

Charles H. Jolissaint
INTASA
Menlo Park, California

N.V. Arvanitidis
INTASA
Menlo Park, California

David G. Luenberger
Stanford University
Stanford, California

Abstract — Guided by the experience that real power is most sensitive to nodal phase angles and reactive power to voltage magnitudes, a new method of power flow optimization is developed which exploits this decoupling and consequently reduces problem size. This decomposition of the real and reactive equations results in the formation of two subproblems—one corresponding to the real power equations and the other to the reactive power equations. In practice, the two subproblems are alternatively solved until the desired accuracy is attained. The method has considerable potential for real time applications such as voltage control, economic dispatch, security analysis, etc., where in many situations the solution of only one subproblem will be required. The algorithm was originally developed for applications related to system security and for this purpose linear programming is used for optimizing each of the subproblems. A test case study of a 38-node network demonstrates the convergence properties of the algorithm.

INTRODUCTION

Present power flow optimization methods can be classified in one of two categories—one providing exact solutions and the other approximate solutions. In this distinction, exact methods 1,2,3 take into account both real and reactive power flows in obtaining the solution, whereas approximate methods 4,5,6 achieve simplified representations and possibly computational efficiencies by ignoring either the real or reactive equations. Approximate models are normally tailored to particular applications and do not have the generality inherent in the exact models. The reduced gradient method of Dommel and Tinney,¹ the Fletcher-Powell method as developed by Sasson for power flow applications,² and Carpentier's method based on satisfying the Kuhn-Tucker conditions³ are all extremely accurate and widely applicable, while some have been developed to become computationally very efficient. A number of approximate models dealing exclusively with either the real or reactive equations have also been developed over the past several years. Hano, *et al.*⁴, and Kumai, *et al.*⁵, used primarily the reactive equations to develop a method for real time control of voltages and reactive powers. The dc power flow models dealing with only the real equations have been extensively used for transmission planning and for obtaining approximate solutions to problems related to system security. Arvanitidis and Rosing⁶ improved the accuracy of the conservative class of real power flow models by developing a linear but lossy model. Exploiting the linearity, they used the linear class of objective functions thus aiding dispatching decisions associated with transmission system security. Subsequently, the need to adjust voltage magnitudes and reactive powers at times so that the overall solution is operationally implementable led to the consideration of the reactive equations. The desire to also retain the basic formulation, which made the approach a valuable system security tool, motivated the decomposition method as presented in the current paper.

The decomposition method fills the existing gap in the current power flow optimization techniques by providing the generality and accuracy attained by exact models while retaining much of the simplicity and computational efficiency attributed to approximate models. That is, the major limitation common to all approximate methods is the basic inability to account for the errors induced by the ignored

set of power flow equations and to account for the effect of these errors on the accuracy and, most important, on the operational implementation of the overall solution. The decomposition method overcomes these difficulties. This means that in system security applications, explored in Ref. 6, the subproblem involving just real power flow can be used to obtain a first solution as before. However, if the resulting reactive power flows have excessive error, then the second subproblem resulting from considering the reactive equations and the inherent coupling to the first subproblem can be solved to reduce these errors. Iteratively solving the two subproblems with updated parameters results in any desired degree of accuracy in the complete set of the power flow equations. In real time applications decomposition has the obvious advantage of processing half the problem size at a time and, therefore, provides savings in resident program storage and possibly in computer processing time. This operational flexibility can also be used to achieve the specified accuracy for a particular application (e.g., voltage-reactive control) by requiring strict accuracy in the relevant set of power flow equations (reactive) and considerably less in the other. The fact that real time applications provide a good initial estimate of all electrical variables assures that the solution to only one subproblem will often be sufficient.

This paper develops the decomposition approach to optimal power flow problems and provides the proof that the decomposition algorithm will converge to the optimum solution attainable if the entire problem were solved by any of the exact methods. First, the decomposed subproblems are formulated on the basis of satisfying the Kuhn-Tucker optimality conditions for the original problem. Then the role of the existing decoupling between real and reactive equations is discussed and its relation to the convergence properties of the algorithm is established. Subsequently, computational techniques for the subproblem optimizations are reviewed and the computer program based on using linear programming for subproblem optimization is described. Finally, the results of applying the program to a 38-node test case demonstrate the convergence and accuracy characteristics of the decomposition method.

NOMENCLATURE

The superscript M used with a variable denotes the maximum value and m the minimum. Subscripts k are used to denote variable values at node k; the subscript km associates the variable with the branch connecting nodes k and m.

Vector Variables

- P+jQ, (C+jD) correspond to the power generation (consumption) at the nodes.
- V, θ represent nodal voltage magnitudes and phases, respectively.
- δ represents the vector of branch phase angles.
- λ, μ are Lagrange multipliers to be associated, respectively, with the real and reactive power constraints.

In addition,

$$x = (P, C, \theta)$$

$$y = (Q, D, V)$$

Paper 71 TP 113-PWR, recommended and approved by the Power System Engineering Committee of the IEEE Power Engineering Society for presentation at the IEEE Winter Power Meeting, New York, N.Y., January 31-February 5, 1971. Manuscript submitted September 14, 1970; made available for printing November 17, 1970.

$$z = (y, \mu)$$

Network Parameters

$$Z_{km} = R_{km} + jX_{km} \quad \text{is impedance of branch } km.$$

$$Z_{kk} = R_{kk} + jX_{kk} \quad \text{is node } k \text{ impedance to ground.}$$

Functional Relationships

$f(P, C)$ is the objective function to be optimized. The real power flow equations expressed in terms of nodal phases are denoted by

$$I(\theta, V) = P - C \quad (1)$$

and the reactive power by

$$K(\theta, V) = Q - D \quad (2)$$

The expression

$$|\theta_k - \theta_m| \leq \delta_{km}^M \quad \text{all } km \quad (3)$$

represents constraints assuring that nodal phase differences satisfy branch phase bounds. We also define the following convenient notations:

$$s(v_k, v_m, \delta_{km}) = v_k^2 - v_k v_m \cos \delta_{km} \quad (4)$$

$$t(v_k, v_m, \delta_{km}) = v_k v_m \sin \delta_{km} \quad (5)$$

$$A(x, y) = I(\theta, V) - (P-C) \quad (6)$$

$$B(x, y) = K(\theta, V) - (Q-D) \quad (7)$$

FORMULATION OF THE DECOMPOSITION METHOD

The classical statement of the optimal power flow problem consists of minimizing an objective function, normally the transmission losses, subject to the real and reactive power flow equations and inequality constraints:

$$\min f(P, C) \quad (8)$$

subject to Equations (1), (2), (3), and bounds on the variables. In contrast, the decomposition approach results in two smaller subproblems which are solved sequentially, first one and then the other, in the process of converging to the optimum. The formulation of each subproblem insures that when the successive solutions converge, the result is the optimum solution of the original problem. That convergence does occur follows from the decoupling between the real and reactive power flow equations which will be explored in detail in the next section.

Associated with the optimization of (8) are the set of equations representing the Kuhn-Tucker necessary conditions for optimality.⁷ Carpentier's work³ showed that the solution of the Kuhn-Tucker conditions for (8) resulted in a locally optimal solution. Carpentier's approach was then used to find a solution to the Kuhn-Tucker conditions of the original problem. In this respect, the decomposition approach proposed can be interpreted in a similar way. The objective, however, is to construct two optimization subproblems which accomplish this in terms of the Kuhn-Tucker conditions. The subproblems are to be constructed so that the collective set of Kuhn-Tucker conditions from each

subproblem optimization forms exactly the Kuhn-Tucker conditions for the original problem. This insures that a solution optimizing each subproblem will locally optimize the original. The first subproblem is obtained by considering the real power equations with the nodal voltage magnitudes and reactive powers held constant. Using the previous notation the first subproblem is formulated as

$$\min \{f(P, C) + \mu_0 [K(\theta, V_0) - (Q_0 - D_0)]\} \quad (9)$$

subject to Equations (1) and (3) and bounds on real demands and real generations at the nodes.⁶ The zero subscripts denote variables held fixed during the optimization. The second subproblem is obtained by considering the reactive power equations with the phase angles and real powers held constant. That is,

$$\min \lambda_0 [I(\theta_0, V) - (P_0 - C_0)] \quad (10)$$

subject to Equation (2) and bounds on voltage magnitudes, reactive demands and reactive generations at the nodes. The method of solution, therefore, consists of fixing the voltage magnitudes V and the reactive dual variable μ and solving the real power subproblem as stated in Equation (9). Then, using the new values for the phase angles θ and the real power dual variables λ obtained from solving (9), the reactive subproblem (10) is set up and solved. Performing the real subproblem again with the updated values for V and μ fixed, establishes the iterative technique for converging to the solution. Optimizing each subproblem results in a solution that satisfies the Kuhn-Tucker conditions for that subproblem. Appendix I presents the proof that when a solution satisfies the Kuhn-Tucker conditions for both subproblems then this solution will also satisfy the Kuhn-Tucker conditions for the original problem. In addition, Appendix I provides the proof that the successive optimization of the subproblems as described above will converge to a solution that satisfies the Kuhn-Tucker conditions for both subproblems which, therefore, is the optimal solution to the original problem. Analytical requirements as to the amount of decoupling necessary for convergence are also developed in this appendix. The proof is given for unbounded variables but the extension to bounded variables clearly follows. If a variable does not hit its bounds then the associated inequality can be dropped from the problem formulation. Otherwise, it can be thought of as a constant and the resulting problem piecewise reduces to the unbounded form.

The physical interpretation of the coupling accounted for between the real and reactive subproblems follows from considering the coupling term in problem (9),

$$\mu_0 [K(\theta, V_0) - (Q_0 - D_0)]$$

In this term, the μ_0 represents the Lagrange multiplier of the reactive power equations and it measures the sensitivity of the objective function to changes in reactive power. Since $K(\theta, V_0)$ is equal to the net reactive injections, the entire product then represents the effect on the objective function of reactive power changes caused by changes in phase angles. Similarly, the coupling term

$$\lambda_0 [I(\theta_0, V) - (P_0 - C_0)]$$

represents the product of the real power Lagrange multiplier λ_0 and the change in net real injections. The λ_0 indicates how the objective varies as a function of real power and $I(\theta_0, V)$ indicates the variation of real power with voltage. This product represents how the objective changes because of voltage effects on real power.

ROLE OF THE REAL AND REACTIVE POWER FLOW DECOUPLING

The decoupling that exists between real and reactive power equations makes the decomposition approach possible. In Appendix I under

"Convergence of the Decomposition Algorithm," the process is shown to converge provided that sufficient decoupling exists. The decoupling requirement for convergence derived in Appendix I is that the real power equations $A(x,y)$ be more sensitive to phases than voltage magnitudes and, conversely, that the reactive power equations $B(x,y)$ be more sensitive to voltage magnitudes than phases. The importance of this decoupling leads us to explore the physical explanation for it and to interpret the manner in which it affects the rate of convergence.

The decoupling can be explained through the manner in which the power flow equations depend on the voltage sensitive term $s(V_k, V_m, \delta_{km})$ as given by

$$P_k - C_k = \frac{R_{kk}}{Z_{kk}^2} V_k^2 + \sum_{m \neq k} \left[\frac{R_{km}}{Z_{km}^2} s(V_k, V_m, \delta_{km}) + \frac{X_{km}}{Z_{km}^2} t(V_k, V_m, \delta_{km}) \right] \quad (11)$$

$$Q_k - D_k = \frac{X_{kk}}{Z_{kk}^2} V_k^2 + \sum_{m \neq k} \left[\frac{X_{km}}{Z_{km}^2} s(V_k, V_m, \delta_{km}) - \frac{R_{km}}{Z_{km}^2} t(V_k, V_m, \delta_{km}) \right] \quad (12)$$

Sensitivity relates the effects of changes in one quantity on another and is given by the partial derivative of the second with respect to the first. The various sensitivities of the terms

$$s(V_k, V_m, \delta_{km})$$

and

$$t(V_k, V_m, \delta_{km})$$

are given by

$$s(V_k, V_m, \delta_{km}) : \quad \frac{\partial s}{\partial V_k} = 2V_k - V_m \cos \delta_{km} \quad (13)$$

$$\frac{\partial s}{\partial V_m} = -V_k \cos \delta_{km}$$

$$\frac{\partial s}{\partial \delta_{km}} = V_k V_m \sin \delta_{km}$$

$$t(V_k, V_m, \delta_{km}) : \quad \frac{\partial t}{\partial V_k} = V_m \sin \delta_{km} \quad (14)$$

$$\frac{\partial t}{\partial V_m} = V_k \sin \delta_{km}$$

$$\frac{\partial t}{\partial \delta_{km}} = V_k V_m \cos \delta_{km}$$

Since the branch phase angles are relatively small and the per-unit nodal voltages are approximately one, it is seen from Equation (13) that the term $s(V_k, V_m, \delta_{km})$ is more sensitive to voltages than to phase angles. That is,

$$\left| \frac{\partial s}{\partial V_k} \right|$$

and
are larger than

$$\left| \frac{\partial s}{\partial V_m} \right|$$

$$\left| \frac{\partial s}{\partial \delta_{km}} \right|$$

Similarly, the term $t(V_k, V_m, \delta_{km})$ is more sensitive to the phase angles. For 115 kV networks or higher, the reactance X_{km} is usually much larger than the line resistance R_{km} . Therefore, the sensitivity of real powers in Equation (11) is dominated by the second term in the summation, while that of the reactive is dominated by the first term. The power equations are thus decoupled by their respective sensitivities to the coupling variables. Furthermore, this decoupling is related to the impedance characteristics of the transmission lines and the operating conditions that tend to keep the branch phase angles small and the per-unit nodal voltages close to unity. For low voltage transmission lines of less than 115 kV, the decoupling between real and reactive power flows may need to be improved before applying the decomposition approach. This occurs when line resistances are greater than line reactances. An approach for accomplishing this will be discussed later.

TECHNIQUES FOR SOLVING THE SUBPROBLEMS

The decomposition formulation presented is completely general in that it does not restrict the method of solution which can be applied to solving the subproblems. The necessary Lagrange multipliers must be determined with each optimization and will usually be a by-product of the optimization rather than require additional calculations. Two optimization techniques will be discussed—the reduced gradient method used by Dommel and Tinney and successive linear programming.

Reduced Gradient Approach

One approach to optimizing the subproblems would be to use a reduced gradient method similar to that proposed by Dommel and Tinney¹ for solving the power flow problem in its original formulation. Applying the reduced gradient method to a particular subproblem optimization parallels the procedure developed by Dommel and Tinney where the constraints and objective function in their original formulation are replaced by those of the subproblem to be optimized. Carrying out the steps in their method for an N node network now results in the inversion of an $N \times N$ matrix in each subproblem solution as opposed to the inversion of the original $2N \times 2N$ matrix. To attain the same accuracy, however, the subproblems may need to be set up and solved a number of times as compared to only once for the $2N \times 2N$ matrix. The $N \times N$ matrix resulting from decomposition would be more sparse than a scaled down version of the original matrix. Optimally-ordered Gaussian elimination and triangularized product techniques⁸ could be applied to effect computationally efficient solutions. The necessary Lagrange multipliers for inclusion in the coupling terms are a normal by-product of the reduced gradient approach and do not require additional calculations.

Linear Programming

Another method of optimization for each of the subproblems is to use successive linear programming.⁶ In this approach, the nonlinear subproblem to be optimized is linearized about some operating set of values for the variables. The resulting linear program is solved using linear programming techniques to obtain the optimum. The objective function and constraints are then relinearized about the new values. The new linearization represents an improved approximation to the nonlinear subproblem. Successive relinearizations can be made until sufficient accuracy in the subproblem solution is achieved. The ad-

constraints and variable bounds efficiently. A disadvantage is that it still requires considerable effort to reach the computational efficiencies of the reduced gradient approach. This will be discussed further in connection with the computer program that has been developed based on successive linear programming.

COMPUTER PROGRAM BASED ON SUCCESSIVE LINEAR PROGRAMMING

A computer program, implementing the decomposition approach and using linear programming, has been developed. Successive linear programming was chosen over the reduced gradient method because of its applicability to system security problems as demonstrated in Reference 6. The problem size limit of the present program is a network with 80 buses and 120 transmission lines for which the storage requirements are 32,000 decimal words. The basis matrix inverse in the linear program is the limiting factor with regard to storage requirements. At present, the sparsity characteristics of this matrix have not been used to expedite the inversion. This feature could be incorporated in it to improve computational efficiencies using concepts similar to those developed by Ogbuobiri⁹ and described in Reference 10.

The basic operation of the computer program is indicated by the flow chart in Figure 1. In anticipation of security analysis applications the subproblems have been expressed in terms of branch phase angle variables rather than nodal phase angles which facilitates the inclusion of line-current ratings and stability limits. Bounds on branch phase angles are now easily handled with the Dantzig upper bounding technique of linear programming.¹¹ The variable transformation from nodal to branch phase angles has no effect on the decomposition other than to change the internal representation of each subproblem. In regard to the real power subproblem, this variable transformation causes the inclusion of the loop equations to preserve the fact that the sum of the phases around any closed loop is zero.⁶ A description of the blocks comprising this flow chart gives insight to the operational characteristics of the decomposition approach.

Master Routing Program

Using successive linear programming to solve each of the non-linear subproblems requires repeated linear optimizations to obtain an accurate solution of the one subproblem before undertaking the solution of the other. This is accomplished by proper routing between the linear programs associated with the real and reactive power subproblems using the Master Routing Program. In practice, this routing was not found to affect the convergence ability of the decomposition algorithm. The major consequence of improved routing was in avoiding wasteful iterations within each subproblem.

This functional block performs data input operations and calculates good initial estimates for branch phase angles based on optimizing the linear conservative model. The setup for the optimization of the conservative model is similar to that for the real power subproblem where the reactive dual variable μ is set equal to zero. The real power flow equations are approximated by using linear average power flows which result in neglecting the transmission losses. For this purpose, two variables were required for each branch phase angle. For an N-node and M-branch network the conservative model linear program will have, at most, $2N + 2M$ decision variables, $2N$ for real powers generated and consumed and $2M$ for the branch variables. The number of equality constraints is $N + L = M + 1$ where N represents the number of real power equations and L represents the necessary number of independent loop equations. The bounds on the variables introduce at most $2N + 2M$ inequality constraints.^{6,10} The results obtained from solving the conservative model include initial estimates of the branch phase angles and of the dual variable λ associated with the real power equations. The conservative model retains the insensitivity of real power to voltage magnitudes and, therefore, the phase angles determined should be quite insensitive to the initial voltages used.

Linearized Real Power Subproblem

When control is passed to the real power subproblem two steps are performed. First, the linearized form of the real power subproblem equations are calculated and set up in a tableau for solution by the linear programming algorithm.¹⁰ Second, the linear programming optimization is performed. In the linearization, the latest value of the variables for branch phase angles, nodal voltages and reactive power dual variables are used. The program optimizes with respect to real power injections and branch phase angles, while the new linearization improves the approximation to the real power equations. The output includes new values for the branch phase angles δ and real power dual variables λ . The number of variables in this real power subproblem formulation is substantially less than that in the conservative model since each branch phase angle can now be represented by one variable in place of the two variables required in the conservative model. Otherwise, the size of the problem is the same and there are $M + 1$ equality constraints with less than $2N + M$ inequalities representing bounds on all free variables.

Linearized Reactive Power Subproblem

The linearized reactive power subproblem block in the flow diagram involves linearizing and solving the linear program associated with reactive power. The effect of this block is to improve the

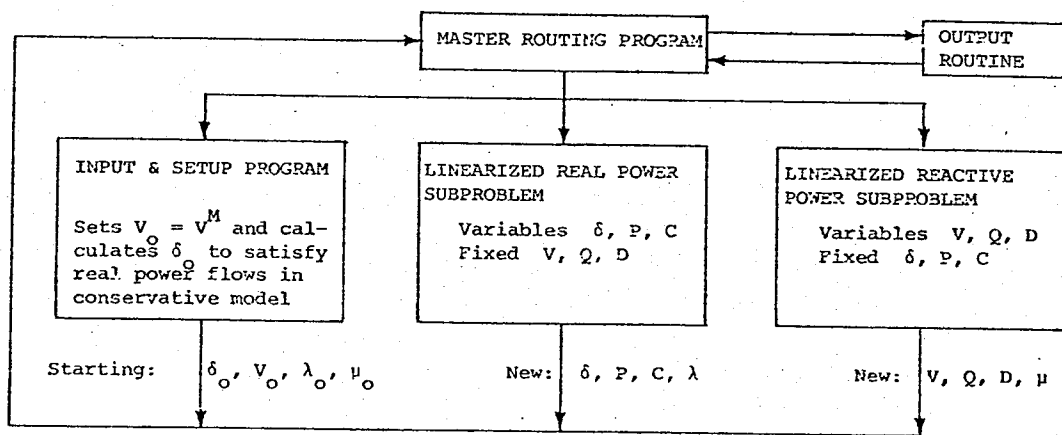


Fig. 1. Flow chart for operational procedure

the voltages and reactive power sources while holding the branch phase angles and the real dual variables fixed. The output includes new values of voltages V and reactive dual variables μ . The number of variables in the reactive subproblem is less than $3N$, with only N equality constraints.

TEST CASE STUDY

The decomposition approach has been applied to an actual case study using the Seattle area in the Pacific Northwest.¹⁰ This area includes 38 nodes and 55 branches. The problem variables include all branch phase angles, all nodal voltages, the real and reactive power generation at the slack bus and reactive power generations at nine additional nodes. Static capacitors and tap-changing transformers were kept fixed throughout the optimization. The objective function chosen was to minimize transmission losses. Convergence to .0035 per unit accuracy with starting voltages equal to maximum values occurred after successively solving eight linear programs for the real and reactive equations. The results were compared with and found almost identical to those obtained using conventional programs in the Bonneville Power Administration. Minor differences can be explained by the relative accuracies of the two programs used. Table I gives the CDC 6600 CP time for solving each of the linear subproblems. The setup time refers to the time required in calculating the entries for the linear programming tableau corresponding to each subproblem.

Table I Run Times (CP sec)

| Subproblem | Setup | Linear Program |
|-------------------------------------|-------|----------------|
| Conservative: Real Power Flow Model | 0.27 | 1.110 |
| Subproblem I: Real Power | 0.27 | 0.930 |
| Subproblem II: Reactive Power | 0.35 | 0.510 |

A discussion of the convergence behavior gives insight into the coupling effect between the two subproblems. It also indicates what might result from using only reactive or real power equations by themselves in cases of large perturbations. For the purpose of illustrating the general trend of the effect of solving one subproblem on the other, we examine a typical single node, BELHM2 in Figure 2. The real and reactive power consumption at this load bus were specified constant at 88 MW and 40 Mvar, respectively. The convergence trend illustrated in Figure 2 shows that solving the linearized real subproblem reduces the error in real power while causing a small increase in the reactive power subproblem. The errors are given in MW (Mvar) as the difference between the specified real (reactive) power and that obtained by the program. One graph in the figure shows the real power errors for each step; the other, the corresponding reactive errors. It is therefore observed that optimizing the conservative model or the linearized real power subproblem tightens the real power equations, whereas optimizing the linearized reactive subproblem tightens the reactive power equations.

The Seattle area case study shows the method of optimization through decomposition to be effective and operationally practical. The decoupling present resulted in very fast convergence of the algorithm. The linear approximations made for the linear programming formulation were seen to be quite accurate even in the initial optimization stages.

FURTHER RESEARCH

Current efforts involve reducing computer storage and run-time requirements and also investigating techniques for reducing the coupling between the two subproblems to speed up convergence for networks in which the real and reactive power equations do not have good decoupling. Basically what is done is to transform the real and reactive equations in such a way that the coupling between these equations is

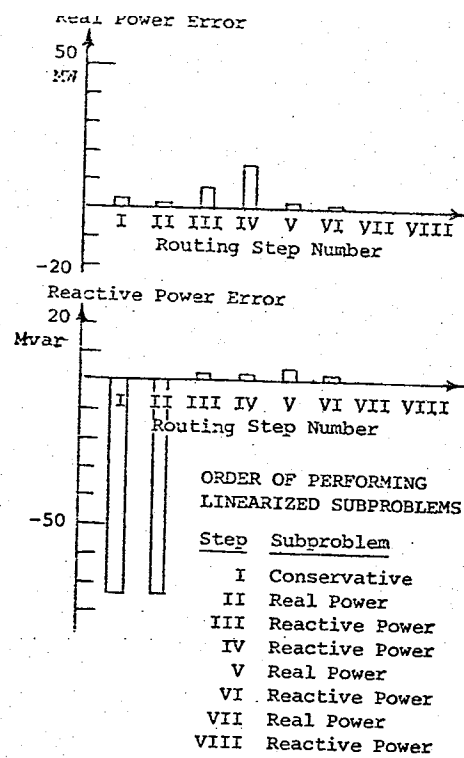


Fig. 2. Nodal power convergence behavior at the bus BELHM2

shifted from terms involving voltage magnitudes and branch phase angles to real and reactive powers, respectively. The advantage of such a shift is that most of the real and reactive injections in practical problems are fixed and therefore the coupling is reduced by such a transformation. The approach utilizes the transmission line property that for each node k the ratio of line resistance to reactance, i.e., R_{km}/X_{km} , is fairly constant for all m , representing connected nodes. Assuming that they were all equal to the same constant denoted by β_k , the following transformation applied to Equations (11) and (12) could be used to improve decoupling.

$$(P_k - C_k) - \beta_k (Q_k - D_k) = \sum_{m \neq k} \frac{X_{km} + \beta_k R_{km}}{Z_{km}^2} t(V_k, V_m, \delta_{km}) \quad (15)$$

$$\beta_k (P_k - C_k) + (Q_k - D_k) = \frac{\beta_k R_{kk} + X_{kk}}{Z_{kk}^2} V_k^2 + \quad (16)$$

$$+ \sum_{m \neq k} \frac{\beta_k R_{km} + X_{km}}{Z_{km}^2} s(V_k, V_m, \delta_{km})$$

The effect of the β_k in expression (15) is to eliminate the voltage sensitive $s(V_k, V_m, \delta_{km})$ term from what becomes the transformed power equation while increasing the sensitivity to the phase dominant term, $t(V_k, V_m, \delta_{km})$. On the other hand, in expression (16) which represents the new reactive power expression, the term $t(V_k, V_m, \delta_{km})$ is eliminated whereas $s(V_k, V_m, \delta_{km})$ is increased. The result of this transformation is two equations which have reduced coupling in the phase angle and voltage magnitude variables. Further study is necessary to assess the contribution such decoupling enhancement can make to the decomposition approach and to determine values for the β_k 's when not all resistance to reactance ratios of the connecting lines are the same.

A method for power flow optimization through the decomposition of real and reactive equations has been developed. The decomposition formulation resulted in two reduced optimization problems with one based on the real power equations and the other on the reactive. The objective function for each of the reduced problems was shown to include a coupling term accounting for the interdependence between real and reactive power flows. The two reduced problems were solved alternatively and this process of performing successive solutions was shown to converge to any required accuracy. Analytical results for the rate of convergence were related to the amount of decoupling present between the real and reactive equations. The expected conclusion that better decoupling leads to faster convergence was rigorously established and gave insight into future methods for improving the decoupling. The practical importance of the approach stems from the computational flexibility afforded for real time applications and, in particular, linear programming. The complete method was applied to a 38-node, 55-branch network demonstrating its rapid convergence and accuracy. With the use of successive linear programming it provides a powerful tool for system security analysis.

ACKNOWLEDGMENT

This work was supported by the Bonneville Power Administration, U.S. Department of the Interior under Contract No. 14-03-9190N. The assistance received from personnel of the Branch of System Operations, in particular from Mr. H.K. Eddington, throughout the project, was essential for the successful completion of the study.

Appendix I
FORMULATION AND CONVERGENCE OF DECOMPOSITION

The proof of decomposition logically consists of two separate parts. The first deals with the formulation of the subproblems so that the solution to both of them satisfies the Kuhn-Tucker necessary conditions of the original problem. For convex problems such a solution will also optimize the original problem. The second step is to prove that given sufficient decoupling, the successive solutions obtained by optimizing one subproblem and then the other will converge to a solution optimizing both subproblems. It then follows from the formulation that this solution satisfies the Kuhn-Tucker conditions for the original problem.

Following the notation developed, the original problem is stated as

$$\begin{aligned} & \min_{x,y} [f(x)] \\ & \text{subject to} \\ & A(x,y) = 0 \end{aligned} \tag{17}$$

and

$$B(x,y) = 0 \tag{18}$$

The two decomposed problems can be written with Subproblem I expressed as

$$\min_x [f(x) + \mu B(x,y)] \tag{19}$$

subject to Equation (17), and Subproblem II as

$$\min_y [\lambda A(x,y)] \tag{20}$$

subject to Equation (18). The dual variables λ and μ in the subproblem formulation are associated respectively with the equations $A(x,y)$ and $B(x,y)$.

Equivalence Theorem

Theorem 1. Given a solution $x^*, \lambda^*, \mu^*, y^*$ which satisfies the Kuhn-Tucker conditions for each subproblem then it satisfies the Kuhn-Tucker necessary conditions for optimality of the original problem. Furthermore, if the original problem is convex, the $x^*, \lambda^*, \mu^*, y^*$ is an optimal solution.

Proof follows from inspection of the Kuhn-Tucker conditions for each subproblem. For the original problem, the Kuhn-Tucker conditions include Equations (17), (18) and

$$f_x + \lambda A_x + \mu B_x = 0 \tag{21}$$

$$\lambda A_y + \mu B_y = 0 \tag{22}$$

Deriving the related Kuhn-Tucker conditions for Subproblem I, one obtains Equations (17) and (21), and for Subproblem II, Equations (18) and (22). These conditions considered collectively are the Kuhn-Tucker conditions of the original problem and the proof is established.

Convergence of Decomposition Algorithm

The main result of this section is the proof that by successively solving first one subproblem and then the other, we converge to a solution of both. Assumptions on the amount of coupling that can exist between the subproblems are necessary and several preliminary concepts must be developed before proving the main convergence theorem.

Let's examine what happens in solving Subproblem I. Choose a value of y and μ and for these fixed values obtain a solution in terms of λ and x . Let

$$w = \begin{bmatrix} x \\ \lambda \end{bmatrix}$$

and

$$z = \begin{bmatrix} y \\ \mu \end{bmatrix}$$

then, assuming that for each value of z there exists a unique value of w , one can write w as a function of z , i.e., $w(z)$. The Kuhn-Tucker conditions for Subproblem I also define an implicit functional relationship between w and z which we shall denote by $F(w,z) = 0$. In addition, if $\det |F_w| \neq 0$ and all functions in the original problem and their derivatives are continuous, then by the Implicit Function Theorem $w(z)$ and $w_z(z)$ exist and are continuous. In particular, we have that $w_z(z) = -F_w^{-1} F_z$. From the Kuhn-Tucker equations for Subproblem I, it is easily seen that

$$F_w = \begin{bmatrix} f_{xx} + \lambda A_{xx} + \mu B_{xx} & A_x^T \\ A_x & 0 \end{bmatrix} \tag{23}$$

$$F_z = \begin{bmatrix} \lambda A_{xy} + \mu B_{xy} & B_x^T \\ A_y & 0 \end{bmatrix} \tag{24}$$

and, therefore, (25)

$$\|w_z(z)\| = \left\| \begin{bmatrix} f_{xx} + \lambda A_{xx} + \mu B_{xx} & A_x^T \\ A_x & 0 \end{bmatrix}^{-1} \begin{bmatrix} \lambda A_{xy} + \mu B_{xy} & B_x^T \\ A_y & 0 \end{bmatrix} \right\|$$

The decoupling between the subproblems is expressed by the assumption that $A(x,y)$ is more sensitive to x than it is to y . That is, the derivatives of A with respect to x are larger than with respect to y .

derivatives of B with respect to y are larger than those with respect to x . Referring to the expression for $\|w_z(z)\|$ it follows that as coupling decreases the $\|w_z(z)\| \rightarrow 0$. What will be assumed is that the coupling is small enough that $\|w_z(z)\| = C < 1$.

Lemma 1. It can be shown that

$$\|w(z_1) - w(z_2)\| \leq C \|z_1 - z_2\|$$

Proof. Let

$$v = [w(z_1) - w(z_2)] / \|w(z_1) - w(z_2)\|$$

and let

$$\phi(\alpha) = w[\alpha z_1 + (1-\alpha)z_2] \cdot v$$

where ϕ is a functional of the scalar α . By the mean value theorem there exists $0 \leq \alpha \leq 1$ such that

$$\phi(1) - \phi(0) = \phi_\alpha(\bar{\alpha})$$

but substituting for ϕ one has:

$$\|w(z_1) - w(z_2)\| = w_z[\bar{\alpha}z_1 + (1-\bar{\alpha})z_2] \cdot (z_1 - z_2) \cdot v \quad (26)$$

By the Cauchy-Schwartz Inequality and the bound on $\|w_z\|$ it follows that

$$|w_z[\bar{\alpha}z_1 + (1-\bar{\alpha})z_2] \cdot (z_1 - z_2) \cdot v| \leq C \|z_1 - z_2\| \quad (27)$$

and, therefore,

$$\|w(z_1) - w(z_2)\| \leq C \|z_1 - z_2\| \quad (28)$$

The results so far have concentrated only on the first subproblem. If we perform an exactly analogous analysis of the second subproblem where the implicit function obtained from the Kuhn-Tucker conditions is denoted by $H(w,z) = 0$ and where

$$D = \|H_z^{-1} H_w\| < 1$$

the following inequality holds:

$$\|z(w_1) - z(w_2)\| \leq D \|w_1 - w_2\|$$

Theorem. The method of successively solving the subproblems converges to a solution. Letting

$$w^n = w(z^{n-1})$$

and

$$z^n = z(w^n)$$

the sequence (w^n, z^n) converges to solution w^*, z^* that satisfies the Kuhn-Tucker conditions of the original problem; the rate of this convergence is determined by the magnitude of the coupling term

$$\|F_w^{-1} F_z\| \|H_z^{-1} H_w\|$$

Proof. Considering the sequence w^0, w^1, w^2, \dots , one has that

$$\begin{aligned} \|w^n - w^{n-1}\| &= \|w(z^{n-1}) - w(z^{n-2})\| \leq \\ C \|z^{n-1} - z^{n-2}\| &= C \|z(w^{n-1}) - z(w^{n-2})\| \leq \\ CD \|w^{n-1} - w^{n-2}\| \end{aligned}$$

Therefore,

$$\|w^n - w^{n-1}\| \leq CD \|w^{n-1} - w^{n-2}\| \quad (29)$$

where $CD < 1$.

This last statement implies (w^n) is a Cauchy sequence and hence converges to some w^* . A similar analysis for z^n shows it also converges to some z^* . By the continuity of the Kuhn-Tucker equations, w^* and z^* solve them and, therefore, the successive solutions to the decomposed problems converge. From the proof it is obvious that the rate of convergence is given by

$$CD = \|F_w^{-1} F_z\| \|H_z^{-1} H_w\| \quad (30)$$

Summary. The method of decomposition was shown to converge to a solution of the Kuhn-Tucker necessary conditions for the original problem. If the functions are convex then the solution must also be optimal. The decomposition method involving the successive optimization of each of the subproblems converges if

$$\|F_w^{-1} F_z\| \|H_z^{-1} H_w\| = CD < 1$$

for all feasible z and w . Furthermore, the rate of convergence is related to the decoupling through this term.

REFERENCES

- [1] H.W. Dommel and W.F. Tinney, "Optimal Power Flow Solutions," *IEEE Trans. (Power Apparatus and Systems)*, vol. PAS-87, pp. 1866-1876, October 1968.
- [2] A.M. Sasson, "Combined Use of the Powell and Fletcher-Powell Nonlinear Programming Methods for Optimal Load Flows," *IEEE Trans. (Power Apparatus and Systems)*, vol. PAS-88, pp. 1530-1537, October 1969.
- [3] J. Carpentier, "Contribution a l'etude du dispatching economique," *Bull. Soc. Franc. Elec.*, Vol. 3, pp. 431-447, August 1962.
- [4] I. Hano, Y. Tamura, S. Narita and K. Matsumoto, "Real Time Control of System Voltage and Reactive Power," *IEEE Trans. (Power Apparatus and Systems)*, Vol. PAS-88, pp. 1544-1559, October 1969.
- [5] K. Kumai and K. Oda, "Power System Voltage Control by Using a Process Control Computer," *IEEE Trans. (Power Apparatus and Systems)*, Vol. PAS-87, pp. 1985-1990, December 1968.
- [6] N.V. Arvanitidis and J. Rosing, "The Use of Objective Functions in Real Power Dispatching," paper accepted for the 1971 IEEE Winter Power Meeting, New York, New York, January 31-February 5, 1971.
- [7] H.W. Kuhn and A.W. Tucker, "Nonlinear Programming," Proc. Second Berkeley Symposium on Math. Stat. and Prob., University of California Press, Berkeley, California, 1951.
- [8] W.F. Tinney and J.W. Walker, "Direct Solution of Sparse Network Equations by Optimally Ordered Triangular Factorization," *Proc. IEEE*, Vol. 55, pp. 1801-1809, November 1967.
- [9] E.C. Ogbuobiri, "Sparsity Techniques in Power-System Grid-Expansion Planning," Proc. Oxford Conference on Large Sparse Sets of Linear Equations, Southend-on-Sea, England, 1970.
- [10] C.H. Jolissaint, N.V. Arvanitidis, D.G. Luenberger and H. Duran, "Power Flow Optimization through the Decomposition of Real and Reactive Power Equations," Final Report prepared for Bonneville Power Administration, Portland, Oregon, under Contract No. 14-03-9190N, INTASA, Inc., Menlo Park, Calif., June 1970.
- [11] G.B. Dantzig, *Linear Programming and Extensions*, Princeton University Press, Princeton, New Jersey, 1963.

S.S. Sachdeva and R. Billinton (University of Saskatchewan, Saskatoon, Canada): The authors are to be congratulated on providing a readable and interesting analytical and mathematical description of the decomposition optimization technique. This technique provides the following three main advantages:

- (1) Reduced computer storage requirement.
- (2) Reduced processing time for each iteration.
- (3) Efficient utilization of the power system functional property by processing the closely coupled variables, reactive power and phase angle separately.

The sequential solution of the two subproblems of real and reactive optimal power flows can be utilized in a number of practical situations, as noted by the authors.

Work has been done in this area at the University of Saskatchewan using a non-linear programming approach to the problem of system optimal and suboptimal operation. Separate real and reactive power optimization will lead to a suboptimal operating point. Successive evaluation by an iterative process will provide an optimal solution at the point where both the real and reactive power suboptimization solutions coincide. The authors have formulated their approach by considering first the real power subproblem, then the reactive power subproblem. For a particular load condition the system loss is a function of reactive flow. If the minimization of loss as an objective function is to be carried out then it would be better to rotate the problem sequence, i.e., first carry out the reactive power optimization, then the real power optimization. Reactive power optimization will then provide the voltage and reactive power variables required in the real power optimization subproblem to obtain the real power and phase angle values. With proper allocation of constraints, an efficient iterative routine can be developed. In this way the power system functional property that losses at a particular load are more sensitive to voltage than to phase angle can be utilized.

Using this approach equations (9) and (10) should read as follows:

$$\min \{f(V, \theta_0) + v_0 [k(\theta_0, V) - (Q-D_0)]\}$$

and

$$\min \lambda_0 [I(\theta, V_0) - (P-C_0)].$$

Studies at the University of Saskatchewan have indicated that the decomposition technique becomes very efficient using this approach. The decoupling requirement for convergence is that the real power flow be more sensitive to phase angles than voltages, and the reactive power flow be more sensitive to voltage magnitude than phase angles. The effect of voltage and phase angle can be seen from the reactive power flow equation and provides another interpretation to the decoupling of real and reactive flow.

$$Q_k - D_k = \frac{V_k^2}{Z_{km}} \cos \theta_{km} - \frac{V_k V_m}{Z_{km}} \cos (\theta_{km} - \delta_{km}) \quad (1)$$

for $k \neq m$

where

$$Z_{km} \sin (\pi/2 - \theta_{km}) = \text{line reactance}$$

$$Z_{km} \cos (\pi/2 - \theta_{km}) = \text{line resistance}$$

$$Q_k - D_k = \frac{\cos \theta_{km}}{Z_{km}} [V_k^2 - V_k V_m (\cos \delta_{km} + \tan \theta_{km} \sin \delta_{km})] \quad (2)$$

$$= \frac{x}{z^2} [V_k^2 - V_k V_m (\frac{1}{\tan \delta_{km}} + \tan \theta_{km}) \sin \delta_{km}] \quad (3)$$

With $r \ll x$ and for small values (for normal line loading) of δ_{km} , $1/\tan \delta_{km} \gg \tan \theta_{km}$; therefore $\tan \theta_{km}$ can be neglected in comparison to $1/\tan \delta_{km}$. By substituting $\sin \delta_{km} \approx \tan \delta_{km}$, eq. (3) can be reduced to the following equation:

$$Q_k - D_k = \frac{x}{z^2} [V_k^2 - V_k V_m] \quad (4)$$

This equation is independent of δ_{km} and the reactive power is sensitive to voltage only. In other situations δ_{km} may produce a small effect on the reactive power flow. It can also be shown that real power is sensitive to phase angles.

The authors have used successive linear programming which appears to be very efficient. It would be of interest to know if a comparison of successive linear and non-linear programming has been carried out and if so, what the conclusions were.

F.J. Jaimes, A.H. El-Abiad, and A. Bonaert (Purdue University, Lafayette, Ind. 47907): The authors should be congratulated for their interesting paper on a decomposition algorithm and its application to the optimal power flow problem. The attractive property of this decomposition scheme is that the problem need not be separable.¹ Instead, the problem need only require certain decoupling properties, defined as bounds on the derivatives.

First, we would like to make some comments on the decomposition algorithm itself. The proof for convergence of the method is very elegant indeed. The authors use the Implicit Function Theorem to establish the functional relationship between the variables of w and z , given by the necessary conditions of subproblem I. Based on their definition of decoupling, the derivative of $w(z)$, is found to be bounded; which in turn implies that this function satisfies the Lipschitz condition (Lemma 1). Furthermore, if the Lipschitz constant c is assumed to be less than one, then the map $w(z)$ is a contracting map, which paves the way for convergence to a fixed point, once the analogous properties for the map $z(w)$ (Based on the necessary conditions of subproblem II) are established.

There is one point we would like the authors to clarify. The degree of decoupling is defined dependent on first order derivatives only. However, equation (25) contains both first and second order derivatives with respect to the coupling variables. Therefore, it appears that one should also impose conditions on the cross partial derivatives A_{xy} and B_{xy} . It is clear that when the coupling is zero (i.e., $A(x,y) = A(x)$ and $B(x,y) = B(y)$), $\|w_z(z)\| = 0$. It is not clear how the limiting process based only on first partial derivatives will give zero as a limit. For example, in the particular application of the paper, the dominating terms on $A_{xy} = A\delta_y$ will become dependent on the voltages rather than the angles. For example from equation (14) $\partial^2 t / \partial V_k \partial \delta_{km} = V_m \cos \delta_{km}$. Since $A(x,y)$ is dominated by t , the A_{xy} terms are not negligible. Would the authors care to expand more on this point?

In the paper it is mentioned that although the decomposition theorem is proved for the case with equality constraints only, it also follows for the case with inequalities, specifically with bounds in the variables. We think that the extension is not so straightforward. The proof of the algorithm is based on the Implicit Function Theorem, which requires open sets around points that satisfy $F(w,z) = 0$. With inequality constraints, the conditions for the inverse open mappings do not hold; therefore a more refined proof may be necessary. As a first guess, one would add terms like $\alpha g(x,y)$ to subproblem I, and $\beta h(x,y)$ for subproblem II, where $g(x,y)$ and $h(x,y)$ are the inequality constraints. In addition, they must be susceptible of decoupling in the same sense as the equality constraints, with α and β greater than or equal to zero. This last requirement is the point of difficulty for the proof of convergence. As far as the practical consequences are concerned, this does not make much difference. For example, consider the case of bounds in x and y . When solving subproblem I, if a bound is reached in an optimum solution, say $x_i = x_i^M$, for some i , the α 's associated with the bounds that were not reached will be zero. On the other hand $x_i - x_i^M = 0$, so that the overall effect of $\alpha(x - x^M)$ has no effect for subproblem II. This is similar for subproblem I.

At the end of the appendix there is a summary, that states: "If the functions are convex then the solution must also be optimal." We feel that this may be too strong a statement. We think that if the functions are convex and all λ and μ at the solution are greater or equal to zero, then the above statement is valid; if any of them is negative, then it may or may not be the global optimum.

The authors apply the decomposition technique to optimal power flows. In particular the case of minimum losses (with all real injections specified except at the slack node) is used to illustrate the actual

convergence of the method. Although this particular application may not satisfy the decomposition hypothesis, strictly speaking, the method converged. We would like to speculate around the reactive dispatch case considered by the authors, which may tend to give optimistic results. The solution of the first two optimization subproblems (conservative and real power, figure 2) are determined by the equality constraints, i.e., the cost function is somewhat artificial. In the conservative subproblem a reasonable set of angles are computed. In step II the real power flow equations are linearized at the solution point of step I, and a new improvement on the angles is found. The next step (reactive power subproblem) will, in general, have more than one feasible solution, so that the optimization subproblem is meaningful. In the reactive power subproblem, the deviations in the real power equations, due to changes in voltage magnitudes, are reasonably weighted in the cost function through the Lagrange multipliers λ , do that the optimum of subproblem II does not differ too much from the optimum of the non-decomposed problem. In other words, in the loss minimization, one is solving actually only one optimization subproblem (subproblem II) in which part of the equality constraints are solved directly, and part are weighted through the Lagrange term which in turn is updated by better approximations. However, for the economic dispatch problem (and it may be even worse for some security functions) the iterated solutions of subproblem I are critical for the overall solution. Generally speaking, the Lagrange multipliers λ will be much greater than μ , this has the effect that the term $\mu B(x,y)$ is weighted very lightly with the consequence that wide errors in reactive powers may occur, and in turn, one may not have feasible solutions for subproblem II, or make the convergence highly oscillatory. Have the authors experienced any difficulties along this line?

Finally, we feel that the solution of each subproblem by nonlinear programming techniques can be done effectively.² Other comments on the use of linear programming to solve the nonlinear problem can be found in our discussion to reference [6].

REFERENCES

- [1] Leon S. Lasdon, "Duality and decomposition in mathematical programming," IEEE Trans. on System Science and Cybernetics, vol. SSC-4, no. 2, pp. 86-100, July 1968.
- [2] F. J. Jaimes and A. H. El-Abiad, "Optimization by a sequence of equality constrained problems - Its application to optimal power flows," Submitted for presentation PICA conference, 1971.

Charles H. Jolissaint, N.V. Arvanitidis, and David G. Luenberger: We wish to thank the discussors for their interest in our paper and to apologize for the confusion associated with designating the real power subproblem as the first subproblem in the paper. This was unfortunate since it was intended that for on-line use where one has some last estimate for the variables, either subproblem could be designated as the "first" subproblem based on the application and power flow error considerations. However, for the present we will remain consistent with the paper.

In reply to Sachdeva and Billinton, we would like to emphasize that performing separate real and reactive power optimizations by ignoring the coupling will not only lead to a suboptimal solution, but also to a solution that in general will not satisfy required operational accuracy in the power flow equations and system operating limits; the latter is of greater practical significance. The decomposition method presented is therefore important not only for achieving global optimality, but also for accounting for the coupling in order to achieve an implementable solution, i.e., satisfying any degree of accuracy required in all the constraints.

This point can be made clear by examining the approach proposed by the discussors. Expressing the transmission losses as a function of node voltages and phase angles, $f(V,\theta)$, leads to the following restatement of equations (9) and (10) for the real and reactive subproblems

$$\min_{\theta} \{ f(V_0, \theta) + \mu_0 [K(\theta, V_0) - (Q_0 - D_0)] \} \quad (9)$$

and

$$\min_{V} \{ f(V, \theta_0) + \lambda_0 [I(\theta_0, V) - (P_0 - C_0)] \} \quad (10)$$

Using (9') and (10') as the form of the objective function, the case study in the paper was run with and without accounting for coupling.

Manuscript received March 24, 1971.

Without accounting for coupling was obtained by fixing μ_0 and λ_0 equal to zero in (9') and (10') for all iterative runs. The result was that bus voltage magnitudes were driven towards their lower bounds and line flows were obtained in excess of line ratings, which resulted in large transmission losses. Coupling was accounted for by including the coupling terms in the objectives (9') and (10'); the resulting solution minimized transmission losses and satisfied all operational constraints. Therefore, accounting for coupling not only achieved optimal results, but also operationally implementable results.

As for the remaining discussion, we found the alternative description of the basis for the decoupling quite interesting, and we have not as yet carried out a comparison of successive linear and non-linear programming methods for the subproblem solutions.

Discussors Jaimes, El-Abiad, and Bonaert raise several questions whose answers lie in the methodology applied to developing the decomposition algorithm. The proof given in Appendix I applies to a general system having weak coupling between constraint sets (17) and (18). What was intended was to express the abstract concept of weak coupling in an analytical way which insures convergence and is consistent with our intuitive feelings about what weak coupling should mean. The result was expression (25) for $\|w_2(z)\|$. A problem, therefore, was defined to have weak coupling if (30) based on (25) was less than one in magnitude. The comment that "as coupling decreases the $w_2(z) \rightarrow 0$ " should be interpreted as the limiting case in which no coupling exists; therefore $A(x,y) = A(x)$ and $B(x,y) = B(y)$. This limiting process does not depend on only first derivatives, and requires that the coupling terms involving first and second order cross derivatives (i.e., A_{xy} , A_{xy} , and B_{xy}) decrease to zero. Therefore, the basis for establishing $w_2(z)$ as a measure of decoupling is that its value decreases to zero as one approaches the case with no coupling. The discussors' comments on the A_{xy} terms are more appropriately directed at whether the method developed can be applied successfully to practical electric power systems.

In testing the applicability to the power control problem, we now need to establish that the measure of decoupling, i.e., the value for $w_2(z)$ is less than one in magnitude. To establish this analytically for all power systems, or to establish the applicability of the method expressed by establishing that branch phase angles should be less than some required magnitude would be very desirable. However, at present, there seem to be insurmountable analytical difficulties for obtaining such hard results. Instead, in the section on the "Role of the Real and Reactive Power Flow Decoupling," we attempt to interpret our feelings about the decoupling between real and reactive power flows and tie them in with the decoupling requirements in the Appendix. There is a limit as to how much inference to applicability can be drawn by such subjective reasoning as pointed out by the discussors concerning the behavior of the resulting A_{xy} terms for the power problem. Therefore, applicability must in the end rest on experience in using the method in practical problems. The attractiveness of the method in regard to computational flexibility and reduced core storage requirements, in addition to its intuitive eligibility for exploiting the decoupling found in electric power problems, should provide the incentive for such efforts.

Our experience so far has indicated that none of the convergence difficulties mentioned by the discussors occurs. Runs of the minimizing transmission loss problem described in the paper in which real power at several generators was allowed to vary (in addition to the reactive variables mentioned) caused no convergence difficulties. This was also found to be true of runs using the security function described in Reference [6].

The extension of the equality proof given in Appendix I to the inequality case follows from piecewise considerations. The part of the proof causing difficulty is establishing Lemma 1 in Appendix I. In particular, consider $\Phi(\alpha)$ where α is a scalar varying from 0 to 1. If, in the inequality case, all the inequalities are tight, then it would be equivalent to the equality case for $\alpha \in [0,1]$, and the proof clearly coincides with what is given. Furthermore, if over the entire interval the first j constraints remain strict inequalities and the remaining ones are tight, then we can hypothetically discard the first j constraints in forming the Kuhn Tucker Equations since they have no effect on the solution. The arguments establishing Lemma 1 then follow when applied only to the tight constraints.

The more realistic occurrence is that various constraints will remain tight only over portions of the interval. By breaking up the interval into subintervals, we can reformulate the problem over each subinterval discarding those constraints which remain strict inequalities. This contraction mapping is therefore established over each subinterval. Summing up all subintervals and taking C to be the maximum of the C_i corresponding to each subinterval establishes Lemma 1. The extension therefore follows from piecewise application to the equality results.

In regard to the discussors' comments on our summary statement, "if the functions are convex then the solution must also be optimal" as being too strong, we disagree. If anything, this is too weak a statement. The reason is that since the dual variables are obtained from subproblem optimizations, they must automatically satisfy the necessary sign conventions at the final solution. For convex inequality constraints this is

non-negative, for concave inequality constraints this is non-positive, and for equality constraints the sign is unrestricted. Requiring that the functions be convex was just a simple approach to insuring there would be no local optimum points or saddle points that might prevent the algorithm from reaching the global optimum solution.

We would like to thank our discussers for their constructive comments and hope that most of the issues that might have also confused other readers have been clarified. Before closing we would like to mention a recent application of decoupling enhancement similar to that discussed under "Further Research" which has come to our attention.¹² The application was a power flow solution using a self-adjusting accelerator in a Gauss-Seidel approach developed at Los Angeles Water and Power by Hubert and Hayes. Their use of decoupling enhancement mentioned in the authors' final closure reduced the required iterations in obtaining a power flow solution by a factor of four. For interested readers, the modifications of their Step 8 to obtain decoupling en-

hancement is equivalent to using equations (15) and (16) in our paper with

$$\rho_k = \frac{\sum_m R_{km}}{\sum_m X_{km}}$$

in place of the usual power flow equations and an acceleration factor in Step 8 of $RA(j) = \sum_m X_{km}$. The encouraging results obtained using decoupling enhancement in this application should provide incentive to direct research efforts in this direction.

- [12] F.J. Hubert and D.R. Hayes, "A Rapid Computer Solution for Power System Network Load-Flow," Transactions Paper No. 70 TP685-PWR presented at the IEEE Summer Power Meeting.