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Brief Paper

Dynamic Equilibria for Linear Systems and Quadratic Costs*

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Key Words—Economics; descriptor variables; Riccati equation; equilibrium; linear quadratic control.

Abstract—This paper considers the computation of dynamic equilibria from a systems viewpoint. Attention is focused on an economic system governed by a linear production system, quadratic production costs, and a linear demand function. The resulting necessary conditions for a dynamic equilibrium take the form of a linear two-point boundary value problem similar in structure to the conditions of optimal control. However, the conditions are not equivalent to those of optimal control, and the standard Riccati equation approach cannot generally be applied. A new procedure based on descriptor variable theory provides a general solution, which is a generalization of the Riccati equation approach.

1. *Introduction*

THE CONCEPT of an economic equilibrium is fundamental to the modern theory of microeconomics (Debreu, 1959). The concept is usually considered in a static context, but it is recognized that it is applicable to dynamic situations as well, where an entire trajectory of price vectors must be in equilibrium. Underlying such an equilibrium are processes that govern demand and supply, and in general, these will be dynamic processes subject to control by agents on one side of the market or the other. The equilibrium price trajectory is the trajectory that induces all agents to operate their dynamic processes so that the market clears (that is, supply equals demand) in every period. The equilibrium price trajectory thus coordinates the individual optimal control processes. It is for this reason that such a trajectory is referred to as a *dynamic equilibrium*; it represents economic equilibrium of the dynamic processes.

Explicit calculation of a dynamic equilibrium is inherently much more difficult than the calculation of an optimal control trajectory, although the optimal control trajectories of the agents are a by-product of the calculation of the dynamic equilibrium. On the other hand, the conditions for existence of a dynamic equilibrium are derived from the conditions of optimality of the individual problems, so the methodology for finding dynamic equilibria can be expected to be closely related to the methodology for finding optimal trajectories, but with perhaps some new features. This theme is what is explored in this paper, with particular focus on the linear–quadratic framework. It is seen that the familiar Riccati equation technique is not quite adequate to determine dynamic equilibria, but a suitable extension does exist.

The type of model used is a "rational expectations" model (which in the deterministic case considered here is called a "perfect foresight" model) which is now in common use. (See Hansen and Sargent (1981), for example.) The focus is primarily on the methods for solving rather than formulating such models, and in particular, the development of systematic solution methods using modern control-theoretic concepts. However, some attention is devoted to problem formulation in order to motivate the theory.

In this paper a particularly simple dynamic market structure is assumed in order to easily obtain the kind of problem that requires this new method for solution. Consumers act according to a static linear demand function and supply is generated from a large number of producers with similar dynamic production facilities. Furthermore, it is assumed that the production process is linear and that the production cost is quadratic in form. The linear and quadratic assumptions are important, but other aspects of the structure can be generalized significantly.

Specifically, assume that production is governed by the process

$$x(k+1) = A(k)x(k) - B(k)u(k) + C(k)w(k) + a(k), \qquad (1.1)$$

where

$x(k)$ is an n-dimensional state vector,
$u(k)$ is an s-dimensional market control vector,
$w(k)$ is an r-dimensional process control vector,
$a(k)$ is an n-dimensional deterministic input vector.

The components of the vector $x(k)$ represent commodities that typically move from stage to stage in the production process (livestock of different age groups, for example). The vector $u(k)$ represents market activity, such as selling some commodity. (Since $u(k)$ does typically represent sales that reduce stock, it is convenient to introduce the minus sign as a coefficient of $B(k)$.) Assume that $s \leqslant n$ and that Rank $B(s) = s$. The vector $w(k)$ represents non-market control action in the process. Finally $a(k)$, the deterministic input, represents inputs that are fixed in advance and are outside the scope of control.

In additioin to the dynamic process, there are initial and final conditions specified of the form

$$x(0) = x_0 \qquad (1.2a)$$

$$Gx(N+1) = g, \qquad (1.2b)$$

where G is an $m \times n$ matrix and g is an m-vector. The terminal constraints represent the fact that the finite horizon problem may be only an approximation to a longer horizon and thus restarting conditions should be specified. It is assumed that (1.2) represents feasible constraints.

The system (1.1), (1.2) is actually a representative, or aggregate, of many individual processes of different firms, each having identical dynamic structure and boundary conditions proportional to those of (1.2). Each of these producers operates a process with a quadratic objective, and hence so does the aggregate. Assume the aggregate is operated so as to maximize net profit.

$$J = \sum_{k=0}^{N} \{p(k)'u(k) - c(k)'x(k) - \frac{1}{2}x(k)'Q(k)x(k) - d(k)'w(k) - \frac{1}{2}w(k)'W(k)w(k)\}, \qquad (1.3)$$

where $p(k)$ is the price vector in the market at period k. Each producer is a *price taker* and hence the sequence $\{p(k)\}$ is taken to be given for purposes of defining the optimization problem. The price sequence will be determined later to establish the dynamic equilibrium.

The interpretation of the other terms in the profit function

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should be clear. The negative terms involving $x(k)$ represent quadratic holding and processing costs and the negative terms involving $w(k)$ represent quadratic control costs. The symmetric matrices $Q(k)$ are positive semi-definite and the $W(k)$ are positive definite.

The problem of maximizing (1.3) subject to (1.1) and (1.2) is a standard optimal control problem that must be solved, once the sequence $\{p(k)\}$ is known. This complex process of going from $\{p(k)\}$ to $\{u(k)\}$ is what determines the supply.

The demand is assumed to be governed by a linear demand function of the form

$$v(k) = e(k) - E(k)p(k), \tag{1.4}$$

where $v(k)$ is the s-dimensional demand vector and $e(k)$ and $E(k)$ are given. Once $\{p(k)\}$ is known, the sequence of demands $\{v(k)\}$ is easily determined.

The principal condition for dynamic equilibrium is that supply equal demand; that is, $u(k) = v(k)$ for each k. The basic equilibrium problem is to find a sequence $\{p(k)\}$ such that this condition holds.

Note that this problem is one of *partial* equilibrium. Only the prices of the traded products $u(k)$ need be determined, not the costs of production or inputs. Accordingly, these prices will be determined relative to the given (exogenous) cost structure.

2. *Equilibrium conditions*

The necessary conditions for equilibrium are found by combining the necessary conditions of optimality for the optimal control problem with the conditions for market clearing. First, determine whether the control problem has a solution and characterize the necessary conditions. The necessary conditions for the optimal control problem of maximizing (1.3) subject to (1.1) and (1.2) are based on the Hamiltonian

$$\begin{aligned} H = {} & \lambda(k+1)'A(k)x(k) - \lambda(k+1)'B(k)u(k) \\ & + \lambda(k+1)'C(k)w(k) + \lambda(k+1)'a(k) + p(k)'u(k) \\ & - c(k)'x(k) - \frac{1}{2}x(k)'Q(k)x(k) - d(k)'w(k) \\ & - \frac{1}{2}w(k)'W(k)w(k). \end{aligned} \tag{2.1}$$

The appropriate adjoint equation is

$$\lambda(k) = A(k)'\lambda(k+1) - Q(k)x(k) - c(k) \tag{2.2}$$

with terminal condition

$$\lambda(N+1) = G'v, \tag{2.3}$$

where v is as yet unknown. Additional conditions are found by setting to zero the derivative of the Hamiltonian with respect to u and w, which leads to

$$p(k) = B(k)'\lambda(k+1) \tag{2.4}$$

$$C(k)'\lambda(k+1) = W(k)w(k) + d(k). \tag{2.5}$$

The equations (2.2)–(2.5) together with the original system equations (1.1) and (1.2) constitute a complete set of necessary conditions for optimality.

The optimal control problem is actually *singular* because u appears only linearly in the Hamiltonian. For some values of $\{p(k)\}$, therefore, the optimal value will be infinite. Equation (2.4) puts restrictions on $\{p(k)\}$ so that such infinite values will not occur. (Note in particular that $p(N) = B(N)'\lambda(N+1) = B(N)'G'v$.) Since the problem has a positive semi-definite quadratic objective and linear constraints, any solution to the necessary conditions will be a global maximum. The solution may not be unique; but, nevertheless, the conditions are necessary and sufficient for a solution to the optimal control problem.

To incorporate the market clearing condition, set $u(k) = v(k)$, as determined from (1.4), to write

$$u(k) = e(k) - E(k)p(k). \tag{2.6}$$

Combining (2.6) with the necessary conditions for optimality yields the conditions for dynamic equilibrium. Carrying out this combination leads to the two-point boundary problem

$$\begin{aligned} x(k+1) = {} & A(k)x(k) + [B(k)E(k)B(k)' \\ & + C(k)W(k)^{-1}C(k)']\lambda(k+1) \\ & - C(k)W(k)^{-1}d(k) - B(k)e(k) + a(k) \end{aligned} \tag{2.7a}$$

$$\lambda(k) = A(k)'\lambda(k+1) - Q(k)x(k) - c(k) \tag{2.7b}$$

$$x(0) = x_0 \tag{2.7c}$$

$$Gx(N+1) = g \tag{2.7d}$$

$$\lambda(N+1) = G'v. \tag{2.7e}$$

Once this system is solved, the auxiliary unknowns $p(k)$, $u(k)$, and $w(k)$ can be found from (2.4)–(2.6).

Obviously an important ingredient of the system are the $n \times n$ matrices

$$V(k) = B(k)E(k)B(k)' + C(k)W(k)^{-1}C(k)'. \tag{2.8}$$

These matrices are not necessarily symmetric since the $E(k)$s are not.

If the $E(k)$ matrices (and hence the $V(k)$ matrices) were symmetric and positive definite, the problem could be converted to an equivalent linear–quadratic optimization problem and solved by standard optimization methods. The generality of $E(k)$ in this formulation makes the equilibrium problem more difficult than that of linear–quadratic control.

3. *Solution to the linear–quadratic problem*

The general dynamic equilibrium problem reduces to the solution of the system (2.7), which is expressed more compactly as

$$x(k+1) - V(k)\lambda(k+1) = A(k)x(k) - b(k) \tag{3.1a}$$

$$A(k)'\lambda(k+1) = \lambda(k) + Q(k)x(k) + c(k) \tag{3.1b}$$

$$x(0) = x_0, \quad Gx(N+1) = g, \quad \lambda(N+1) = G'v, \tag{3.1c}$$

where $b(k) = C(k)W(k)^{-1}d(k) + B(k)e(k) - a(k)$. In this form it is evident that this is a descriptor variable system. That is, it is a system in the composite vector $(x(k), \lambda(k))$ and the coefficient matrix on the left is not the identity matrix as in a state space system. In fact under these assumptions this matrix may be singular. If each $V(k)$ were symmetric and positive definite, the solution could be expressed in terms of a discrete Riccati equation. In general, however, more powerful and more general methods are required. The double-sweep method (Luenberger, 1977) will be used here. (For another general approach see Campbell, 1984.)

The double-sweep method works by sweeping conditions forward and then backward. The initial condition $x(0) = x_0$ defines n conditions. These conditions are reflected at each k as conditions on the composite vector $(x(k), \lambda(k))$ of the form $S(k)x(k) + T(k)\lambda(k) = y(k)$, where $y(k)$ depends on x_0 and the inputs to the system. The $n \times 2n$ coefficient matrix $\Gamma(k) = [S(k), T(k)]$ is independent of the inputs and reflects how the structure of the conditions propagate with k. The two-point problem can be solved by sweeping the initial condition to the end in this way.

As is shown later explicitly, this method of sweep can be regarded as a generalization of solution of a discrete Riccati equation, where typically $S(k) = I$. In general, however, such a choice is not possible, and, as shown below, the sweep relations are quite different from simple matrix recursion. The following procedure yields the structure of the conditions as they are swept forward. The resulting matrices are used in the solution method.

Procedure. Let $P(0) = I$, $T(0) = 0$, $R(0) = A(0)$ be $n \times n$ matrices, and define successive values for $k = 1, 2, \ldots, N+1$ as follows.

(a) Find $T(k+1)$ from

$$T(k+1) = -P(k)V(k) + R(k)T(k)A(k)'. \tag{3.2a}$$

(b) Find $R(k+1)$ and $P(k+1)$ satisfying

$$R(k+1)[P(k) - T(k+1)Q(k+1)] = P(k+1)A(k+1) \tag{3.2b}$$

such that $[R(k+1), -P(k+1)]$ has full rank.

One should note the somewhat unusual recursion represented by (3.2b). It can be rewritten as

$$[R(k+1), -P(k+1)]\begin{bmatrix} P(k) - T(k+1)Q(k+1) \\ A(k+1) \end{bmatrix} = 0.$$

This is termed a *nullity relation*, and its solution amounts to finding an independent set of row vectors orthogonal to a given rectangular matrix. Clearly there is not a unique solution since $R(k+1)$ and $P(k+1)$ could be multiplied on the left by an arbitrary non-singular $n \times n$ matrix. Such a relation is easily solved by Gaussian elimination or by the QR matrix factorization algorithm applied to the given rectangular matrix; and overall, the amount of computational effort (as measured by the number of multiplications) is smaller than for the perhaps seemingly simpler recursion (3.2a).

The system (3.1) defining an equilibrium may not always exist. The basic regularity conditions that do guarantee a well-defined unique solution are that the system (3.1a, b) be *solvable* and *conditionable* (Luenberger, 1977; Lewis, 1983). Furthermore the boundary conditions

$$\begin{aligned} x(0) &= x_0 \\ Gx(N+1) &= g \\ F\lambda(N+1) &= f, \end{aligned}$$

where F is a maximum rank matrix with $FG' = 0$, define a unique solution for all x_0, g, f. (The condition $F\lambda(N+1) = 0$ is equivalent to $\lambda(N+1) = G'v$.) If these requirements are met there will be a unique solution for all values of $b(k)$, $c(k)$, $k = 0, 1, \ldots, N$ and all initial conditions. In this case, the system is said to be *standard*.

If this system (3.1) is standard, then the procedure defined above is well defined. The recursions (3.2) can progress from 0 to $N+1$ while satisfying the required rank condition.

The theorem below shows how to obtain a solution to the problem (3.1).

Theorem. If system (3.1) is standard, then the complete solution can be found in terms of the matrices defined above as follows.

(a) Solve the forward recursion

$$y(k+1) = R(k)y(k) - P(k)b(k) + R(k)T(k)c(k) \tag{3.3a}$$

$$y(0) = x_0. \tag{3.3b}$$

(b) Solve the system

$$P(N)x(N+1) + T(N+1)G'v = y(N+1) \tag{3.4a}$$

$$Gx(N+1) = g \tag{3.4b}$$

for $x(N+1)$, v, and put

$$\lambda(N+1) = G'v. \tag{3.5}$$

(c) Solve backward for $x(k)$, $\lambda(k)$ from the equations

$$x(k+1) - V(k)\lambda(k+1) = A(k)x(k) - b(k) \tag{3.6a}$$

$$A(k)'\lambda(k+1) = \lambda(k) + Q(k)x(k) + c(k) \tag{3.6b}$$

$$P(k-1)x(k) + T(k)\lambda(k) = y(k). \tag{3.6c}$$

Proof. The result is derived by adapting the double-sweep method of Luenberger (1977) to this situation and taking advantage of the special structure. Let $S(0) = I$ and $T(0) = 0$ be $n \times n$ matrices. According to the general method, the condition matrix $\Gamma(k) = [S(k), T(k)]$ is updated by the recursions:

(a) Given $\Gamma(k) = [S(k), T(k)]$ find $n \times n$ matrices $R(k)$, $P(k)$ and $U(k)$ such that

$$R(k)[S(k), T(k)] = P(k), U(k)]\begin{bmatrix} A(k) & 0 \\ Q(k) & I \end{bmatrix} \tag{3.7}$$

and such that $[R(k), -P(k), -U(k)]$ has full rank. It must follow that $[P(k), U(k)$ has full rank if the system is solvable.

(b) Find $\Gamma(k+1) = [S(k+1), T(k+1)]$ from

$$[S(k+1), T(k+1)] = [P(k), U(k)]\begin{bmatrix} I & -V(k) \\ 0 & A(k)' \end{bmatrix}. \tag{3.8}$$

These yield the relations

$$R(k)S(k) = P(k)A(k) + U(k)Q(k) \tag{3.9a}$$

$$R(k)T(k) = U(k) \tag{3.9b}$$

$$S(k+1) = P(k) \tag{3.9c}$$

$$T(k+1) = -P(k)V(k) + U(k)A(k)'. \tag{3.9d}$$

One has $\Gamma(0) = [I, 0]$ representing the fact that $x(0)$ is specified and $\lambda(0)$ is not, which is why $S(0) = I$, $T(0) = 0$ was taken. Then (3.7) may be solved for $k = 0$ by setting $R(0) = A(0)$, $P(0) = I$, $U(0) = 0$. Thus initial values have been obtained for all matrices.

Since (3.7) has already been solved for $k = 0$, we are one-half step ahead in the recursion. Proceed by solving (3.8) followed by (3.7) for k one step larger. Thus the recursions are solved in the form

$$S(k+1) = P(k) \tag{3.10a}$$

$$T(k+1) = -P(k)V(k) + U(k)A(k)' \tag{3.10b}$$

$$\begin{aligned} R(k+1)S(k+1) = P(k+1)A(k+1) \\ + U(k+1)Q(k+1) \end{aligned} \tag{3.10c}$$

$$R(k+1)T(k+1) = U(k+1). \tag{3.10d}$$

Then eliminate $S(k+1)$ by using (3.10a) and eliminate $U(k)$ by using (3.10d) (for k rather than $k+1$). This yields the relations (3.2) of the Procedure defined earlier.

The rank conditions must now be verified. The general double sweep requires that $[R(k), -P(k), -U(k)]$ be full rank. It must be shown that this implies $[R(k), -P(k)]$ is full rank so that the simpler rank test stated in the Procedure can be used. Assume that $[R(k), -P(k)]$ is not full rank. Then there is a row vector $\lambda \neq 0$ such that $\lambda R(k) = \lambda P(k) = 0$. From (3.9b) it follows that $\lambda U(k) = 0$ which means that $[R(k), -P(k), -U(k)]$ is not full rank—which is a contradiction. Thus the smaller dimensional relation (3.2b) is appropriate.

The functional form of the swept condition is

$$S(k)x(k) + T(k)\lambda(k) = y(k). \tag{3.11}$$

It is clear that $y(0) = x_0$ since $\Gamma(0) = [I, 0]$. The recursion for $y(0)$ is found from the general double-sweep method [3] to be $y(k+1) = R(k)y(k) - P(k)b(k) + U(k)c(k)$. Using $U(k) = R(k)T(k)$ yields (3.3).

The set of all terminal conditions, including the initial conditions swept forward are

$$S(N+1)x(N+1)+T(N+1)\lambda(N+1)=y(N+1) \tag{3.12a}$$

$$Gx(N+1)=g \tag{3.12b}$$

$$\lambda(N+1)=G'v. \tag{3.12c}$$

Eliminating $\lambda(N+1)$ from this system gives

$$S(N+1)x(N+1)+T(N+1)G'v=y(N+1) \tag{3.13a}$$

$$Gx(N+1)=g. \tag{3.13b}$$

These are (3.4), and (3.5) follows from (3.12c).

Once $y(N+1)$, $x(N+1)$, $\lambda(N+1)$ are known, the system equations can be solved backward as shown in (3.6). They will be consistent and lead to $x(0)=x_0$ because the system is standard and because of the choice of terminal conditions. ■

The nature of these results is illustrated by considering some special cases.

Case 1. *Pure linear case* ($Q(k)\equiv 0$, $C(k)\equiv 0$). This is a highly degenerate case. In this case set

$$P(k)\equiv I, \qquad R(k)=A(k), \qquad T(0)=0.$$

The rank condition in (3.2b) is always satisfied. Then

$$T(k+1)=A(k)T(k)A(k)'-B(k)E(k)B(k)'. \tag{3.14}$$

Thus

$$T(N+1)=-\sum_{k=0}^{N}\Phi(N+1,k+1)B(k)E(k)B(k)'\Phi(N+1,k+1)', \tag{3.15}$$

where

$$\Phi(N+1,k+1)=\begin{cases}A(N)A(N-1)\ldots A(k+1) & k<N\\ I & k=N.\end{cases}$$

Let $T(N+1)=-C$. Then, for (3.4) to have a solution, the matrix

$$\begin{bmatrix} I & -CG' \\ G & 0\end{bmatrix}$$

must be invertible, which is true iff GCG' is invertible. The backward equations (3.6) are always of full rank. Hence, there is a solution iff GCG' is invertible.

Case 2. *Control theory* ($V(k)$ symmetric and positive semi-definite). In this case a solution of the form

$$P(k)\equiv I.$$

is tried. Then

$$T(k+1)=R(k)T(k)A(k)'-V(k),$$

but using $R(k)=A(k)[I-T(k)Q(k)]^{-1}$ yields

$$T(k+1)=A(k)[I-T(k)Q(k)]^{-1}T(k)A(k)'-V(k), \tag{3.16}$$

which is a discrete Riccati equation. Since in this case $V(k)$ and $Q(k)$ are symmetric, it is easy to show by induction that $T(k)$ is symmetric. Also since $V(k)$ and $Q(k)$ are positive semi-definite, it follows that $T(k)$ is negative semi-definite for all k so the required inverse in the recursion exists. For general $V(k)$ this would not work.

4. *Backward sweep*

The procedure developed above is the simplest form of solution for the market equilibrium problem. It has the disadvantage, however, that it does not yield a feedback solution in the case of optimal control. For this reason a backward sweep of conditions, rather than a forward sweep as developed above, is desirable. The development is only sketched since it is a relatively minor variation of the forward sweep. However, the initial conditions of the backward sweep are somewhat more complicated than in the forward sweep.

Consider the system (3.1) in the form

$$A(k)x(k)=x(k+1)-V(k)\lambda(k+1)+b(k) \tag{4.1a}$$

$$Q(k)x(k)+\lambda(k)=A(k)'\lambda(k+1)-c(k) \tag{4.1b}$$

$$x(0)=x_0, \qquad Gx(N+1)=g, \qquad \lambda(N+1)=G'v \tag{4.1c}$$

to establish the backward direction of the sweep. Also it is convenient to offset the indices k by one unit in the definition of conditions. Define the conditions in the form

$$[S(k),T(k)]\begin{bmatrix} x(k+1)\\ \lambda(k+1)\end{bmatrix}=y(k+1). \tag{4.2}$$

Let F be an $n\times(n-m)$ matrix of full rank such that $FG'=0$. Then the terminal conditions of the system are

$$Gx(N+1)=g \tag{4.3a}$$

$$F\lambda(N+1)=0. \tag{4.3b}$$

Now the condition $\Gamma(k)=[S(k),T(k)]$ must be swept backward. The analogue to (3.7) and (3.8) is thus

$$R(k)[S(k),T(k)]=[P(k),U(k)]\begin{bmatrix} I & -V(k)\\ 0 & A(k)'\end{bmatrix}, \tag{4.4}$$

where $[R(k),-P(k),-U(k)]$ is full rank, and

$$[S(k-1),T(k-1)]=[P(k),U(k)]\begin{bmatrix} A(k) & 0\\ Q(k) & I\end{bmatrix}. \tag{4.5}$$

This set of relations is initiated by setting

$$S(N)=\begin{bmatrix} G\\ 0\end{bmatrix}, T(N)=\begin{bmatrix} 0\\ F\end{bmatrix}$$

and then solving (4.4) for $R(N)$, $P(N)$, $U(N)$.

The two equations (4.4) and (4.5) can be written out as

$$R(k)S(k)=P(k) \tag{4.6a}$$

$$R(k)T(k)=-P(k)V(k)+U(k)A(k)' \tag{4.6b}$$

$$S(k-1)=P(k)A(k)+U(k)Q(k) \tag{4.6c}$$

$$T(k+1)=U(k). \tag{4.6d}$$

Then eliminate $T(k)$ and $P(k)$ to obtain the recursion

$$S(k-1)=R(k)S(k)A(k)+U(k)Q(k) \tag{4.7a}$$

$$R(k-1)[U(k)+S(k-1)V(k-1)]=U(k-1)A(k-1)', \tag{4.7b}$$

where $[R(k-1),-U(k-1)]$ has full rank. (Note: the condition that $[R(k-1),-U(k-1)]$ be full rank is equivalent to the condition that $[R(k-1),-P(k-1),-U(k-1)]$ be full rank because of (4.6a).) The recursion (4.7) is the basic recursion for backward sweep of the terminal conditions.

If the full-rank condition cannot be satisfied at any k, then the system is not solvable or the terminal conditions are not feasible.

Once this recursion is complete, solve the recursion

$$y(k) = R(k)y(k+1) + R(k)S(k)b(k) - U(k)c(k) \tag{4.8}$$

$$y(N+1) = (g, 0).$$

Then using the known value of $x(0)$, solve

$$U(0)\lambda(0) = y(0) - S(-1)x(0) \tag{4.9}$$

for $\lambda(0)$, which will be possible if the boundary conditions are compatible. From this, the complete forward solution can be found.

Special case. Control theory ($V(k)$ is symmetric and positive semi-definite). In this case, try a solution to the recursion (4.7) of the form $U(k) = I$. Then

$$S(k-1) = R(k)S(k)A(k) + Q(k) \tag{4.10}$$

but using $R(k) = A(k)'[I + S(k)V(k)]^{-1}$, this becomes

$$\begin{aligned} S(k-1) = A(k)'[I + S(k)V(k)]^{-1}S(k)A(k) \\ + Q(k), \end{aligned} \tag{4.11}$$

which is a discrete Riccati equation. This is the analogue to the forward Riccati equation (3.16). $S(k)$ will be symmetric and positive semi-definite.

In this case the basic entry (4.2) becomes

$$\lambda(k+1) = y(k+1) - S(k)x(k+1). \tag{4.12}$$

Using the system equation (4.1a) this becomes

$$\begin{aligned} \lambda(k+1) = y(k+1) - S(k)A(k)x(k) \\ - S(k)V(k)\lambda(k+1) + S(k)b(k) \end{aligned}$$

or, equivalently,

$$\lambda(k+1) = [I + S(k)V(k)]^{-1}\{y(k+1) - S(k)A(k)x(k) + S(k)b(k)\}. \tag{4.13}$$

This last equation can be used directly in the optimality relations

$$u(k) = e(k) - E(k)B(k)'\lambda(k+1) \tag{4.14a}$$

$$w(k) = W(k)^{-1}[C(k)'\lambda(k+1) - d(k)] \tag{4.14b}$$

to obtain a feedback solution for the optimal inputs in terms of $x(k)$.

It can be noted that the solution presented here is actually simpler than the standard solution to the linear–quadratic control problem with terminal constraints. In the standard solution the terminal condition on the Riccati equation is zero, but an additional linear matrix equation must be solved backward (Bryson and Ho, 1969). In the solution presented here, only a single discrete Riccati equation need be solved.

5. *Conclusions*

This paper has explored the dynamic systems aspects of dynamic equilibria, and has shown that such problems are similar to but inherently more difficult to solve than optimal control problems. An effective method of solution, based on descriptor variable theory, was developed.

There are several possible directions for future research. The producers might be dissimilar, and the consumers might have a more complex, even dynamic, demand function. These changes would lead to larger problems. A very important direction for further research is the consideration of stochastic variables. Many dynamic markets play an important function in distribution of risk due to uncertainty in production, demand, government policy etc. The inclusion of stochastic variables would make the theory initiated in this paper much more realistic.

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