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Differential Games with Imperfect State Information

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Abstract—**Nondeterministic differential games of imperfect information are considered, with particular emphasis on the case of a linear system, a quadratic cost functional, and independent white Gaussian noises additively corrupting the observable output measurements. Solutions are presented for a number of particular cases of this problem, including those in which one of the two controllers has either no information or, under certain additional restrictions, perfect measurements of the state vector. In each case the optimal control for each controller is shown to be closely related to that which would result by assuming a separation theorem to hold. Furthermore, the various terms in the resulting optimal cost are shown to be readily assignable to the appropriate contributing source, such as the optimal cost that would result if the problem were instead a deterministic one with perfect information, the effect of estimation errors, or the effect of measurement errors.**

I. INTRODUCTION

CONSIDERABLE attention has been given recently to the study of differential games, e.g., [1], [2], [5], [7], [8], [12]. Most of the literature in this area has been concerned with the development of a theory of deter-

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ministic differential games of perfect information, i.e., differential games in which both the controllers or "players" are assumed to know exactly at each time the total system state.

In this paper consideration is given to differential games in which each controller has access only to noise-corrupted measurements of some (not necessarily the same) linear function of the system state. Attention is directed primarily towards the special case of a linear system, a quadratic cost functional, and independent white Gaussian noises additively corrupting the output measurements. The principal purpose here is to present solutions to a number of particular cases of this problem, demonstrating that in each case considered the optimal closed-loop solution for each controller is closely related to that which would result from a "separation theorem" or "certainty equivalence principle" analogous to that which holds in the corresponding one-sided optimal control situation [6], [9], [11], [14]. Furthermore, it is shown that in each case the necessary state estimates may be generated either from linear dynamic systems of the Kalman filter type [10] or from singular estimators.

In the nondeterministic one-sided optimal control problem with imperfect information, the assumption of a linear system, a quadratic cost functional, and white Gaussian noise additively corrupting the output measurements is sufficient to insure that the optimal closed-loop control is given by a separation theorem or certainty equivalence principle [6], [9], [11], [14], i.e., the optimal closed-loop control may be implemented in two distinct steps: first, the best estimate of the system state (in the least-mean-square-error sense) is formed from the available noisy measurements, and second, this best estimate is used in place of the (unknown) state in the optimal feedback control law for the corresponding deterministic optimal control problem of perfect information. In this case, moreover, the best estimate of the system state may be generated in real time using a Kalman filter [10]. In the case of nondeterministic differential games of imperfect information, however, the corresponding assumption of a linear system, a quadratic cost functional, and independent white Gaussian noises additively corrupting the output measurements is not sufficient for the optimal closed-loop controls to be given by a separation theorem. Indeed, it appears that even under these assumptions a solution to the nondeterministic differential game problem will require in general infinite-dimensional state estimators or, equivalently, the preservation and use in the control at each time of the entire output function measured up to that time. One is, therefore, led to consider a number of particular cases of this linear system and quadratic cost functional differential game problem with the output measurements additively corrupted by independent white Gaussian noises. Since each controller may be allowed either noisy output measurements, perfect state measurements, or no measurements at all, nine separate particular cases of this problem may be posed; due to considerations of symmetry, however, only six of these nine problems are basically different. Five of these six distinct problems are considered in this paper, while the remaining one is the subject of a forthcoming paper.

In each of these cases the optimal closed-loop control for each controller is closely related to a separation theorem. Since under a separation theorem the implementation of the optimal controls involves the optimal control laws for the corresponding deterministic problem of perfect information, the next section is devoted to a review of this problem in the case where the system is linear and the cost functional quadratic. Our solution to each of the nondeterministic differential game problems indicated rests upon a theorem which gives sufficient conditions for a pair of closed-loop control laws to be optimal. Since this theorem may be proved under considerably less restrictive conditions than those of a linear system, a quadratic cost functional, and independent white Gaussian noise corrupting the output measurements, we proceed to do so by first considering in Section III a more general problem (Problem 2). Following the sufficiency theorem and its proof, Section III is completed by the formulation of a basic problem involving a linear system, a quadratic cost functional, and independent white Gaussian noises additively corrupting the measurements. The remaining sections of the paper are then devoted to an examination of the previously indicated particular cases of this basic problem.

The notation used conforms closely to that which is common in optimal control theory. Matrices are denoted by upper case letters, vectors are denoted by lower case letters, while time is denoted by t, τ, and, occasionally, s. The notation for functions and their values is as follows: $z(\cdot)$ denotes a function, $z(t)$ its value at time t, and z a value in the range of $z(\cdot)$. Euclidean n space is denoted E^n, the superscript 1 being omitted in the case of the reals. Transposition of a matrix or vector is denoted by a prime, while the trace of a matrix B is denoted $\operatorname{tr}\{B\}$. Finally, the time derivative of an entity ( ) is denoted $(\dot{\ })$.

II. Review of the Deterministic Differential Game Problem for Linear System and Quadratic Cost Functional

The deterministic differential game problem of perfect information involving a linear system and a quadratic cost functional has been examined by a number of authors, e.g., [1], [7]. We present here a statement of the problem and give its solution as a proposition.

Problem 1

Consider a linear continuous-time system governed by the vector differential equation

$$\dot{x} = \frac{dx}{dt} = F(t)x + G_1(t)u(t) + G_2(t)v(t), \quad x(t_0) = x_0 \quad (1)$$

where the n vector $x(t)$ is the system state; the control vectors $u(t)$ and $v(t)$ are of dimension p and q, respectively; and the matrices $F(t)$, $G_1(t)$, and $G_2(t)$ have the appropriate dimensions. Consider also a quadratic cost functional

$$J_\tau[u,v] = \int_\tau^T (u'Q_1u + v'Q_2v)dt + x'(T)C'Cx(T) \quad (2)$$

where the matrices $Q_1(t)$ and $Q_2(t)$ are symmetric and, respectively, positive definite and negative definite; the matrix C has dimension $m \times n$ with $m \leq n$; and the final time T is fixed and finite. The objective is to find (if they exist) closed-loop control laws $u^0(\cdot,\cdot): X \times [t_0,T] \rightarrow E^p$ and $v^0(\cdot,\cdot): X \times [t_0, T] \rightarrow E^q$ which are optimal in the sense that, for any closed-loop control laws $u(\cdot,\cdot)$ and $v(\cdot,\cdot)$, there holds

$$J_{t_0}[u^0,v] \leq J_{t_0}[u^0,v^0] \leq J_{t_0}[u,v^0] \quad (3)$$

i.e., we seek, if it exists, a saddle point of $J_{t_0}[u,v]$.

It is noted that the analysis and discussion that follow may be readily extended to the case where terms of the form

$$\int_\tau^T [x'R_1x + (x-d)'R_2(x-d)]dt + [x(T) - d(T)]'R_3[x(T) - d(T)] \quad (4)$$

are added to the cost functional (2), where R_1, R_2, and R_3 are symmetric non-negative definite matrices, and $d(\cdot)$ is a prescribed n-vector function defined over $[t_0,T]$.

The solution to Problem 1 is given by the following proposition.

Proposition 1: The closed-loop control laws which solve Problem 1 exist and are unique under conditions to be defined later and are given by

$$u^0(t) = -Q_1^{-1}(t)G_1'(t)P(t)x(t) \quad (5a)$$

$$v^0(t) = -Q_2^{-1}(t)G_2'(t)P(t)x(t) \quad (5b)$$

where the $n \times n$ matrix $P(t)$ is the symmetric non-negative definite solution to the matrix Riccati equation

$$\dot{P} + PF(t) + F'(t)P - P[G_1(t)Q_1^{-1}(t)G_1'(t) + G_2(t)Q_2^{-1}(t)G_2'(t)]P = 0 \quad (6)$$

with boundary condition

$$P(T) = C'C. \quad (7)$$

The corresponding optimal cost is

$$J_{t_0}[u^0,v^0] = x'(t_0)P(t_0)x(t_0). \quad (8)$$

A necessary and sufficient condition for the closed-loop controls given by (5) to be the unique well-defined solutions to Problem 1 is that a unique non-negative definite matrix $P(t)$ solving (6) with boundary condition (7) exists for all $t \in [t_0,T]$. This is equivalent to the condition that there are no conjugate points in the interval $[t_0,T]$, which is, in turn, equivalent to the condition that the matrix

$$K(t) = I + C \int_t^T \Phi(T,\tau)[G_1(\tau)Q_1^{-1}(\tau)G_1'(\tau) + G_2(\tau)Q_2^{-1}(\tau)G_2'(\tau)]\Phi'(T,\tau)d\tau\, C' \quad (9)$$

is positive definite for all $t \in [t_0,T]$.

A detailed discussion of the problem and its solution as they are formulated here may be found in [13]. Ho, Bryson, and Baron [7] and Baron [1] contain a solution for the corresponding pursuit–evasion problem which is the counterpart of that given here. The pursuit–evasion problem is the special case of Problem 1 in which the system (1) represents two separate systems, corresponding to two maneuverable adversaries, and the matrix C in (2) is of the form $C = [A \vdots -A]$, i.e., the third term in (2) depends only on the difference between the final states of the two adversaries. Since these structural properties of the pursuit–evasion game were exploited to reduce the dimensions of various vectors by both Ho *et al.* [7] and Baron [1] early in their analyses, the transformation of their solutions into the form presented here requires some algebraic manipulations. On the other hand, to simply identify the terms and expressions in their solutions which correspond to those given here is almost immediate.

III. General Problem and Sufficiency Theorem

In this section we first formulate a somewhat general differential game problem with imperfect information and state and prove a theorem which gives sufficient conditions for a pair of closed-loop control laws to be optimal. It is this theorem that forms the basis for our solutions to the various problems considered in later sections.

Problem 2

Consider a system described by the vector differential equation

$$\dot{x} = f(x,u(t),v(t),t) \quad (10)$$

to which Controller 1, controlling u, has available measurements of the form

$$y_1(t) = h_1(x(t),w_1(t)) \quad (11a)$$

while Controller 2, controlling v, has available measurements

$$y_2(t) = h_2(x(t),w_2(t)). \quad (11b)$$

The n vector $x(t)$ is the system state, $u(t)$ and $v(t)$ are the control vectors (dimensions p and q, respectively), and $y_1(t)$ and $y_2(t)$ are the measured output vectors (dimensions $m_1 \leq n$ and $m_2 \leq n$, respectively). The functions $f(\cdot,\cdot,\cdot,\cdot)$, $h_1(\cdot,\cdot)$, and $h_2(\cdot,\cdot)$ are assumed sufficiently smooth. The random disturbances $w_1(t)$ and $w_2(t)$ are vectors whose statistics are assumed known by both controllers, and the initial state $x(t_0)$ is assumed by each controller to be a random vector with prescribed statistics. It is assumed that each controller knows his opponent's assumptions on the a priori distribution of the initial state $x(t_0)$. Both controllers make the same assumptions on the statistics of the random disturbances w_1 and w_2. Consider also the cost functional

$$J_\tau[u,v] = \int_\tau^T g(x,u,v,t)dt + L(x(T)) \tag{12}$$

where $t_0 \leq \tau \leq T$, the final time T is fixed, and the functions $g(\cdot,\cdot,\cdot,\cdot)$ and $L(\cdot)$ are continuous in all variables. Denote by $Y_i(t)$, $i = 1,2$, the output function measured by Controller i over the interval $[t_0,t)$, i.e.,

$$Y_i(t) = \{(y_i(s),s)\colon \quad s \in [t_0,t)\}. \tag{13}$$

It is understood that the measurement set $Y_i(t)$ includes the a priori information available to Controller i so that, in particular, $Y_i(t_0)$ is simply this a priori information. Denoting by $\{Y_i\}$ the set of possible $Y_i(T)$, find (if they exist) the closed-loop control laws

$$u^0(\cdot,\cdot)\colon \{Y_1\} \times [t_0,T] \rightarrow E^p \tag{14a}$$

$$v^0(\cdot,\cdot)\colon \{Y_2\} \times [t_0,T] \rightarrow E^q \tag{14b}$$

which are optimal in the sense that, for any closed-loop control laws $u(\cdot,\cdot)$ and $v(\cdot,\cdot)$, there holds for all $\tau \in [t_0,T]$

$$E\{J_\tau[u^0,v^0]|Y_1(\tau)\} \leq E\{J_\tau[u,v^0]|Y_1(\tau)\} \tag{15a}$$

$$E\{J_\tau[u^0,v]|Y_2(\tau)\} \leq E\{J_\tau[u^0,v^0]|Y_2(\tau)\}. \tag{15b}$$

It is understood that the map (14a) is to be interpreted to mean that $u^0(t)$ will be of the form $u^0(Y_1(t),t)$; a similar interpretation applies to (14b).

It should be noted at the outset that the formulation of this problem differs in an important respect from that of Problem 1. This is the replacement of the double inequality (3) by the pair of inequalities (15), a modification made necessary by the fact that each of the two controllers conditions his expectation of the cost on his individual noisy measurements and prior statistics of the initial state, both of which may differ in general from those of his opponent. The game is thus in some sense nonzero sum. The first inequality (15a) represents the pessimistic or defensive viewpoint of Controller 1, the second the pessimistic viewpoint of Controller 2. Together they constitute an optimality criterion that may be interpreted intuitively to mean that although the two controllers will not in general "agree" on the optimal cost, they are required to "agree" on a pair of (defensive) optimal control laws $u^0(\cdot,\cdot)$ and $v^0(\cdot,\cdot)$.

In the following theorem, Z is a space of vectors z which are suitably identified in each of the subsequent applications of the theorem. In all these applications $z(t)$ is either an estimate of the state $x(t)$, the error in a certain state estimate, or identically the null vector (in which case the space Z may be suppressed). In any event it is assumed that the vector function $z(\cdot)$ defined over $[t_0,T]$ satisfies a differential equation of the form

$$\dot{z} = r(x(t),z,u(t),v(t),t)$$

with the function $r(\cdot,\cdot,\cdot,\cdot,\cdot)$ sufficiently smooth.

Theorem 1

A sufficient condition for the closed-loop control laws $u^0(\cdot,\cdot)$ and $v^0(\cdot,\cdot)$ to solve Problem 2 is that there exist two real-valued functions $I_1(\cdot,\cdot,\cdot)\colon X \times Z \times [t_0,T] \rightarrow E$ and $I_2(\cdot,\cdot,\cdot)\colon X \times Z \times [t_0,T] \rightarrow E$, each differentiable in each variable, which together with $u^0(\cdot,\cdot)$ and $v^0(\cdot,\cdot)$ satisfy for all $t \in [t_0,T]$ the following five conditions. Defining for all t the scalar function $S_1(\cdot,\cdot,\cdot,\cdot,\cdot)$, $i = 1,2$, by

$$S_i(x,z,u,v,t) = I_{it}(x,z,t) + I_{ix}(x,z,t)f(x,u,v,t) + I_{iz}(x,z,t)r(x,z,u,v,t) + g(x,u,v,t) \tag{16}$$

the five conditions are

1) $\min\limits_{u} E\{S_1(x(t),z(t),u,v^0(Y_2(t),t),t)|Y_1(t)\} = 0$

2) $E\{S_1(x(t),z(t),u^0(Y_1(t),t),v^0(Y_2(t),t),t)|Y_1(t)\} = 0$

3) $\max\limits_{v} E\{S_2(x(t),z(t),u^0(Y_1(t),t),v,t)|Y_2(t)\} = 0$

4) $E\{S_2(x(t),z(t),u^0(Y_1(t),t),v^0(Y_2(t),t),t)|Y_2(t)\} = 0$

5) $I_i(x,z,t) = L(x), \quad i = 1, 2.$

Proof: For any u we have from 1)

$$E\{S_1(x(t),z(t),u,v^0(Y_2(t),t),t)|Y_1(t)\} \geq 0.$$

Taking an additional expectation conditioned on $Y_1(\tau)$, $\tau \leq t$, and integrating with respect to t over the interval $[\tau,T]$, we have

$$\int_\tau^T E[E\{S_1(x(t),z(t),u,v^0(Y_2(t),t),t)|Y_1(t)\}|Y_1(\tau)]dt \geq 0.$$

Cancelling the inner expectation since $Y_1(\tau)$ is a subset of $Y_1(t)$ for $\tau \leq t$, reversing the order of expectation and integration, and substituting (16), we have

$$E\left[\int_\tau^T I_{1t}(x(t),z(t),t) + I_{1x}(x(t),z(t),t)f(x(t),u,v^0\,(Y_2(t),t),t) + I_{1z}(x(t),z(t),t)r(x(t),z(t),u,v^0\,(Y_2(t),t),t)dt\ |Y_1(\tau)\right]$$

$$\geqq -E\left[\int_\tau^T g(x(t),u,v^0(Y_2(t),t),t)dt\ |Y_1(\tau)\right].$$

Noting that the integrand of the left-hand side is (d/dt) $I_1(x(t),z(t),t)$, it then follows that

$$E\left[I_1(x(T),z(T),T) - I_1(x(\tau),z(\tau),\tau) + \int_\tau^T g(x(t),u,v^0(Y_2(t),t),t)dt\ |Y_1(\tau)\right] \geq 0.$$

Using condition 5), we note that the first and third terms are, from (12), $J_\tau[u,v^0]$. Transposition of terms then gives

$$E\{I_1(x(\tau),z(\tau),\tau)|Y_1(\tau)\} \leq E\{J_\tau[u,v^0]\,|\,Y_1(\tau)\}. \tag{17}$$

It is readily checked that, using at each time t the control $u^0(Y_1(t),t)$ defined by condition 2), equality holds throughout the preceding algebra, i.e.,

$$E\{I_1(x(\tau),z(\tau),\tau)|Y_1(\tau)\} = E\{J_\tau[u^0,v^0]|Y_1(\tau)\}. \quad (18)$$

Combining (17) and (18) we thus have

$$E\{J_\tau[u^0,v^0]|Y_1(\tau)\} \leq E\{J_\tau[u,v^0]|Y_1(\tau)\} \quad (15a)$$

which is seen to hold for all τ since τ may be chosen arbitrarily in $[t_0,T]$. A symmetrical argument using conditions 3) through 5) completes the proof.

It is seen from (18) that, from the viewpoint of Controller 1, the conditional expectation of $I_1(x(t),z(t),t)$ gives the conditional expectation of the optimal cost $J_t[u^0,v^0]$ from time t. Similarly, it is seen from (16) that, from the viewpoint of Controller 1, the conditional expectation of $S_1(x(t),z(t),u(t),v^0(Y_2(t),t),t)$ is the conditional expectation of the time rate of change of $I_1(x(t), z(t),t)$ plus the conditional expectation of the time rate of decrease of the cost $J_t[u,v^0]$ at time t when control $u(t) = u(Y_1(t),t)$ is applied. Conditions 1) and 2) of Theorem 1 state that, from the viewpoint of Controller 1, for any control $u(t) = u(Y_1(t),t)$ the conditional expectation of the time rate of change of $I_1(x(t),z(t),t)$ is not less than the expected time rate of increase in cost $J_t[u,v^0]$, while for the choice $u(Y(t),t) = u^0(Y(t),t)$, the two are equal. Entirely analogous remarks apply to the interpretation of $I_2(x(t),z(t),t)$ and $S_2(x(t),z(t),u^0(Y_2(t),t),v(t),t)$ from the viewpoint of Controller 2.

Although Problem 2 and the sufficient conditions for its solution given by Theorem 1 are of considerable generality, it appears possible to find functions $I_1(\cdot,\cdot,\cdot)$ and $I_2(\cdot,\cdot,\cdot)$ with corresponding control laws $u^0(\cdot,\cdot)$ and $v^0(\cdot,\cdot)$, which together satisfy the hypotheses of the theorem only in a number of special cases involving a linear system, a quadratic cost functional, and white additive Gaussian noises. This is indeed not altogether surprising in view of the currently available results in the corresponding stochastic optimal control situation [6], [9], [11], [14.

Accordingly, we now formulate a basic differential game problem of imperfect information for which the system is linear, the cost functional quadratic, and independent white zero-mean Gaussian noises additively corrupt the measurements. All problems considered in later sections are then special cases of this basic problem.

Problem 3

Consider the special case of Problem 2 involving the linear system described by the vector differential equation

$$\dot{x} = F(t)x + G_1(t)u(t) + G_2(t)v(t) \quad (1)$$

and the quadratic cost functional

$$J_\tau[u,v] = \int_\tau^T (u'Q_1u + v'Q_2v)dt + x'(T)C'Cx(T). \quad (2)$$

The matrices $F(t)$, $G_1(t)$, and $G_2(t)$ have the appropriate dimensions; the matrices $Q_1(t)$ and $Q_2(t)$ are symmetric and, respectively, positive definite and negative definite; the matrix C has dimension $m \times n$ with $m \leq n$; and the final time T is fixed. The measurements available to Controller 1 are of the form

$$y_1(t) = H_1(t)x(t) + w_1(t) \quad (19a)$$

while Controller 2 has available measurements

$$y_2(t) = H_2(t)x(t) + w_2(t) \quad (19b)$$

where the matrices $H_1(t)$ and $H_2(t)$ are, respectively, $m_1 \times n$ and $m_2 \times n$ with m_1, $m_2 \leq n$. The noises $w_1(t)$ and $w_2(t)$ are assumed to be independent, white, zero mean, and Gaussian with covariances

$$\text{cov}\,[w_1(t),w_1(\tau)] = W_1(t)\delta(t-\tau) \quad (20a)$$

$$\text{cov}\,[w_2(t),w_2(\tau)] = W_2(t)\delta(t-\tau). \quad (20b)$$

It is assumed for simplicity that both controllers consider the initial state $x(t_0)$ to be a Gaussian random vector uncorrelated with $w_1(t)$ and $w_2(t)$ for all t and having mean $\bar{x}_0$ and covariance

$$\text{cov}\,[x(t_0),x(t_0)] = M_0. \quad (21)$$

Extension to the case of different a priori estimates of the mean and different assumed covariances is in most cases straightforward. Again, note that the analyses and discussion that follow may be readily extended to the case where terms of the form (4) are added to the cost functional (2). Furthermore, it is formally straightforward to extend Theorem 1 to the situation where, in addition to noises corrupting the measurements, noise is also present in the system dynamics (1), and to extend the analysis in Section IV to the case where this noise in the dynamics is additive, white, Gaussian, and independent of the measurement noises. As in the corresponding stochastic optimal control situation, noises in the dynamics or measurements which are Markov with rational spectra instead of white may be included in the preceding formulation by the standard procedure of defining additional state variables. Finally, we note that the extension of Theorem 1 to the nonzero sum game in which the two controllers wish to extremize different cost functionals (i.e., where two different cost functionals of the form (2) are involved) is trivial.[1]

IV. PROBLEMS IN WHICH ONE CONTROLLER HAS NO MEASUREMENTS

The first case of Problem 3 to be considered is that in which only Controller 1 has available noise-corrupted output measurements, while Controller 2 has no measurements available to him and is, therefore, restricted to using only a priori information. This type of problem may represent, for instance, the encounter of an intercepting missile, guided by radar, and a blind incoming missile.

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[1] We do not mean to imply that the nonzero sum game is a trivial extension of zero sum games.

We introduce the notation

$$\hat{x}_i(t|t) = E\{x(t)|Y_i(t)\}, \quad i = 1,2$$

and

$$\tilde{x}_i(t|t) = x(t) - \hat{x}_i(t|t), \quad i = 1,2$$

and note that since in all cases considered in this paper the state $x(t)$ and the measurements $y_i(\tau)$ are Gaussian random vectors, $\hat{x}_i(t|t)$ is a linear function of $Y_i(t)$. It is also the minimal-mean-square-error estimate of $x(t)$ given $Y_i(t)$.

Problem 4

Consider the special case of Problem 3 in which Controller 1 has available the measurements (19a) while Controller 2 has no measurements available to him and is thus restricted to using only the a priori information, i.e., referring to (19b), $H_2(t) \equiv 0$ (null matrix) and $Y_2(t) \equiv Y_2(t_0)$ for all t.

A solution to this problem is available using Theorem 1 and is given by the following proposition.

Proposition 2: For Problem 4, the following functions $I_1(\cdot,\cdot,\cdot)$ and $I_2(\cdot,\cdot,\cdot)$ and the control laws $u^0(\cdot,\cdot)$ and $v^0(\cdot,\cdot)$ satisfy the hypotheses of Theorem 1, so that $u^0(\cdot,\cdot)$ and $v^0(\cdot,\cdot)$ solve Problem 4:

$$\begin{aligned} I_1(x(t),\tilde{x}_2(t|t),t) &= I_2(x(t),\tilde{x}_2(t|t),t) \\ &= x'(t)P(t)x(t) - \tilde{x}_2'(t|t)N(t)\tilde{x}_2(t|t) + b(t) \end{aligned} \tag{22}$$

$$\begin{aligned} u^0(t) = -Q_1^{-1}(t)G_1'(t)\,[P(t)\hat{x}_1(t|t) \\ + N(t)\,(\hat{x}_2(t|t) - \hat{x}_1(t|t))] \end{aligned} \tag{23a}$$

$$v^0(t) = -Q_2^{-1}(t)G_2'(t)P(t)\hat{x}_2(t|t) \tag{23b}$$

where the symmetric matrix $P(t)$ satisfies the Riccati equation (6) with boundary condition (7), while the symmetric matrix $N(t)$ satisfies the differential equation

$$\begin{aligned} &\dot{N} + NF(t) + F'(t)N \\ &\quad - P(t)\,[G_1(t)Q_1^{-1}(t)G_1'(t) + G_2(t)\,Q_2^{-1}(t)G_2'(t)\,]P(t) \\ &\quad + (P(t) - N)G_1(t)Q_1^{-1}\,(t)G_1'(t)(P(t) - N) = 0 \end{aligned} \tag{24}$$

with boundary condition

$$N(T) = 0. \tag{25}$$

The scalar $b(t)$ is the solution to the differential equation

$$\begin{aligned} \dot{b} = -\mathrm{tr}\{M(t)\,[P(t) - N(t)\,]G_1(t)Q_1^{-1}(t)G_1'(t) \\ \cdot [P(t) - N(t)]\} \end{aligned} \tag{26}$$

with boundary condition

$$b(T) = 0 \tag{27}$$

where the symmetric matrix $M(t)$ defined by

$$M(t) = E\{\tilde{x}_1(t|t)\tilde{x}_1'(t|t)|Y_1(t)\} \tag{28}$$

satisfies the differential equation

$$\dot{M} = F(t)M + MF'(t) - MH_1'(t)W_1^{-1}(t)H_1(t)M \tag{29}$$

with boundary condition [recall (21)]

$$M(t_0) = \mathrm{cov}\,[x(t_0),x(t_0)] = M_0. \tag{30}$$

A proof of this proposition is given in the Appendix. We content ourselves here with the identification of the necessary state estimators, a heuristic derivation of the result given by the proposition, and an interpretation of the solution.

Substitution of (23) into the system equation (1) gives

$$\begin{aligned} \dot{x} = Fx - G_1Q_1^{-1}G_1'[P\hat{x}_1 + N(\hat{x}_2 - \hat{x}_1)] \\ - G_2Q_2^{-1}G_2'Px_2. \end{aligned} \tag{31}$$

It then follows that the optimal estimate of the state by Controller 2, given only a priori information, i.e., no output measurements, is given by

$$\begin{aligned} \dot{\hat{x}}_2 &= F\hat{x}_2 - G_1Q_1^{-1}G_1'[P\hat{x}_2 + N(\hat{x}_2 - \hat{x}_2)] - G_2Q_2^{-1}G_2'P\hat{x}_2 \\ &= (F - G_1Q_1^{-1}G_1'P - G_2Q_2^{-1}G_2'P)\hat{x}_2 \end{aligned} \tag{32}$$

with initial condition the a priori initial state estimate, i.e., $\hat{x}_2(t_0|t_0) = \bar{x}_0$. We note that this estimate is independent of the system state and the measurements and estimates made by Controller 1. It may, therefore, be computed in advance by both Controller 1 and Controller 2 and considered by both simply as a known time function. Controller 1's optimal estimate $\hat{x}_1(t|t)$ of the system state $x(t)$, given the measurement set $Y_1(t)$, is thus from standard filtering theory [10] given by

$$\begin{aligned} \dot{\hat{x}}_1 = F\hat{x}_1 - G_1Q_1^{-1}G_1'[P\hat{x}_1 + N(\hat{x}_2 - \hat{x}_1)] \\ - G_2Q_2^{-1}G_2'Px_2 + MH_1'W_1^{-1}(y_1 - H_1\hat{x}_1) \end{aligned} \tag{33}$$

and the error $\tilde{x}_1(t|t)$ in this estimate satisfies

$$\dot{\tilde{x}}_1 = F\tilde{x}_1 - MH_1'W_1^{-1}(H_1\tilde{x}_1 + w_1). \tag{34}$$

It then follows that the matrix $M(t)$ defined by (28) satisfies the differential equation (29) with boundary condition (30).

The optimal control laws established previously can be obtained quite simply from the following heuristic argument which demonstrates the close connection of this result with the classical separation theorem. Since Controller 2 has no available measurements, the control law

$$v^0(t) = -Q_2^{-1}(t)G_2'(t)P(t)\hat{x}_2(t|t) \tag{23b}$$

with P satisfying (6) is on the basis of Proposition 1 the obvious candidate for his solution. Controller 1's control law can be argued to have the linear form

$$u^0(t) = A(t)\hat{x}_1(t|t) + B(t)\,[\hat{x}_2(t|t) - \hat{x}_1(t|t)\,]$$

and it remains to find A and B. If it happened that $\hat{x}_1(t|t) = \hat{x}_2(t|t)$ throughout the game (due to a particularly fortuitous noise sequence), then the game would essentially reduce to that of Problem 1 and hence the choice

$$A(t) = -Q_1^{-1}(t)G_1'(t)P(t)$$

is obvious. In addition, the special case $\hat{x}_2(t|t) = 0$, which obtains if $\bar{x}_0 = 0$, reduces the whole problem to an optimal control problem for Controller 1 without the term $v'Q_2v$ in the cost functional (2). Thus Controller 1 will in this case select

$$u^0 = [A(t) - B(t)]\hat{x}_1(t|t)$$

with

$$A(t) - B(t) = -Q_1^{-1}(t)G_1'(t)D(t)$$

where D satisfies the Riccati equation

$$\dot{D} + DF(t) + F'(t)D - DG_1(t)Q_1^{-1}(t)G_1'(t)D = 0$$

with boundary condition

$$D(T) = P(T) = C'C.$$

It can be shown by direct substitution that this equation for D is equivalent to (24) and (25) together with the identification $D = P - N$. Hence we have heuristically derived the control laws (23).

Comparison of (23b) and (5b) shows that for this problem the optimal control for Controller 2 satisfies a separation theorem in the sense that the unknown state in the solution to the deterministic problem is replaced by the best estimate $\hat{x}_2(t|t)$ of $x(t)$, given the available information to time t (in this case just a priori information). Comparison of (23a) and (5a) shows that the optimal control for Controller 1 is composed of two terms: the first is one satisfying a separation theorem, while the second is an additional term depending on the known (to Controller 1) difference between the two state estimates. Comparison of (22) and (8) shows that what, in view of (18), might be called by abuse of language "the optimal cost from time t" is in the present case the sum of three terms: the first term $x'(t)P(t)x(t)$ is the deterministic cost, i.e., that which would result if both controllers had perfect state information and controlled optimally; the second term $-\tilde{x}_2'(t|t)N(t)\tilde{x}_2(t|t)$ is the contribution due to the error at time t in Controller 2's state estimate; the third term $b(t)$ is, from (26), due to the noise in Controller 1's measurements and the error in his state estimate $\hat{x}_1(t|t)$. The solution to this game of imperfect information thus bears a close relationship to that given by a separation theorem.

In the special case of Problem 4 in which Controller 1 has perfect state measurements [i.e., referring to (19a), $H_1(t) = I$ (identity matrix) and $w_1(t) \equiv 0$], we note that, for all t, $\hat{x}_1(t|t) = x(t)$ so that $\tilde{x}_1(t|t) \equiv 0$ and both $M(t) \equiv 0$ and $b(t) \equiv 0$. In the special case in which Controller 2, as well as Controller 1, has no measurements at all available to him, the solution may be obtained formally from Proposition 2 by setting $w_1(t) = \infty$ and noting that, since the two controllers are now restricted to using only a priori information which is the same for both, the two state estimates will be the same, i.e., $\hat{x}_1(t|t) = \hat{x}_2(t|t)$, so that, in particular, the term in (23a) involving $x_2(t|t) - \hat{x}_1(t|t)$ vanishes and with it the dependence of the optimal controls on the matrix $N(t)$. Both of these special cases may be verified in detail by the construction of an appropriate function $I(\cdot,\cdot,\cdot)$.

V. Problem in Which One Controller Has Perfect State Measurements

In this section we examine the special case of Problem 3 in which one controller has perfect information about the system state while the other has available only noise-corrupted output measurements. To avoid the need for infinite-dimensional state estimators (or preservation of the entire measured output function), we assume that, in addition, conditions are such that the controller with perfect state information can deduce exactly the state of his opponent's state estimator.

A problem closely related to this has been solved by Behn and Ho [3] using the Euler–Lagrange approach of the classical calculus of variations. The present treatment provides a solution based on the application of Theorem 1 which may be viewed as a two-sided stochastic analog of the Hamilton–Jacobi equations in the calculus of variations.

Problem 5

Consider the special case of Problem 3 in which Controller 1 has available perfect measurements of the system state, while Controller 2 has available noisy output measurements of the form (19b), i.e., $H_1(t) \equiv I$ (unit matrix) and $w_1(t) \equiv 0$. It is also assumed that conditions are such that at each time t Controller 1 can deduce exactly his opponent's estimate $\hat{x}_2(t|t)$ of the system state $x(t)$.

Proposition 3: A solution to Problem 5 is given by the control laws $u^0(\cdot,\cdot)$ and $v^0(\cdot,\cdot)$ defined for all $t \in [t_0,T]$ by

$$u^0(t) = -Q_1^{-1}(t)G_1'(t)[P(t)x(t) + N(t)\tilde{x}_2(t|t)] \quad (35a)$$

$$v^0(t) = -Q_2^{-1}(t)G_2'(t)P(t)\hat{x}_2(t|t) \quad (35b)$$

where the symmetric matrix $P(t)$ satisfies the Riccati equation (6) with boundary condition (7) and the symmetric matrix $N(t)$ satisfies the differential equation

$$\begin{aligned}\dot{N} &+ NF(t) + F'(t)N \\ &+ P(t)[G_1(t)Q_1^{-1}(t)G_1'(t) + G_2(t)Q_2^{-1}(t)G_2'(t)]P(t) \\ &- (P(t) + N)G_1(t)Q_1^{-1}(t)G_1'(t)(P(t) + N) \\ &- NM(t)H_2'(t)W_2^{-1}(t)H_2(t) \\ &- H_2'(t)W_2^{-1}(t)H_2(t)M(t)N = 0 \end{aligned} \quad (36)$$

with boundary condition

$$N(T) = 0. \quad (37)$$

The symmetric matrix $M(t)$ satisfies

$$\dot{M} = A(t)M + MA'(t) - MH_2'(t)W_2^{-1}(t)H_2(t)M \quad (38)$$

with boundary condition [recall (21)]

$$M(t_0) = M_0 \quad (39)$$

where the matrix $A(t)$ is defined by

$$A(t) = F(t) - G_1(t)Q_1^{-1}(t)G_1'(t)[P(t) + N(t)]. \quad (40)$$

It is noted that (36) and (38) are coupled, so that solution of this problem involves the solution of the two-point boundary-value problem given by these equations with boundary conditions (37) and (39). It is seen, however, that the matrices $M(t)$, $N(t)$, and $P(t)$ may be computed in advance.

The state estimator which generates $\hat{x}_2(t|t)$ may be derived as follows. Substitution of (35) into the system equation (1) gives

$$\dot{x} = (F - G_1Q_1^{-1}G_1'P - G_2Q_2^{-1}G_2'P)x + (G_2Q_2^{-1}G_2'P - G_1Q_1^{-1}G_1'N)\tilde{x}_2.$$

Standard filtering theory [10] then shows that the best estimate $\hat{x}_2(t|t)$ is generated by

$$\dot{\hat{x}}_2 = (F - G_1Q_1^{-1}G_1'P - G_2Q_2^{-1}G_2'P)\hat{x}_2 + MH_2'W_2^{-1}(y_2 - H_2\hat{x}_2)$$

with boundary condition

$$\hat{x}_2(t_0|\,t_0) = \bar{x}_0 = \text{a priori estimate of } x(t_0)$$

so that the estimation error $\tilde{x}_2(t|t)$ satisfies

$$\dot{\tilde{x}}_2 = A\tilde{x}_2 - MH_2'W_2^{-1}H_2\tilde{x}_2 - MH_2'W_2^{-1}w_2 \quad (41)$$

where the matrix $M(t)$ defined by

$$M(t) = E\{\tilde{x}_2(t|\,t)\tilde{x}_2'(t|\,t)|\,Y_2(t)\}$$

satisfies (38) with boundary condition (39).

A proof of Proposition 3 is quite straightforward and proceeds in two parts.

1) The proof that $u^0(\cdot,\cdot)$ and $v^0(\cdot,\cdot)$ given by (35) satisfy (15a) proceeds directly from the observation that, for the given $v^0(\cdot,\cdot)$, the problem from the viewpoint of Controller 1 may be reduced to a standard problem in optimal control theory involving a linear system (the $2n$-dimensional state of which is the combined vector $[x'(t), x_2'(t|t)]'$), a quadratic cost functional, additive white Gaussian noise in the dynamics, and, by assumption, perfect information of the total combined state. The minimizing control law for this problem is that given by (35a), while the corresponding minimum expected cost is

$$\begin{aligned} I(x(t),\tilde{x}_2(t|t),\, t) &\triangleq \min_{u(\cdot,\cdot)} E\{J_t[u,v^0]|\,x(t),\tilde{x}_2(t|t)\} \\ &= E\{J_t[u^0,v^0]|\,x(t),\tilde{x}_2(t|t)\} \\ &= x'(t)P(t)x(t) + \tilde{x}_2'(t|t)N(t)\tilde{x}_2(t) \\ &\quad + \operatorname{tr}\left\{\int_t^T N(s)M(s)H_2'(s) \right. \\ &\quad \left. \cdot W_2^{-1}(s)H_2(s)M(s)ds\right\}. \end{aligned} \quad (42)$$

2) The proof that (15b) is satisfied by the control laws $u^0(\cdot,\cdot)$ and $v^0(\cdot,\cdot)$ given by (35) follows directly from a straightforward verification of the satisfaction of conditions 3) through 5) of Theorem 1 by $u^0(\cdot,\cdot)$ and $v^0(\cdot,\cdot)$, with the function $I(\cdot,\cdot,\cdot)$ defined by (42).

The details of part 2) of the proof are sufficiently straightforward and similar in structure and argument to the proof of Proposition 2 in the Appendix that they are not included here. The details of part 1) of the proof along the lines indicated in 1) may be found in [3] which also contains an alternative proof of the satisfaction of (15b).

The proof indicated here also forms the basis for a heuristic derivation of the result, since the choice of $v^0(\cdot,\cdot)$ given by (35b) is the obvious candidate for Controller 2's solution. The control for Controller 1 is then deduced as indicated in part 1), and the heuristic derivation becomes a complete proof by using the steps indicated in part 2).

It is seen from (35) that the optimal control for Controller 2 satisfies a separation theorem, while that for Controller 1 is the sum of a term which satisfies a separation theorem and a term which is a linear function of the known (to Controller 1) error in his opponent's state estimate. Also, as in Section IV, the "optimal cost from time t," $I(x(t),\tilde{x}_2(t|t),t)$, is seen from (42) to be the sum of the optimal cost $x'(t)P(t)x(t)$ for the corresponding deterministic problem, a term $\tilde{x}_2'(t|t)N(t)\tilde{x}_2(t|t)$ which is quadratic in the error in Controller 2's state estimate at time t, and a term due to the noise in Controller 2's measurements.

We now turn to a discussion of the conditions under which Controller 1 can deduce exactly at each time t the error $\tilde{x}_2(t|t)$ in his opponent's state estimate. We adopt an inductive type of argument, assuming that at time t Controller 1 knows exactly the estimation error $\tilde{x}_2(t|t)$ and demonstrating conditions under which he may deduce exactly its time rate of change $\dot{\tilde{x}}_2(t|t)$.

An alternative realization scheme included in Behn and Ho [3] requires that the dimension of the control vector $v(t)$ equal that of the system state. This requirement becomes not unreasonable only when consideration is restricted, as in [3], to pursuit–evasion games, whose additional structure can be exploited to define a reduced system state of substantially smaller dimension than the total system state. The realization scheme indicated below removes this dimensionality requirement by introducing an additional derivative.

By twice differentiating the perfectly measured $x(t)$, Controller 1 knows at time t the values $x(t)$, $\dot{x}(t)$, $\ddot{x}(t)$, and, by assumption, $_2(xt|t)$. Differentiating the system equation (1) after substituting (35), we have

$$\begin{aligned} \ddot{x} = {} & (F - G_1Q_1^{-1}G_1'P - G_2Q_2^{-1}G_2'P)\dot{x} \\ & + (G_2Q_2^{-1}G_2'P - G_1Q_1^{-1}G_1'N)\dot{\tilde{x}}_2 \\ & + \frac{d}{dt}(F - G_1Q_1^{-1}G_1'P - G_2Q_2^{-1}G_2'P)\cdot x \\ & + \frac{d}{dt}(G_2Q_2^{-1}G_2'P - G_1Q_1^{-1}G_1'N)\cdot\tilde{x}_2. \end{aligned}$$

Noting that the term on the left and the first, third, and fourth terms on the right are known, substitution of (41) gives

$$(G_2Q_2{}^{-1}G_2{}'P - G_1Q_1{}^{-1}G_1{}'N)[A\tilde{x}_2 - MH_2{}'W_2{}^{-1}(H_2\tilde{x}_2 + w_2)] = \gamma_1$$

where γ_1 is a known n vector. The only unknown in this equation is w_2, so that we have

$$(G_2Q_2{}^{-1}G_2{}'P - G_1Q_1{}^{-1}G_1{}'N)MH_2{}'W_2{}^{-1}w_2 \triangleq Dw_2 = \gamma_2$$

where γ_2 is a known n vector. Recalling that w_2 is a vector of dimension m_2, we thus have that, provided the $n \times m_2$ matrix D has full rank m_2, the noise w_2 may be deduced exactly, and so, therefore, from (41), may $\dot{\tilde{x}}_2$.

It does not appear to be possible to derive any sufficient condition for the matrix $D(t)$ to have full rank for all t in $[t_0,T]$. We note, however, that $D(t)$ may be computed in advance from the entities given in the problem statement and checked for full rank before commencement of the game itself; i.e., the matrix $D(t)$ does not depend on any random entities which are generated in a given play of the game.

Finally, we note that the special case of Problem 5 in which both controllers have perfect state measurements is solved immediately once the material in Section II is at hand. For even if the initial state is imperfectly known a priori by the controllers, it becomes known exactly and thereafter remains so, immediately following the first measurement made by each controller; the problem thus reduces to the deterministic differential game whose solution is given in Section II.

VI. CONCLUDING REMARKS

Attention has been given to a number of differential games with imperfect information and involving a linear system, a quadratic cost functional, and independent white Gaussian noises additively corrupting the output measurements. A solution has been presented to five of the six distinct problems that arise when each controller is allowed either perfect information, noise-corrupted information, or no information at all. These five problems correspond to the situations where only one of the two controllers has noisy measurements, the other having either no measurements or perfect state information, and the attendant special cases. The remaining problem in which both controllers have available only noise-corrupted measurements requires special consideration and is the subject of a forthcoming paper.

For each problem considered in this paper, the optimal control for each controller has been shown to be closely related to that which would result by assuming a separation theorem to hold. This close relationship is emphasized further by the heuristic derivations of the optimal controls included in the paper. Furthermore, the various terms in the optimal cost are shown to be readily assignable to the appropriate contributing source, such as the optimal cost that would result if the problem were instead a deterministic one, the effect of estimation errors, or the effect of measurement errors.

APPENDIX

PROOF OF PROPOSITION 2

Substitution of (23b) into the system equation (1) gives

$$\dot{x} = Fx + G_1u - G_2Q_2{}^{-1}G_2{}'P\hat{x}_2. \tag{43}$$

Subtraction of (32) then yields

$$\dot{\tilde{x}}_2 = F\tilde{x}_2 - G_1Q_1{}^{-1}G_1{}'P\tilde{x}_2 + G_1u + G_1Q_1{}^{-1}G_1{}'Px. \tag{44}$$

We have from (2), (16), (22), (23), (43), and (44),

$$\begin{aligned} S(x,\tilde{x}_2,u,v^0,t) = {} & x'\dot{P}x - \tilde{x}_2{}'N\tilde{x}_2 + \dot{b} + u'Q_1u \\ & + \hat{x}_2{}'PG_2Q_2{}^{-1}G_2{}'P\hat{x}_2 \\ & + 2x'P(Fx + G_1u - G_2Q_2{}^{-1}G_2{}'P\hat{x}_2) \\ & - 2\tilde{x}_2{}'N(F\tilde{x}_2 + G_1u + G_1Q_1{}^{-1}G_1{}'P\hat{x}_2). \end{aligned} \tag{45}$$

It then follows immediately that

$$\begin{aligned} & \arg\min_u E\{S(x,\tilde{x}_2,u,v^0,t) \mid Y_1(t)\} \\ & \quad = -Q_1{}^{-1}G_1{}'E\{Px - N\tilde{x}_2 \mid Y_1(t)\} \\ & \quad = -Q_1{}^{-1}G_1{}'E\{(P - N)x + N\hat{x}_2 \mid Y_1(t)\} \\ & \quad = -Q_1{}^{-1}G_1{}'[(P - N)\hat{x}_1 + N\hat{x}_2] = u^0. \end{aligned} \tag{23a}$$

Substitution of (23a) into (45) then gives, after some algebra,

$$\begin{aligned} & \min_u E\{S(x,\tilde{x}_2,u,v^0,t) \mid Y_1(t)\} \\ & \quad = E\{S(x,\tilde{x}_2,u^0,v^0,t) \mid Y_1(t)\} \\ & \quad = E\{x'(\dot{P} + PF + F'P - PG_1Q_1{}^{-1}G_1{}'P \\ & \qquad - PG_2Q_2{}^{-1}G_2{}'P)x - \tilde{x}_2{}'(\dot{N} + NF + F'N \\ & \qquad - P(G_1Q_1{}^{-1}G_1{}' + G_2Q_2{}^{-1}G_2{}')P)\tilde{x}_2 \\ & \qquad + \tilde{x}_2{}'(P - N)G_1Q_1{}^{-1}G_1{}'(P - N)\tilde{x}_2 \\ & \qquad + E[x_1{}'(P - N)G_1Q_1{}^{-1}G_1{}'(P \\ & \qquad - N)\tilde{x}_1 \mid Y_1(t)] + \dot{b} \mid Y_1(t)\}. \end{aligned} \tag{46}$$

The second to last term within the outer expectation of (46) is

$$\begin{aligned} & \operatorname{tr}[E\{\tilde{x}_1\tilde{x}_1{}' \mid Y_1(t)\}(P - N)G_1Q_1{}^{-1}G_1{}'(P - N)] \\ & \qquad = \operatorname{tr}[M(t)(P - N)G_1Q_1{}^{-1}G_1{}'(P - N)] \end{aligned}$$

so that, using (6), (24), and (26) with boundary conditions given by (7), (25), and (27), we thus have

$$E\{S(x,\tilde{x}_2,u^0,v^0,t) \mid Y_1(t)\} = \min_u E\{S(x,\tilde{x}_2,u,v^0,t) \mid Y_1(t)\} = 0$$

so that conditions 1), 2), and 5) of Theorem 1 are satisfied. The proof that conditions 3), 4), and 5) are met proceeds analogously. After showing that

$$\arg\max_{v} E\{S(x,\tilde{x}_2,u^0,v,t) \mid Y_2(t)\} = -Q_2^{-1}G_2'P\hat{x}_2 = v^0$$

substitution shows that $\max_v E\{S(x,\tilde{x}_2,u^0,v,t) \mid Y_2(t)\}$ is given by the right-hand side of (46) with the outer expectation conditioned on $Y_2(t)$ instead of $Y_1(t)$. The inclusion in the second last term of an additional expectation conditioned on $Y_1(t)$ even though the outer expectation is now conditioned on $Y_2(t)$ is valid, since $Y_2(t) = Y_2(t_0)$ (the a priori information available to both controllers) is a subset of $Y_1(t)$, so that

$$E\{E[\cdot \mid Y_1(t)] \mid Y_2(t)\} = E\{\cdot \mid Y_2(t)\}.$$

It then follows, using (6), (24), and (26) with boundary conditions (7), (25), and (27), that

$$E\{S(x,\tilde{x}_2,u^0,v^0,t) \mid Y_2(t)\} = \max_{v} E\{S(x,\tilde{x}_2,u^0,v,t) \mid Y_2(t)\} = 0$$

as required, and the proof is complete.

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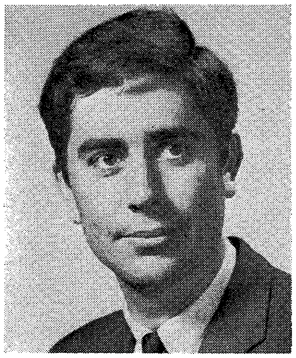

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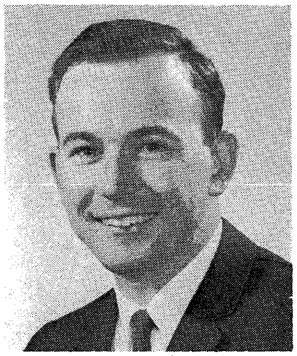

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