of the vector $x(k)$ is $n$, and the system (1) is assumed to operate over the period $k=0, 1, \ldots, N-1$, then for a specified set of final demands the set of equations represents a set of $nN$ equations in $n(N+1)$ unknowns. A solution to this set is called a *trajectory* of the Leontief system.

From a purely computational viewpoint, it would be desirable to be able to calculate trajectories recursively. And, indeed, if the matrix $B$ were invertible the system (1) could be transformed to the form

$$x(k+1)=B^{-1}\{[I-A+B]x(k)-d(k)\}, \tag{2}$$

yielding a direct forward recursive solution, starting with an arbitrarily specified initial $x(0)$. Unfortunately, the assumption of $B$ being nonsingular is usually not justified. The element $b_{ij}$ of the matrix $B$ represents the amount of stock of commodity $i$, as a capital good, that sector $j$ must have on hand for each unit of production. Since not every sector produces significant capital goods (agriculture being a typical example in many models), it is common for some rows of the matrix $B$ to contain only zero elements. Thus, the very structure of capital requirements in a multisector economy often dictates that $B$ be singular. In fact, the rank of $B$ may be much smaller than $n$, the number of sectors. Therefore, the transformation to (2) cannot be used to develop a forward recursive procedure.

The problem of computing Leontief trajectories when $B$ is singular has been addressed by several authors. Perhaps the simplest technique is that proposed by Leontief [**2**], whereby if the matrix $I-A+B$ is assumed to be nonsingular, the system (1) can be transformed to

$$x(k)=[I-A+B]^{-1}\{Bx(k+1)+d(k)\}. \tag{3}$$

This system can be solved recursively *backward* in time, provided that $x(N)$ is specified. The disadvantage of this procedure, of course, is that in problems of analysis it is almost always $x(0)$ that is known, not $x(N)$. Other authors have developed forward recursive procedures under alternative assumptions [**1**, **2**, and **4**].

The issue of the existence of *forward* recursive solution techniques is much more than simply a matter of computational convenience—it relates fundamentally to the question of whether (1) is a meaningful representation of a dynamic economy. The economy itself presumably determines the level of outputs recursively, moving from one year to the next. It does not require information about future demand in order to determine current output levels. And if the economy can determine its behavior recursively, while being consistent with (1), it logically follows that there must be a transformation of (1) to recursive form.

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[2] The results of this paper can all be easily extended to the case where $A$ and $B$ vary with $k$.

If a system can be transformed to forward recursive form (and a precise interpretation of this is given later), then the system is said to be *regular*. In terms of this definition, the issue that is to be addressed, motivated both from philosophic and computational perspectives, is that of deducing conditions that characterize regular dynamic Leontief systems.

This paper presents necessary and sufficient conditions for a dynamic Leontief system to be regular. Not surprisingly, the mathematical formulation of these conditions has strong intuitive interpretations in terms of the underlying economic structure. It is shown, for example, that if the system is regular, the vector of capital stock serves as a state vector for the underlying dynamic system. Thus, the singularity condition on $B$ serves to decrease the dynamic order—essentially making the system simpler, rather than more complex.

## 2. RECURSIVE SOLUTIONS

By defining $R = I - A + B$, which is consistent with standard convention, the original equation (1) can be expressed equivalently as

$$Bx(k+1) = Rx(k) - d(k). \tag{4}$$

If such an equation is to represent a dynamically evolving economy, containing an underlying explicit dynamic structure, some assumptions must be imposed on the matrices $B$ and $R$. In order to state the appropriate conditions, it is useful to first perform some elementary operations on (4). By performing row operations on the $n$ equations comprising (4), if necessary, it is always possible to write (4) in the partitioned form

$$\begin{bmatrix} T \\ 0 \end{bmatrix} x(k+1) = \begin{bmatrix} G \\ H \end{bmatrix} x(k) - \begin{bmatrix} C \\ D \end{bmatrix} d(k), \tag{5}$$

where the matrix $T$ has full rank $m \leq n$. The other matrices are partitioned accordingly into $m \times n$ and $(n-m) \times n$ submatrices. Since the only operations required to produce (5) from (4) are row operations, the definition of the vector $x(k)$ remains unchanged. The matrix

$$\begin{bmatrix} C \\ D \end{bmatrix},$$

being the result of the row transformations applied to what was the identity matrix, is itself nonsingular. Most Leontief systems will already be in the form (5) in which case, of course, no such transformation is required. It is the form (5) that is used throughout the remainder of the development.

The following assumption on the defining matrices in (5) is fundamental to the existence of recursive solution procedures.

ASSUMPTION: The equation (5) is said to satisfy the *regularity assumption* if the $n \times n$ matrix

$$\begin{bmatrix} T \\ H \end{bmatrix}$$

is nonsingular.

To derive the recursive form of (5), and thereby establish that the regularity assumption introduced above is essentially equivalent to the general notion of regularity discussed in the introduction, define

$$y(k) = Tx(k). \tag{6}$$

Also, based on the regularity assumption, let

$$\begin{bmatrix} T \\ H \end{bmatrix}^{-1} = [E\ F], \tag{7}$$

where $E$ and $F$ are $n \times m$ and $n \times (n-m)$ matrices, respectively. Then the vector $x(k)$ can be determined uniquely from $y(k)$ and $d(k)$ through solution of the equations

$$Tx(k) = y(k)$$

and

$$Hx(k) = Dd(k).$$

Thus,

$$x(k) = Ey(k) + FD\, d(k). \tag{8}$$

Substitution of this result in (5), and again using (6), leads immediately to the recursive equation for $y(k)$,

$$y(k+1) = GEy(k) + [GFD - C]\, d(k) \tag{9}$$

with

$$y(0) = Tx(0).$$

System (9) is the dynamic system underlying the original equations, defined by (1), (4), or (5). The dimension of this system is equal to the rank of the original $B$ matrix. It can be solved forward in time once the initial conditions are specified. And, once this dynamic system is solved, the output vector can be determined from (8).

It is possible, of course, to combine (8) and (9) to develop a single forward recursion for the output vector $x(k)$. Indeed, direct substitution of (9) into (8) yields immediately

$$\begin{aligned} x(k+1) &= Ey(k+1) + FD\,[(k+1) \\ &= EGETx(k) + E[GFD - C]\, d(k) + FD\, d(k+1). \end{aligned} \tag{10}$$

For computational purposes, this single $n$th order recursive expression may be most convenient. However, it somewhat masks the fact that the underlying dynamic structure is actually only of order $m \leq n$.

The transformation (8), (9) or combined version (10), is simpler in form than earlier proposed transformations [**1**, **2**, and **4**], and requires weaker assumptions. Indeed, it can be shown [**5**] that the regularity assumption is both necessary and sufficient for the set of dynamic equations to possess a forward recursion of the form (8), (9). Thus, the solution proposed here is not only the simplest available, it is also the most general.

## 3. INTERPRETATION

The transformation developed in the previous section has a simple economic interpretation. Suppose, as is the usual case, that the capital coefficient matrix $B$ has the form

$$B = \begin{bmatrix} T \\ 0 \end{bmatrix}$$

with $T$ of full rank $m$. Thus the dynamic Leontief system (1) when written in partitioned form, becomes

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} x(k) = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} x(k) + \begin{bmatrix} T \\ 0 \end{bmatrix} [x(k+1) - x(k)] + \begin{bmatrix} d_1(k) \\ d_2(k) \end{bmatrix} \tag{11}$$

where $I_1$, $A_1$, and $d_1$ denote the top $m$ rows of $I$, $A$, and $d$, while $I_2$, $A_2$, and $d_2$ denote the bottom $n-m$ rows. In terms of our previous notation, $G=I_1-A_1+T$, $H=I_2-A_2$, $C=I$, $D=I$.

The vector $y(k)=Tx(k)$ is the vector representing the total stock of capital goods required to produce the output $x(k)$. That is, its $i$th component is the total required capital stock of commodity $i$ summed over all sectors which employ this commodity as capital. Since the Leontief equation implicitly assumes available capital is fully utilized, $y(k)$ can be regarded as the total capital stock on hand in period $k$.

The assumption that

$$\begin{bmatrix} T \\ H \end{bmatrix}$$

is nonsingular has a simple interpretation in terms of the economy's treatment of capital goods. By definition, $Tx(k)=y(k)$.

On the other hand, the bottom part of (11) is equivalent to $Hx(k)=d_2(k)$. Thus, combining these two equations, there results

$$\begin{bmatrix} T \\ H \end{bmatrix} x(k)=\begin{bmatrix} y(k) \\ d_2(k) \end{bmatrix}. \tag{12}$$

Therefore, the regularity assumption is equivalent to the statement that: *current total output in all sectors is uniquely determined from the total stock of capital goods, and the current demand for noncapital goods.*

The somewhat indirect method of determining output by use of (12) is at the root of the dynamic structure and, therefore, deserves further elaboration. Suppose first that there is zero demand for noncapital goods. Then, according to the regularity assumption, it is (uniquely) possible given any vector of available capital, to determine how to divide this capital among the various industries in such a way that each industry fully utilizes its capital resources, but net production in noncapital goods is zero. The relation between resulting total output and capital can be described by $x(k)=Ey(k)$. If, on the other hand, there is nonzero demand for the noncapital goods but no available capital, the output required to generate the demand is $Fd_2(k)$. Of course, it should be noted that $E$ and $F$ in general will not be nonnegative matrices.

With these economic interpretations, it is possible to directly deduce the dynamic equations underlying a Leontief system. Given the capital stock $y(k)$, the new stock $y(k+1)$ is equal to (1) the old stock; plus (2), the difference between output and input of the current year's production activity; minus (3), the current consumption demand for capital goods. That is,

$$y(k+1)=y(k)+[I-A]_1x(k)-d_1(k) \tag{13}$$

where $[I-A]_1$ and $d_1(k)$ denote the top partition (of height $m$) of $I-A$ and $d(k)$, respectively. However, from the previous discussion, $x(k)=Ey(k)+Fd_2(k)$, and so (13) becomes

$$y(k+1)=y(k)+[I-A]_1Ey(k)+[I-A]_1Fd_2(k)-d_1(k)$$

which can be written as

$$y(k+1)=[I-A+T]_1Ey(k)+[I-A+T]_1Fd_2(k)-d_1(k) \tag{14}$$

which agrees with the earlier, mathematically derived, (9).

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