Design of multivariable feedback systems

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Synopsis

A canonical form for a multivariable linear control system is described. This canonical form is important because it enables a linear-feedback law to be chosen to produce arbitrary characteristic modes in the closed-loop system. The computational difficulties and methods suggested for minimising them are discussed.

List of symbols

$x, \hat{x}$ = state-variable vector in original- and transformed-co-ordinate space
u, u_0 = system inputs
y = system output
$A, \hat{A}$ = state-transition matrix in original- and transformed-co-ordinate space
$B, \hat{B}$ = distribution matrix in original- and transformed-co-ordinate space
$H, \hat{H}$ = output matrix in original- and transformed-co-ordinate space
$W(s)$, $W_c(s)$ = system-transfer-function matrix before and after feedback
$K, \hat{K}$ = feedback law in original- and transformed-co-ordinate space
T = basis-transformation matrix
$b_i, \hat{b}_i$ = ith column of $B, \hat{B}$
f_i = ith basis vector

A superscript prime will be used to denote matrix transposition

1 Introduction

Much control engineering is concerned with the choice of a feedback law to achieve desired objectives, such as optimisation with respect to some performance index, minimisation of the effects of noise, or reduction of the sensitivity of the system to plant-parameter variations. The objective we consider here is none of these: it is to achieve arbitrary dynamics of the system, or, in other words, arbitrary pole positions of the transfer-function matrix.

This problem has been discussed in detail for the case where there is a single input and the system states are available,[1] and the multiple-input problem has also been considered in an abstract fashion.[2] The dual problem of constructing an asymptotic estimator of the states with arbitrary poles has been considered[3, 4] for the multiple-output case. The first effective computational method for arbitrarily locating the poles of a feedback controller, or for a state estimator, is to be found in hitherto unpublished work;* this foreshadows some of the material presented here and presents an alternative solution to the problem.

In Section 2 of the paper, a canonical form for multiple-input, linear, time-invariant dynamical systems is presented, and, using this canonical form, it is shown in Section 3 how essentially arbitrary system dynamics can be achieved when the states are available. Appendix 7 deals with some of the computational difficulties occurring, and explains how to minimise them. Most of these difficulties arise from the necessity of determining possible linear dependences among a set of vectors.

In Section 4 the results are discussed; in particular, a comparison is made between the design method of this paper and methods derived from other approaches.

While the main contribution of the paper is to set down a computational method, it seems necessary to stress the apparently little known fact that the computations necessary to achieve arbitrary dynamics for a plant can, at least in theory, be performed, provided that the plant is 'completely controllable'. The method can also be used to stabilise plants with right-halfplane poles. Even if the states are not directly observable, but the plant is 'completely observable', there exists a controller, a cascade of a state estimator and a linear-feedback law, so that the combined plant and controller have arbitrary dynamics. Furthermore, the estimator may be designed to deal with and minimise the effects of noisy measurements.[5] We hope that these facts will soon become as much a part of control system theory as, for example, the fact that a damping ratio of 0·7 in a quadratic-denominator transfer function is, in some sense, an optimum.

2 Canonical form

We consider linear, time-invariant, dynamical systems described by the equations

$$\dot{x} = Ax + Bu \tag{1a}$$

$$y = H'x \tag{1b}$$

where x is an $n \times 1$ state vector, assumed fully measurable
u is an $r \times 1$ input vector
y is an $m \times 1$ output vector
A is an $n \times n$ transition matrix
B is an $n \times r$ distribution matrix, and
H' is an $m \times n$ output matrix

The transfer-function matrix relating y to u is given by[1]

$$W(s) = H'(sI - A)^{-1}B \tag{2}$$

The poles of $W(s)$ are therefore determined by the eigenvalues of the matrix A.

The simplementation of a feedback law K means that we ensure that

$$u = Kx + u_0 \tag{3}$$

where u_0 is an input to the new system, and u is the input of the old system; note that we are feeding back linear combinations of the states rather than the outputs. The case where outputs only are available can be solved by cascading an estimator with the controller designed according to the method of this paper.

The system with feedback is

$$\dot{x} = (A + BK)x + Bu_0 \tag{4a}$$

$$y = H'x \tag{4b}$$

and it evidently has a transfer function relating u_0 to y (which is the closed-loop transfer function of the system of eqn. 1 with feedback law K) of the form

$$W_c(s) = H'(sI - A - BK)^{-1}B \tag{5}$$

The problem of achieving arbitrary pole locations for $W_c(s)$ is thus the problem of selecting K, so that the matrix $A + BK$ has arbitrary eigenvalues.

For the purpose of computing the matrix K, it will be convenient to change the basis of the state space. This change of basis does not alter the open-loop or closed-loop response

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* LUENBERGER, D. G.: 'Canonical forms for multivariable systems', submitted for publication

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Paper 5201 C, first received 28th April and in final revised form 8th November 1966

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of the system, but makes the replacements

$$\left.\begin{aligned} A &\rightarrow \hat{A} = TAT^{-1} \\ B &\rightarrow \hat{B} = TB \\ H' &\rightarrow \hat{H}' = H'T^{-1} \\ K &\rightarrow \hat{K} = KT^{-1} \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (6)$$

where T is the matrix relating the state vectors in the two co-ordinate systems by

$$\hat{x} = Tx \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (7)$$

It is easy to verify that the expressions on the right-hand sides of eqns. 2 and 5 are invariant when the transformation of eqn. 6 is made.

Since, from the point of view of closed-loop response, the particular co-ordinate basis used in eqn. 1 is not important, it is permissible to replace A, B and H in eqn. 1 by $\hat{A}$, $\hat{B}$ and $\hat{H}$, and to choose a $\hat{K}$ to yield arbitrary dynamics. This is, in fact, what we shall do: selecting a nonsingular T yielding $\hat{A}$ and $\hat{B}$ that permit the choice of $\hat{K}$ by inspection, and then, by using eqn. 4, we can derive the feedback law necessary in the original basis.

In this Section, we shall be restricting attention to that part of the system relating the input to the states, i.e. that part of the system described by eqn. 1a. The system is termed 'completely controllable'[1] if the $n \times nr$ matrix

$$[B, AB, A^2B, \dots, A^{n-1}B] \quad . \; . \; . \; . \; . \; . \; . \quad (8)$$

has rank n. As pointed out in Reference 2, if it is possible to achieve arbitrary system dynamics, a necessary condition is that the system be completely controllable. We shall assume this to be the case, or, in other words, we shall assume that there are n linearly independent vectors in the following array:

$$\left.\begin{array}{llll} b_1 & Ab_1 & A^2b_1 \dots & A^{n-1}b_1 \\ b_2 & Ab_2 & A^2b_2 \dots & A^{n-1}b_2 \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ b_r & Ab_r & A^2b_r \dots & A^{n-1}b_r \end{array}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (9)$$

Here b_i is the ith column of the matrix B. The vectors in the array 9 are precisely the set of columns of the matrix $[B, AB, \dots, A^{n-1}B]$.

We can now proceed with the derivation of the canonical form. From the array 9, we shall derive a sequence of basis vectors for a transformed-co-ordinate system, and this sequence will be given in reverse order. That is, we shall list a series of vectors $f_n, f_{n-1}, \dots, f_1$, all linearly independent, so that f_j' in the transformed system becomes $f_j' = [0, 0, \dots, 0, 1, 0, \dots, 0]$ (a vector possessing zeros in every position except the jth). We note that the ordering of the b_i in array 9 is immaterial; a renumeration of them merely amounts to reordering the components of the input vector. We note further that we can assume $b_i \neq 0$, for, in the contrary case, the ith input will serve no purpose.

We consider the set of vectors b_1, Ab_1, A^2b_1 etc. Let r_1 be the least integer r, so that A^rb_1 is a linear combination of the earlier members of the sequence. Then we may write

$$A^{r_1}b_1 + \alpha_1 A^{r_1-1}b_1 + \dots + \alpha^{r_1}b_1 = 0 \quad . \; . \quad (10)$$

Such an r_1 must exist, for $r_1 > n$ would imply the existence of more than n linearly independent $n$vectors. Questions concerning the checking of linear independence are discussed in Appendix 7.

We then choose the last r_1 basis vectors as

$$\left.\begin{aligned} f_n &= b_1 \\ f_{n-1} &= Ab_1 + \alpha_1 b_1 \\ f_{n-2} &= A^2b_1 + \alpha_1 Ab_1 + \alpha_2 b_1 \\ &\cdot \\ &\cdot \\ &\cdot \\ f_{n-r_1+1} &= A^{r_1-1}b_1 + \alpha_1 A^{r_1-2}b_1 + \dots + \alpha_{r_1-1}b_1 \end{aligned}\right\} \quad . \quad (11)$$

If $r_1 = n$, there is no need to proceed further, since, at this stage, the full number n of basis vectors has been determined. In general, however, there remain some basis vectors yet to be defined. Before defining them, we draw attention to the constraints that the selection of eqn. 11 imposes on $\hat{A}$ and $\hat{B}$. First, f_n transforms into $\hat{f}_n$, with $\hat{f}_n' = [0, 0, \dots, 0, 1]$. Since f_n is the same as b_1 (see eqn. 11), we shall have

$$\hat{b}_1' = [0, 0, \dots, 0, 1] \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (12)$$

Secondly, we note that the equation $f_{n-1} = Ab_1 + \alpha_1 b_1$ transforms into $\hat{f}_{n-1} = \hat{A}\hat{f}_n + \alpha_1 \hat{f}_n$. Our knowledge of f_n and f_{n-1} then allows us to conclude that the last column of $\hat{A}$ has all elements zero, except the last two, which are 1 in the second-to-last place and $-\alpha_1$ in the last place.

Thus

$$\hat{A} = \begin{array}{c} \begin{array}{cc} n-1 & \quad 1 \end{array} \\ \left[\begin{array}{cccc|c} \times & . & . & \times & 0 \\ . & & & . & . \\ . & & & . & . \\ . & & & . & . \\ \times & . & . & \times & 0 \\ \times & . & . & \times & 1 \\ \times & . & . & \times & -\alpha_1 \end{array}\right] \end{array} n \quad . \; . \; . \; . \; . \; . \quad (13a)$$

where the $\times$ symbols to the left of the vertical line indicate elements which are, at the moment, unspecified.

In a similar fashion, the remaining relations in eqn. 11 serve to determine completely the last r_1 columns of $\hat{A}$, and it may be verified that

$$\hat{A} = \begin{array}{c} \begin{array}{cc} n - r_1 & \qquad\qquad r_1 \end{array} \\ \left[\begin{array}{cccc|ccccc} \times & . & . & \times & & & 0 & & \\ \hline . & & & . & 0 & 1 & & & 0 \\ . & & & . & & 0 & & . & . \\ . & & & . & . & . & & . & . \\ \times & . & . & \times & . & . & & . & 1 \\ \times & . & . & \times & -\alpha_{r_1} & -\alpha_{r_1-1} & . & . & -\alpha_1 \end{array}\right] \end{array} n \quad . \quad (13b)$$

Consider again the array 9. We have assigned $r_1 < n$ basis elements using only the entries in the first row of the array. Thus there are certainly other vectors in the array which are linearly independent of the first row, and so we examine the second row. If b_2 is linearly dependent on the vectors of the first row, so is every other vector of the second row. Consequently, if the second row possesses any vectors at all which are linearly independent of the first row, b_2 is one of them. If b_2 is linearly dependent, we pass on to the third row and examine b_3, and the fourth etc., if necessary. But note that a row must be reached whose first entry is linearly independent of the first row of the array, since the array contains n linearly independent vectors, as compared with the r_1 of the first row.

Without loss of generality, suppose b_2 is linearly independent of the first row (by reordering of the inputs, if necessary).

Now, starting with b_2, we form successively Ab_2, A^2b_2 etc., until $A_{r_2}b_2$, which is the first vector of the sequence which is linearly dependent on the earlier sequence members and the vectors A^ib_1 (or equivalently $f_n, f_{n-1}, \dots, f_{n-r_1+1}$). We may write

$$A^{r_2}b_2 + \beta_1 A^{r_2-1}b_2 + \dots + \beta_{r_2}b_2 \\ + m_1 f_{n-r_1+1} + \dots + m_{r_1}f_n = 0 \quad . \quad (14)$$

Continuing to select basis vectors in reverse order, we set

$$\left.\begin{aligned} f_{n-r_1} &= b_2 \\ f_{n-r_1-1} &= Ab_2 + \beta_1 b_2 \\ f_{n-r_1-2} &= A^2b_2 + \beta_1 Ab_2 + \beta_2 b_2 \\ &\cdot \\ &\cdot \\ &\cdot \\ f_{n-r_1-r_2+1} &= A^{r_2-1}b_2 + \beta_1 A^{r_2-2}b_2 + \dots + \beta_{r_2-1}b_2 \end{aligned}\right\} \quad (15)$$

Owing to the way r_2 is defined, all these vectors must be linearly independent. The selection of these vectors further defines $\hat{A}$ and $\hat{b}_2$. Thus $\hat{b}_2'$, being the same as $\hat{f}_{n-r_1}'$, by eqn. 15 is

$$\hat{b}_2' = [0, 0, \dots, 0, 0, 1, 0, \dots, 0] \quad . \; . \; . \; . \; . \quad (16)$$

the 1 being in the $(n - r_1)$th entry. Eqns. 15 also serve to to determine further columns of $\hat{A}$. It is not difficult to show by replacing eqn. 15 by the corresponding equations in the transformed-co-ordinate system, that

$$\hat{A} = \begin{array}{cc}
\begin{array}{ccc} n-r_1-r_2 & \qquad r_2 & \qquad\qquad r_1 \end{array} & \\
\left[\begin{array}{c:cccc:cccc}
\times\,.\,.\,\times & & 0 & & & & 0 & & \\ \hdashline
\cdot & 0 & 1 & & 0 & & & & \\
\cdot & & 0 & \cdot & \cdot & & & & \\
\cdot & \cdot & \cdot & & 1 & & 0 & & \\
 & -\beta_{r_2} & -\beta_{r_2-1} & \cdot\,\cdot & -\beta_1 & & & & \\ \hdashline
 & -m_1 & & & & 0 & 1 & & 0 \\
 & -m_2 & & & & & 0 & \cdot & \cdot \\
 & \cdot & & 0 & & \cdot & \cdot & \cdot & \cdot \\
 & \cdot & & & & \cdot & \cdot & & 1 \\
\times\,.\,.\,\times & -m_{r_1} & & & & -\alpha_{r_1} & -\alpha_{r_1-1} & \cdot\,\cdot & -\alpha_1
\end{array}\right] &
\begin{array}{c} n-r_1-r_2 \\ \\ \\ r_2 \\ \\ \\ \\ r_1 \\ \\ \end{array}
\end{array} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (17)$$

If, at this stage, the unknown part of $\hat{A}$ has not disappeared, controllability assures us that there exists b_3 (perhaps after reordering) in the array 9, which is independent of the basis vectors hitherto selected. Then b_3 will be used to generate another group of basis vectors, so that $\hat{A}$ can be further defined, and $\hat{b}_3$ is a vector consisting entirely of zeros, except for 1 in the $(n - r_1 - r_2)$th entry.

The complete-controllability assumption allows the continuation of this process until the unknown part of $\hat{A}$ disappears, and $\hat{A}$ is 'almost' the direct sum of a number of companion matrixes. $\hat{B}$ meanwhile contains some columns which have all zero entries except for one entry, these columns having a 1 in the last position, r_1 positions before the last, $(r_1 + r_2)$ positions before the end, and so on.

The matrix T (see eqn. 7), relating the two co-ordinate systems, may be written almost immediately as

$$T^{-1} = [f_1, f_2, \ldots, f_n] \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (18)$$

The matrix $\hat{A} = TAT^{-1}$ has the explicit form

$$\hat{A} = \begin{array}{cc}
\begin{array}{ccc} \qquad\qquad r_3 & \qquad\qquad\qquad r_2 & \qquad\qquad\qquad r_1 \end{array} & \\
\left[\begin{array}{c:cccc:cccc:cccc}
\cdot & & 0 & & & & 0 & & & & 0 & & \\ \hdashline
\cdot & 0 & 1 & & 0 & & & & & & & & \\
 & & 0 & \cdot & \cdot & & & & & & & & \\
\cdot\;\cdot\;\cdot & \cdot & \cdot & \cdot & \cdot & & 0 & & & & 0 & & \\
 & \cdot & \cdot & & 1 & & & & & & & & \\
 & -\gamma_{r_3} & -\gamma_{r_3-1} & \cdot\,\cdot & -\gamma_1 & & & & & & & & \\ \hdashline
 & -n_1 & 0 & \cdot\,\cdot & & 0 & 1 & & 0 & & & & \\
 & -n_2 & & & & & 0 & \cdot & \cdot & & & & \\
 & \cdot & & & & \cdot & \cdot & \cdot & \cdot & & & & \\
\cdot\;\cdot\;\cdot & \cdot & & & & \cdot & \cdot & & 1 & & & & \\
 & \cdot & & & & & & & & & 0 & & \\
 & -n_{r_2} & & & & -\beta_{r_2} & -\beta_{r_2-1} & \cdot\,\cdot & -\beta_1 & & & & \\ \hdashline
 & -n_{r_2+1} & 0 & \cdot\,\cdot & & -m_1 & 0 & \cdot\,\cdot & & 0 & 1 & & 0 \\
 & -n_{r_2+2} & 0 & \cdot\,\cdot & & -m_2 & 0 & \cdot\,\cdot & & & 0 & \cdot & \cdot \\
 & \cdot & & & & \cdot & & & & \cdot & \cdot & \cdot & \cdot \\
\cdot\;\cdot\;\cdot & \cdot & & & & \cdot & & & & & & & \\
 & \cdot & & & & \cdot & & & & \cdot & \cdot & & 1 \\
 & -n_{r_2+r_1} & 0 & \cdot\,\cdot & & -m_{r_1} & 0 & \cdot\,\cdot & & -\alpha_{r_1} & -\alpha_{r_1-1} & \cdot\,\cdot & -\alpha_1
\end{array}\right] &
\begin{array}{c} \\ \\ \\ r_3 \\ \\ \\ \\ \\ \\ r_2 \\ \\ \\ \\ \\ \\ r_1 \\ \\ \\ \end{array}
\end{array} \quad . \quad (19)$$

The matrix $\hat{B} = TB$ has the form

$$\hat{B} = \begin{array}{c}
\begin{array}{cc} \qquad p & \qquad\qquad r-p \end{array} \\
\left[\begin{array}{ccccc:cccc}
0 & 0 & 0 & \cdot & \cdot & \times & \cdot & \cdot & \times \\
0 & 0 & 0 & & & & & & \\
\cdot & \cdot & \cdot & & & & & & \\
\cdot & \cdot & 1 & & & \cdot & & & \cdot \\
 & 0 & 0 & & & \cdot & & & \cdot \\
 & 1 & \cdot & & & \cdot & & & \cdot \\
 & 0 & \cdot & & & & & & \\
0 & \cdot & \cdot & \cdot & \cdot & & & & \\
1 & 0 & 0 & \cdot & \cdot & \times & \cdot & \cdot & \times
\end{array}\right]
\end{array} \quad . \quad (20)$$

All columns to the left of the broken line will have all elements, except one, zero, and the number of such prescribed columns is the same as the number of diagonal blocks of A; this is, in turn, equal to the number of rows of the array 9 whose elements go towards making up the new co-ordinate basis. This number will be called p.

Example 1

Consider the following system:

$$\dot{x} = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 0 & -2 \\ 4 & 2 & -5 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u \quad . \; . \; . \; . \quad (21)$$

It can be easily verified that this system is not completely controllable from either u_1 or u_2 alone, but that it is completely controllable from the two together.

We have

$$b_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \; Ab_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \; A^2b_1 = \begin{bmatrix} -3 \\ -2 \\ -6 \end{bmatrix} \quad . \; . \; . \quad (22)$$

from which it follows that

$$A^2b_1 + 3Ab_1 + 2b_1 = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \quad (23)$$

Following the procedure of Section 2, we set

$$f_3 = b_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (24a)$$

$$f_2 = Ab_1 + 3b_1 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (25)$$

Carrying out similar calculations for b_2:

$$b_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \; Ab_2 = \begin{bmatrix} -2 \\ -2 \\ -5 \end{bmatrix} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (26)$$

Thus $Ab_2 + b_2 + 2f_2 - 4f_3 = 0 \quad . \; . \; . \; . \; . \; . \; . \quad (27)$

and, according to the procedure:

$$f_1 = b_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (28)$$

The resulting system equations under the transformation

$$\hat{x} = Tx \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (7)$$

with

$$T^{-1} = [f_1\, f_2\, f_3] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 3 & 1 \\ 1 & 2 & 0 \end{bmatrix}$$

and $$T = \begin{bmatrix} -2 & 0 & 1 \\ 1 & 0 & 0 \\ -3 & 1 & 0 \end{bmatrix} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (29)$$

are

$$\dot{\hat{x}} = \begin{bmatrix} -1 & 0 & 0 \\ -2 & 0 & 1 \\ 4 & -2 & -3 \end{bmatrix} \hat{x} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} u \quad . \; . \; . \; . \quad (30)$$

which is the canonical form for this system.

3 Selection of the feedback law

Before describing the way in which the feedback law is calculated to achieve arbitrary dynamics, we indicate how the dynamics of the system can be found readily from $\hat{A}$.

As will be recalled, the poles are determined as those points where $(sI - \hat{A})$ is singular, in other words, where

$$\det (sI - \hat{A}) = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (31)$$

For convenience, let us denote the diagonal blocks appearing in $\hat{A}$ as $A_1, A_2, \ldots, A_p$, starting with that block in the lower-right corner and working upwards towards the left. It is not hard to establish, using elementary properties of determinants, that

$$\det (sI - \hat{A}) = \prod_{i=1}^{p} \det (sI - A_i) \quad . \; . \; . \; . \; . \quad (32)$$

Moreover, an easy calculation yields

$$\det (sI - A_1) = s^{r_1} + \alpha_1 s^{r_1 - 1} + \ldots + \alpha_{r_1} \quad . \quad (33)$$

and similar expressions for $\det (sI - A_i)$.

Since the zeros of $\det (sI - \hat{A})$ are the zeros of the various $\det (sI - A_i)$, we can control the former through the latter. Thus if, with the aid of feedback, we can change $\alpha_1, \alpha_2, \ldots, \alpha_{r_1}$ and $\beta_1, \beta_2, \ldots, \beta_{r_2}$ etc. to arbitrary values, we can effectively achieve any desired set of zeros of $\det (sI - \hat{A})$. We say effectively, because there is a simple but minor restriction: each factor $\det (sI - A_i)$ of odd degree must contain at least one real zero, if we restrict ourselves to real coefficients. As a consequence, $\det (sI - \hat{A})$ must contain a minimum number of real zeros equal to the number of determinantal factors $\det (sI - A_i)$ which are of odd degree. Note that this number will depend on the exact basis selected for the transformed-co-ordinate system.

It will now be shown that the coefficients α_j in eqn. 33 can be varied arbitrarily by feedback. At the same time, the coefficients in the expansion of $\det (sI - A_i)$ $(i = 2, 3, \ldots)$ can be varied arbitrarily by feedback in a similar way.

We choose a feedback law $\hat{K}$ of the form

$$\hat{K} = \begin{array}{c} \\ \\ p \\ \\ \\ \\ r-p \end{array} \left[\begin{array}{ccccccccccccccccc} 0 & 0 & . & & & & & & & . & . & 0 & \kappa_{r_1} & \kappa_{r_1-1} & . & . & \kappa_1 \\ 0 & 0 & . & & . & . & 0 & \lambda_{r_2} & \lambda_{r_2-1} & . & . & \lambda_1 & 0 & . & . & . & 0 \\ 0 & 0 & \mu_{r_3} & \mu_{r_3-1} & . & . & \mu_1 & 0 & . & & & & & & . & . & 0 \\ & & & & & & & . & & & & & & & & & \\ & & & & & & & . & & & & & & & & & \\ & & & & & & & . & & & & & & & & & \\ \hdashline & & & & & & & 0 & & & & & & & & & \end{array}\right] \quad . \; . \; . \; . \; . \; . \quad (34)$$

Then $\hat{A} + \hat{B}\hat{K}$ is the same as $\hat{A}$, except that α_j is replaced by $\alpha_j - \kappa_j$, β_j by $\beta_j - \lambda_j$ etc., and

$$\begin{aligned} \det (sI - \hat{A} - \hat{B}\hat{K}) & \\ &= [s^{r_1} + (\alpha_1 - \kappa_1)s^{r_1 - 1} + \ldots + (\alpha_{r_1} - \kappa_{r_1})] \\ &\times [s^{r_2} + (\beta_1 - \lambda_1)s^{r_2 - 1} + \ldots + (\beta_{r_2} - \lambda_{r_2})] \\ & \quad . \; . \; . \; . \quad (35) \end{aligned}$$

The effect of the feedback law $\hat{K}$ is thus to vary the α_j, β_j etc., in a predictable fashion. The $\hat{K}$ necessary to achieve desired dynamics is immediately evident from eqns. 34 and 35, once the α_j, β_j etc. are known. Without the canonical form developed, the way in which $\hat{K}$ should be chosen is not clear.

In the original co-ordinate system, the feedback law is

$$K = \hat{K}T \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (36)$$

The matrix T can be derived from eqn. 19.

The preceding manipulations may become clearer if the structure of the canonical form is viewed in terms of a collection of coupled subsystems. Each companion matrix can be regarded as a separate subsystem of the complete dynamical system. The entries in the system matrix which are not within the individual companion matrixes represent coupling between the subsystems. By the nature of the construction of the canonical form, the coupling between two subsystems is in one direction only. Thus the first subsystem has output feeding the inputs of all other subsystems, but none of them feeds the first subsystem. Since the first subsystem is unaffected by the others, its poles can be chosen arbitrarily by appropriate self feedback. Similar arguments apply to the other subsystems. This is merely an exploitation of the well known fact that, as far as pole locations are concerned, two subsystems are independent, provided that they are coupled in one direction only.

In addition to the arbitrariness in K resulting from the arbitrariness of the canonical form, an additional arbitrariness arises, for the following reason. In achieving the desired zeros of $\det (sI - \hat{A} - \hat{B}\hat{K})$, it does not matter by which polynomial factor on the right-hand side of eqn. 35 the desired zero is produced. The various possible distributions of the desired zeros among the polynomial factors will yield different $\hat{K}$ (and thus K).

Example 2

The poles of the system in example 1 are easily seen to be $\lambda = -1, \; -1, \; -2$. Suppose we wish to move these to $\lambda = -3, \; -3, \; -3$ by state-vector feedback.

Using the canonical form developed earlier and the results of this Section, we take

$$\hat{K} = \begin{bmatrix} 0 & -7 & -3 \\ -2 & 0 & 0 \end{bmatrix} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (37)$$

whence

$$\hat{B}\hat{K} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -7 & -3 \end{bmatrix} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (38)$$

and $$\hat{A} + \hat{B}\hat{K} = \begin{bmatrix} -3 & 0 & 0 \\ -2 & 0 & 1 \\ 4 & -9 & -6 \end{bmatrix} \quad . \; . \; . \; . \; . \; . \; . \quad (39)$$

which has three poles at -3.

The feedback law in the original-co-ordinate system is

$$u = Kx \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (40)$$

where $$K = \hat{K}T = \begin{bmatrix} 2 & -3 & 0 \\ 4 & 0 & -2 \end{bmatrix} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (41)$$

4 Discussion and conclusions

Unlike the single-input case, the problem of closed-loop pole location by state-vector feedback does not possess a unique solution in the multi-input case. There are a number of possible schemes for choosing the basis vectors that determine the co-ordinate transformation, and a number of ways to distribute the poles among the subsystems determined by the resulting canonical form.[1]

The flexibility of these choices must be exploited, in any particular practical problem, by engineering judgment based on other design objectives.

The approach discussed in this paper may be particularly suited to some kinds of control situations, because the method tends to stress control by only a few of the available inputs. For instance, if the plant is completely controllable from the first input (as well as from all of them), the procedure for selecting the basis vectors will lead to a solution using only that one input for control. Other methods* might include other inputs, each contributing approximately equally to control. Likewise, if the plant is completely controllable from the first two inputs, the approach of this paper would lead to control from these two. In general, the resulting solution depends strongly on the *a priori* ordering of the inputs before the selection process is begun. Thus, the engineer can order the inputs according to desirability (based on costs, reliability etc.). The control design procedure will then tend to emphasise the inputs in that order.

Obviously, this is not always the best procedure, and it is not easy to compare different solutions until they are all designed and analysed.

In the case of plants where the state variables cannot be measured and only noisy outputs are available, the state-estimation technique of Reference 5 will result in some states being more accurately known than others. The choice of a feedback law could then conceivably be based on a consideration of the linear combinations of the states which are found to be the best known.

In any but the smallest systems, computation would presumably be carried out with a computer, with the complexity of the calculations increasing no more than linearly with the dimension of the state space. Moreover, it would seem reasonable, in some situations, to program the computer to compute all possible feedback laws, or at least a large number, and then to carry out comparison of these feedback laws on the basis of some programmed criterion of the type mentioned.

5 Acknowledgment

This research was supported by Joint Services Contract Nonr 225 (83).

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7 Appendix

Use of the Gram–Schmidt procedure for checking independence

The main computational difficulty associated with the method proposed in this paper is the determination of linear dependence among a set of vectors. In general, this type of problem is computationally difficult, no matter what approach is employed, owing to the inherent ambiguity in distinguishing zero from nonzero numbers in the presence of roundoff errors. With this apparently ubiquitous limitation in mind, it is possible, however, to address the problem of effectively determining a computational scheme for determining the linear dependence required for development of the canonical form in Section 2.

The basic approach suggested here is to apply the Gram–Schmidt procedure to the array 9, moving by rows. Explicitly, the procedure is to define

$$\boldsymbol{e}_k = \frac{v_k - \sum_{i=1}^{k-1} (v_k' \boldsymbol{e}_i)\boldsymbol{e}_i}{\left|\left|v_k - \sum_{i=1}^{k-1} (v_k' \boldsymbol{e}_i)\boldsymbol{e}_i\right|\right|} \qquad (42)$$

where v_k represents the kth element in the array 9 (therefore v_k is of the form $A^j\boldsymbol{b}_l$). If the denominator of eqn. 42 is zero, the vector $\boldsymbol{e}_k$ is taken to be zero.

The result of this procedure is a sequence of orthogonal vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ etc. each having a norm of unity or zero. The first k vectors in the sequence span the same space as the first k vectors in the array 9.

If a vector $\boldsymbol{e}_k$ is found to be zero, the corresponding vector in the array 9 is linearly dependent on the previous vectors. Furthermore, all remaining vectors in that row of the array will also be linearly dependent on the previous vectors.

If $v_k = A^j\boldsymbol{b}_l$ is linearly dependent on the previous vectors, the exact dependency can be found by the equation

$$v_k = \sum_{i=1}^{k-1} (v_k, \boldsymbol{e}_i)\boldsymbol{e}_i \qquad (43)$$

which follows immediately from the orthogonality of the $\boldsymbol{e}_i$s. Since the $\boldsymbol{e}_i$s are themselves linear combinations of previous vectors in the array, eqn. 43 can be used to determine easily the dependency in the form of eqn. 10 or, more generally, of eqn. 14.

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* LUENBERGER, D. G.: *loc. cit.*