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Complete Stability of Noncooperative Games[1]

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Abstract. This paper is concerned with a class of noncooperative games of n players that are defined by n reward functions which depend continuously on the action variables of the players. This framework provides a realistic model of many interactive situations, including many common models in economics, sociology, engineering, and political science. The concept of Nash equilibrium is a suitable companion to such models.

A variety of different sufficient conditions for existence, uniqueness, and stability of a Nash equilibrium point have been previously proposed. By sharpening the noncooperative aspect of the framework (which is really only implicit in the original framework), this paper attempts to isolate one set of "natural" conditions that are sufficient for existence, uniqueness, and stability. It is argued that l_∞ quasicontraction is such a natural condition. The concept of complete stability is introduced to reflect the full character of noncooperation. It is then shown that, in the linear case, the condition of l_∞ quasicontraction is both necessary and sufficient for complete stability.

Key Words. Game theory, stability, contraction mappings.

1. Introduction

There are many situations, involving a group, where the profit, reward, or value accruing to each individual depends jointly both on his own actions and on the actions of others. Such situations include purely competitive systems where a given reward is divided among a number of individuals, and systems where the reward received by each individual increases as any individual effort increases. These situations often can be

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conveniently and realistically modeled in terms of n payoff functions which depend continuously on the value of n action variables. That is, there are n functions $f_i(x_1, x_2, \ldots, x_n)$, $i = 1, 2, \ldots, n$, such that f_i is the reward to the ith individual, and x_i is a variable measuring the activity level of the ith individual. The activity vector x is required to be within a given set Ω. This framework is a special case of a *noncooperative game.*

An important and useful equilibrium concept for noncooperative games of this type is that of the *Nash equilibrium.* In a Nash equilibrium the variables $x_1, x_2, \ldots, x_n$ collectively have values such that, for each $i = 1, 2, \ldots, n$, $f_i(x_1, x_2, \ldots, x_n)$ is maximal with respect to changes in x_i alone. Under relatively mild conditions, it is known that such a Nash equilibrium exists for the framework above, and hence, this solution concept provides a basis for analysis that is both intuitively attractive and mathematically consistent.

Special forms of this structure or of closely related structures have arisen in many contexts. A classic example is the Cournot theory of oligopoly, where the f_i's are firm's profits and the x_i's are production levels (Ref. 1). Another example is provided by the theory of teams of Marschak and Radner (Refs. 2–3), where all f_i's are the same (quadratic) function, representing the objective of the team. In another example, related to the neighborhood effects of housing maintenance, the f_i's correspond to rental values of housing units and the x_i's correspond to maintenance levels. The rental values depend on the maintenance level of all units in the neighborhood (Ref. 4). The theory of anarchy of Bush and Mayer (Refs. 5–6) provides another example, with the f_i's corresponding to utility functions and the x_i's corresponding to stealing effort. The theory of market signaling between an educated potential employee and an employer, developed by Spence (Ref. 7), can be cast into the general framework of this paper with the f_i's being expected net profit and the x_i's being educational level and salary offers, respectively (Ref. 8). Even classroom behavior can be considered in this framework with the f_i's corresponding to each student's reward, which is partly the reward of knowledge and partly a competitively assigned grade, and the x_i's corresponding to individual effort. Finally, there are numerous other models, characterized by sets of equations, that can be easily transformed to the mutual maximization framework and thus fall within the context of this paper. Examples of this type include the characterization of economic equilibria (Ref. 9) and some problems in decentralized control systems (Refs. 10–11).

Three basic analytical issues that arise in virtually every application of this theoretical framework are existence, uniqueness, and stability of equilibrium points. The stability issue is treated, of course, only after specification of an adjustment process (or class of adjustment processes) by which individual participants are assumed to adjust their action variables. The three analytical issues are generally resolved by posing a set of *sufficient* conditions. Thus, depending on the author or the circumstances, conditions of concavity, monotonicity, row diagonal dominance, column diagonal dominance or some form of contraction might be imposed. Each such set of assumptions leads to a "theory" which ensures existence, uniqueness, and stability. For example, see Refs. 10–11 for a sophisticated development of three parallel theories. Although each of these special theories has its merits, there has been no clear indication of which is most natural for the framework.

This paper examines these issues from a perspective that extrapolates the noncooperative viewpoint beyond what exists within the original framework. From this perspective, the desired properties are required to be immune from incidental (implicit) cooperation that might arise because of restrictive definitions of the set Ω or of the adjustment process.

2. Basic Framework

As described in the introduction, let $f_i(x_1, x_2, \ldots, x_n)$, $i = 1, 2, \ldots, n$, be *n reward* (or *payoff*) *functions* defined on a region $\Omega \subset R^n$. It is assumed throughout that each of these functions is continuous, and that Ω is a compact convex set.

Equilibria. A point $(x_1, x_2, \ldots, x_n) \in \Omega$ is an *equilibrium point* for these functions if, for each $i = 1, 2, \ldots, n$,

$$f_i(x_1^*, x_2^*, \ldots, x_n^*)$$
$$= \max_{x_i}\{f_i(x_1^*, x_2^*, \ldots, x_i, \ldots, x_n^*): (x_1^*, x_2^*, \ldots, x_i, \ldots, x_n^*) \in \Omega\}.$$

That is, the function f_i is maximal at the equilibrium point with respect to changes in the ith variable x_i. In terms of a setting for noncooperative games, an equilibrium point is a feasible set of actions with the property that no individual can increase his reward by a unilateral change in his own action.

The following existence theorem can be considered as a foundation for this model. For the proof, see Rosen (Ref. 12).

Theorem 2.1. Let Ω be a compact convex subset of E^n. Let each of the functions f_i, $i = 1, 2, \ldots, n$, be continuous on Ω and concave with respect to x_i for each fixed $x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$. Then, there is an equilibrium point $(x_1^*, x_2^*, \ldots, x_n^*) \in \Omega$.

A basic theoretical tool for the analysis that follows is that of response mappings (often termed a reaction function in economic analysis), which essentially characterize the result of the component-by-component maximization. Specifically the ith *response mapping* h_i, $i = 1, 2, \ldots, n$, is defined by

$$h_i(x_1, x_2, \ldots, x_i, \ldots, x_n) = \max_{\bar{x}_i}{}^{-1}\{f_i(x_1, x_2, \ldots, \bar{x}_i, \ldots, x_n) : (x_1, x_2, \ldots, \bar{x}_i, \ldots, x_n) \in \Omega\}.$$

That is, the ith response mapping gives the value (or values) of the variable $\bar{x}_i$ that maximizes f_i for given fixed values of the $n-1$ other variables. For notational convenience only, the mapping h_i is regarded as dependent on the variable x_i as well as the other $n-1$ variables. In fact, of course, x_i has no influence on h_i, since it is the one free variable in the maximization. The overall *response mapping* is defined as the mapping h, where the ith component of $h(x)$ is the corresponding $h_i(x)$. The response mapping h is therefore a mapping from Ω into Ω. In general, it may be a point-to-set mapping, since some maxima may not be unique. For notational simplicity, it is assumed throughout the remainder of the paper that f_i is strictly concave with respect to x_i so that the maxima are unique, and hence, that the response mapping is single-valued.

An equilibrium point corresponds to a fixed point of the response function. That is, x^* is an equilibrium point, as defined earlier, if $x^* = h(x^*)$. It is therefore clear that properties of the response function are important in questions of existence and uniqueness of equilibria, as well as of convergence of successive approximation procedures.

Contractions. To establish uniqueness of equilibrium, a contraction assumption is imposed. Although there are a number of natural possibilities, based on established conventions, the particular assumption used here, contraction with respect to the l_∞ norm on R^n, is actually motivated by the investigation of dynamic stability presented in the next section. For convenience, the notation

$$|x| = \max_{1 \le i \le n} |x_i|$$

is introduced.

Definition 2.1. The mapping $h : \Omega \to \Omega$ is a *contraction* if

$$|h(x) - h(y)| < |x - y| \tag{1}$$

for all $x, y \in \Omega$, $x \neq y$. The mapping h is a *quasicontraction* mapping if there is a set of positive constants $c_1, c_2, \ldots, c_n$ such that

$$\max_i c_i|h_i(x)-h_i(y)| < \max_i c_i|x_i - y_i|$$

for all $x, y \in \Omega$, $x \neq y$.

The statement that h is a quasicontraction mapping is equivalent to the statement that h is a contraction mapping under the change of variable

$$\xi_i = x_i/c_i.$$

Obviously, a quasicontraction mapping is, in the abstract nonlinear setting, essentially equivalent to a contraction mapping. This generalization is, however, extremely valuable in the context of specific situations, and plays a major role in the linear theory of Section 4. Nevertheless, throughout the remainder of this section explicit reference is made only to contractions, with the understanding that virtually all results extend to quasicontractions as well.

A standard result is the following.

Theorem 2.2. If the response function h is a contraction on Ω then there is at most one equilibrium point.

Proof. Suppose x^* and y^* are solutions, with $x^* \neq y^*$. Then, x^* and y^* are both fixed points of h. Thus,

$$|x^* - y^*| < |x^* - y^*|,$$

which is a contradiction.

Until now, little has been said about the nature of the set Ω, which potentially can play an important role in determining whether a given individual is dominated by the rest of the group. It is sometimes possible, for example, that by moving to the boundary of the set Ω one individual can effectively constrict the action available to another, forcing a loss of individual control. This and other undesirable effects can be traced to the fact that a game having a contractive response function over a set Ω may have a noncontractive response function when restricted to a closed convex subset of Ω. This is often accompanied, of course, by nonuniqueness of equilibria over the subset.

Example 2.1. On R^2, let

$$f_1(x_1, x_2) = -(x_1 - 1)^2 - x_2^2,$$

$$f_2(x_1, x_2) = -x_1^2 - (x_2 - 1)^2.$$

Over the whole of R^2, there is a unique equilibrium

$$x^* = (1, 1).$$

Indeed, the response mapping is

$$h(x) = (1, 1),$$

which is clearly a contraction.

Now, define the set

$$\Omega = \{x : x_1 + x_2 \leq -1, x_1 \leq 0, x_2 \leq 0\}.$$

Over this set, the equilibria are not unique, since any point on the interval

$$x_1 + x_2 = -1, \qquad x_1 \leq 0, \qquad x_2 \leq 0$$

is an equilibrium point. The reponse function in this case is

$$h_1(x) = \min\{0, -1 - x_2\}, \qquad h_2(x) = \min\{0, -1 - x_1\},$$

which is not a contraction (see Fig. 1).

Motivated by these observations, the following result guarantees that the contraction property is preserved when the only additional constraints imposed are those affecting variables individually. This requirement on the constraints is, of course, entirely consistent with the original objective to consider games where individuals operate separately. For this to be meaningful, it is natural to assume that the choice range of a particular variable is not dependent on the values of other variables.

Theorem 2.3. Suppose that the response mapping of a game defined over a closed convex set Ω is contractive. Then, the response mapping for the same game defined with additional constraints of the form $a_i \leq x_i$ or $x_i \leq b_i$, $i = 1, 2, \ldots, n$, is also contractive.

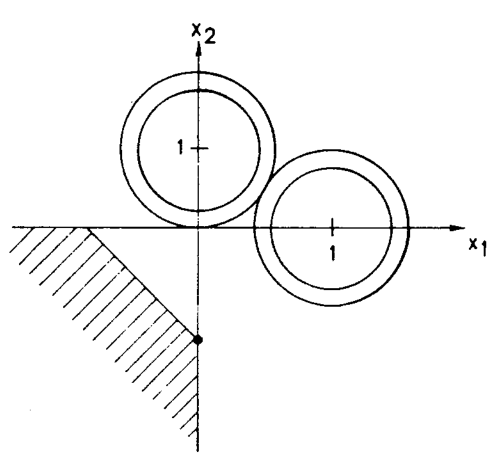

Fig. 1. An example.

Proof. Consider the introduction of the constraints

$$a_i \le x_i, \qquad i = 1, 2, \ldots, n.$$

For each $x \in \Omega$, let A_i denote the segment of the real line,

$$A_i = \{x_i \ge a_i,\ x \in \Omega\}.$$

By the concavity of f_i, it follows that the new ith response mapping $\tilde{h}_i$ is defined by

$$\tilde{h}(x) = \max\{a_i, h_i(x)\}.$$

Then, for any $x, y \in \Omega$, satisfying the additional inequality constraints, it follows that

$$|\tilde{h}(x) - \tilde{h}(y)| < |x - y|.$$

Therefore, $\tilde{h}$ is a contraction. A similar argument applies to the constraints

$$x_i \le b_i, \qquad i = 1, 2, \ldots, n.$$

Differentiable Case. In the case where the functions f_i, $i = 1, 2, \ldots, n$, are differentiable with respect to their corresponding x_i's, it is natural to define the functions

$$w_i(x_1, x_2, \ldots, x_n) = \partial f_i(x_1, x_2, \ldots, x_n)/\partial x_i. \tag{2}$$

In view of the concavity of f_i with respect to x_i, it follows that w_i is a nonincreasing function of x_i for fixed values of the other variables.

In terms of these functions, an interior equilibrium point must satisfy the system of equations

$$w_i(x_1, x_2, \ldots, x_n) = 0, \qquad i = 1, 2, \ldots, n. \tag{3}$$

Or more generally, if the set Ω is defined entirely by inequality constraints,

$$a_i \le x_i \le b_i,$$

then the necessary and sufficient conditions for an equilibrium are satisfaction of

$$\begin{aligned} &w_i(x_1, x_2, \ldots, x_n) = 0 && \text{if } x_i \ne a_i,\ x_i \ne b_i, \\ &w_i(x_1, x_2, \ldots, x_n) \ge 0 && \text{if } x_i = b_i, \\ &w_i(x_1, x_2, \ldots, x_n) \le 0 && \text{if } x_i = a_i. \end{aligned} \tag{4}$$

Conversely, problems of economic equilibria and some problems of decentralized control (Ref. 10) have the form (4). It is possible, if the w_i's are continuous, to put these in the framework of this paper by defining a set of associated reward functions by

$$f_i(x_1, x_2, \ldots, x_n) = \int_{a_i}^{x_i} w_i(x_1, x_2, \ldots, x_{i-1}, \xi_i, x_{i+1}, \ldots, x_n) d\xi_i.$$

If w_i is nonincreasing with respect to x_i, f_i will be concave with respect to x_i.

The response function $h(x)$ is, of course, related to the function $w(x)$. Since it is the latter which has an explicit representation, it is natural to seek conditions on $w(x)$ that imply that $h(x)$ is contracting.

Theorem 2.4. Suppose that, for any $x \in \Omega$, $y \in \Omega$, with $x \neq y$, there holds

$$[w_i(x) - w_i(y)](x_i - y_i) < 0 \tag{5}$$

for all i for which

$$|x_i - y_i| = \max_j |x_j - y_j|.$$

Then, the response function is contracting.

Proof. Given

$$x = (x_1, x_2, \ldots, x_n) \quad \text{and} \quad y = (y_1, y_2, \ldots, y_n),$$

define

$$\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n) \quad \text{and} \quad \bar{y} = (\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_n)$$

by

$$\bar{x} = h(x), \qquad \bar{y} = h(y).$$

Let us consider $i = 1$. If $\bar{x}_1 = a_1$, then the corresponding

$$w_1(\bar{x}_1, x_2, \ldots, x_n) \leq 0.$$

If $\bar{x}_1 = b_1$, then

$$w_1(\bar{x}_1, x_2, \ldots, x_n) \geq 0;$$

and, if $a_1 < \bar{x}_1 < b_1$, then

$$w_1(\bar{x}_1, x_2, \ldots, x_n) = 0.$$

Similar statements apply to $\bar{y}_1$. Assume that $\bar{x}_1 \leq \bar{y}_1$. Then, it is clear from the above that

$$w_1(\bar{x}_1, x_2, \ldots, x_n) \leq w_1(y_1, y_2, \ldots, y_n).$$

Thus, in any case,

$$[w_1(\bar{x}_1, x_2, \ldots, x_n) - w_1(\bar{y}_1, y_2, \ldots, y_n)](\bar{x}_1 - \bar{y}_1) \geq 0.$$

Thus, by hypothesis, it must follow that

$$|\bar{x}_1 - \bar{y}_1| < \max_{j \neq 1} |x_j - y_j|.$$

The same argument shows that, for any $i = 1, 2, \ldots, n$,

$$|\bar{x}_i - \bar{y}_i| < \max_j |x_j - y_j|.$$

Hence, the response mapping is a contraction.

This result has an interpretation in terms of the relative control of the various individuals. If f_i is strictly concave with respect to x_i, it follows that, if x_i alone is changed (by say Δx_i), then the resulting change Δw_i in w_i satisfies

$$\Delta w_i \, \Delta x_i < 0.$$

Now, suppose that x_i is changed by Δx_i and the other variables are allowed to change no more than this (in absolute value). If the change in w_i always remains negative, the condition of the theorem is satisfied. In this case, the influence of x_i on w_i is in a sense greater than that of all other variables.

An important special case, which is discussed in greater detail later, is where the functions w_i are linear (or affine). Suppose that

$$w_i = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n - c_i.$$

Condition (5) is then equivalent to diagonal dominance of the matrix

$$A = [a_{ij}]$$

defined by the condition

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}|$$

for each $i = 1, 2, \ldots, n$.

Summary. The main point of this section is that the condition of l_∞ contraction of the response function provides a sound foundation for further investigation of noncooperative games. It guarantees uniqueness of equilibrium, it is not destroyed by the imposition of separate action variable constraints, and it is implied by a natural dominance condition on the derivative of the reward functions. This provides a good backdrop for the somewhat deeper topic of stability analysis, where the contraction property is again employed.

3. Stability

To address the question of stability, it is necessary to hypothesize a dynamic adjustment mechanism for a noncooperative game (Refs. 13–14). One standard mechanism, for example, is simultaneous displacements, where at every iteration each individual changes his action variable so as to maximize his reward function, assuming the values from the last iteration for the other variables. In reality, of course, the process of adjustment by independent individuals cannot be expected to necessarily follow a fixed known pattern; rather, it is likely that it will be a complex process dependent on many personal factors not explicitly accounted for in the basic framework defining the reward function. One is therefore motivated to define the process as loosely as possible, consistent only with the idea that individuals attempt to improve their own reward, and to seek conditions guaranteeing stability with respect to this broad definition. To carry out this plan, a general adjustment process described as a point-to-set mapping is defined (Ref. 15).

Set Mappings. The metric space (Γ, ρ) is defined to consist of the nonempty, compact subsets of Ω, with metric defined as

$$\rho(U, V) = \inf\{\epsilon > 0 : U + S_\epsilon \supset V \text{ and } V + S_\epsilon \supset U\},$$

where

$$S_\epsilon = \{x : |x| \leq \epsilon\}.$$

It is known (Ref. 16) that this metric space is complete and compact.

Now, suppose that G is a mapping on (Γ, ρ) taking compact subsets into compact subsets. Such a mapping is a *contraction* if

$$\rho(G(U), G(V)) < \rho(U, V)$$

for all $U, V, U \neq V$.

If $h : \Omega \to \Omega$ is a continuous point-to-point mapping, it induces a mapping H on the set of compact subsets, in the obvious way—pointwise. If such a mapping h is a contraction in the normal sense, then it is easy to see that the included mapping H is a contraction on (Γ, ρ).

In a similar manner, an upper semicontinuous mapping T which maps points in Ω to compact subsets of Ω can be regarded as a mapping on (Γ, ρ) by pointwise combination (Ref. 17). Such a point-to-set mapping T is said to be a contraction if

$$\rho(T(x), T(y)) < |x - y|$$

for all $x \neq y$. It is easy to show that, if T is a contraction, then it is upper semicontinuous and the induced mapping on (Γ, ρ) is also a contraction.

A standard result which will be applied to the metric space (Γ, ρ) for various mappings is the following (Ref. 18).

Theorem 3.1. If X is a complete and compact metric space and T is a contraction mapping on X, then T has a unique fixed point in X. Furthermore, the fixed point is the limit point of the sequence $\{x_k\}$ generated by the method of successive approximation

$$x_{k+1} = T(x_k)$$

with x_0 arbitrary in X.

The general mode of analysis employed in the remainder of this section is this. A dynamic process is defined by defining a particular point-to-set mapping. By showing that this mapping induces a contraction on (Γ, ρ), it follows from the above theorem that the process converges to a closed subset of Ω. By relating the original point-to-set mapping to the response mapping h, it can be shown that the limit set actually consists only of the equilibrium point of the game. Essentially, the contraction principle is employed simultaneously in R^n and (Γ, ρ).

Simultaneous Displacements. A natural adjustment process for initial investigation of stability is that of simultaneous displacements mentioned earlier. In economic contexts, this is often referred to as the Cournot process. It has a simple interpretation in terms of individual response, and its convergence is directly related to the contraction property. The process is defined as follows: Given the point

$$x = (x_1, x_2, \ldots, x_n),$$

the successor point

$$\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$$

is defined as the vector of components corresponding to individual maximization under the assumption that all other variables remain fixed; that is,

$$\bar{x} = h(x).$$

It follows as an immediate corollary of Theorem 3.1 that the method of simultaneous displacements converges if the response function is a quasi-contraction.

As a first modification of the pure method of simultaneous displacements, consider the possibility that some individuals, after calculating their optimal immediate response, as in simultaneous displacements, decide to move only some fraction of the distance toward this value from their current position. This is representative of a kind of conservatism in their response, perhaps reflecting the fact that, after they have adjusted part way, there is opportunity to again observe the state of the entire system. In terms of iteration processes, this form of adjustment is referred to as simultaneous *under-relaxation*.

Specifically, given

$$x = (x_1, x_2, \dots, x_n),$$

let

$$\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)$$

be determined by the usual method of simultaneous displacements. Then, for $0 < \alpha \leq 1$, let

$$T_\alpha(x) = \{\bar{x} : \bar{x}_i = x_i + \alpha_i(\tilde{x}_i - x_i),\ 0 < \alpha_i \leq 1,\ i = 1, 2, \dots, n\}.$$

T_α is a point-to-set mapping for $\alpha < 1$. The usual response function h corresponds to T_1.

Theorem 3.2. If h is a contraction with respect to l_∞, then so is T_α for all $\alpha, 0 < \alpha \leq 1$, and simultaneous under-relaxation converges to the unique equilibrium point.

Proof. Let x and y be given with $x \neq y$, and let

$$\bar{x} \in T_\alpha(x).$$

It will be shown that there is a point

$$\bar{y} \in T_\alpha(y)$$

such that

$$|\bar{x} - \bar{y}| < |x - y|.$$

This will establish that

$$\rho(T(x), T(y)) < |x - y|.$$

In particular, select

$$\bar{y} \in T_\alpha(y)$$

to be constructed with the same set of α_i's as is

$$\bar{x} \in T_\alpha(x).$$

Then, for each $i = 1, 2, \ldots, n$,

$$\bar{x}_i - \bar{y}_i = (1-\alpha_i)(x_i - y_i) + \alpha_i[h_i(x) - h(y)],$$

$$|\bar{x}_i - \bar{y}_i| \leq (1-\alpha_i)|x_i - y_i| + \alpha_i|h(x) - h(y)|$$

$$< (1-\alpha_i)|x_i - y_i| + \alpha_i \max_j |x_j - y_j|$$

$$< \max_j |x_j - y_j|.$$

Therefore, T_α is a contraction. It follows that T_α is upper semicontinuous and that the induced mapping on (Γ, ρ) is a contraction. Thus, by Theorem 3.1, the process converges to a point in (Γ, ρ), that is, to a closed set which is fixed under T_α.

Clearly, the equilibrium point is fixed under T_α. Thus, by the uniqueness of the fixed point in (Γ, ρ), the limit set must be the equilibrium point.

The above result on simultaneous displacements with under-relaxation can be regarded as a point-to-set version of the kind of result that is typical in the literature of adjustment processes. It is the discrete-time version, for example, of the "speeds of adjustment" results known for the tâtonnement process (Ref. 9), which considers a fixed set of speeds for each iteration. Results of this type show that contraction provides a sufficient condition for stability of these elementary processes, but they do not really reveal the full and powerful implications of the contraction assumption.

Compound Processes. To consider more complex processes, the notion of a general subgame is introduced. Essentially, a general subgame is obtained by allowing only some variables to change in a given reward function, with the others held fixed. However, the choice of which variables are free and which are fixed is allowed to be different for each reward function, the only requirement being that x_i be free in f_i.

Definition 3.1. Let

$$x = (x_1, x_2, \ldots, x_n) \in \Omega$$

be given. Suppose that, for each i, the set of indices $\{1, 2, \ldots, n\}$ is divided into two parts: I_i and J_i, with $i \in I_i$. The functions

$$g_i(\bar{x}) = f_i(\xi_1, \xi_2, \ldots, \xi_n),$$

where

$$\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$$

and

$$\xi_j = \bar{x}_j \qquad \text{if } j \in I_i,$$

$$\xi_j = x_j \qquad \text{if } j \in J_i,$$

together with the constraint $\bar{x} \in \Omega$, constitute a *general subgame* of the original noncooperative game.

A general subgame can itself be considered a noncooperative game but with somewhat fewer interactions—the J_i variables being considered fixed in f_i.

General subgames essentially represent various information patterns. A given individual may account for changes in the values of only some of the other action variables during the course of his own adjustments. This is, of course, not optimal, but it may be reasonable given his information sources. The motivation for introducing this concept is that, if the game supposedly does not depend on cooperation for its stability, then stability should hold even if some players ignore information. If some individuals refuse to adjust at all, this should not destroy convergence of the resulting subgame. Thus, the general subgame concept provides a means for exploring the hypothesis that stability of the game is not dependent on (indirect) cooperation.

Theorem 3.3. If the response mapping h of a game is a contraction, then the response mapping of any general subgame is also a contraction.

Proof. Let

$$z = (z_1, z_2, \ldots, z_n)$$

define the values of the fixed variables. Consider

$$g_i(x) = f_i(\xi_1, \xi_2, \ldots, \xi_n),$$

$$g_i(y) = f_i(\eta_1, \eta_2, \ldots, \eta_n),$$

where

$$\xi_j = x_j \qquad \text{and} \qquad \eta_j = y_j \qquad \text{if } j \in I_i$$

and where

$$\xi_j = \eta_j = z_j \qquad \text{if } j \in J_i.$$

If $g_i(x)$ and $g_i(y)$ are maximized at $\bar{x}_i$ and $\bar{y}_i$, respectively, then clearly

$$|\bar{x}_i - \bar{y}_i| < \max_k |\xi_k - \eta_k| \le \max_k |x_k - y_k|.$$

The concept of a general subgame is also used to construct a class of general adjustment processes. This construction is based on the observation that the function g_i, $i = 1, 2, \ldots, n$, in a general subgame is concave with respect to the ith variable because f_i is. Thus, by the fundamental existence theorem, there is an equilibrium point $\bar{x}$ for the subgame on the compact convex set Ω. Such a point can be used to define the successor point of the original fixed values.

Definition 3.2. Let G be a noncooperative game. Let I_i, J_i, $i = 1, 2, \ldots, n$, be a given set of decompositions of $\{1, 2, \ldots, n\}$ with $i \in I_i$. For each

$$x = (x_1, x_2, \ldots, x_n) \in \Omega,$$

let G_x be the general subgame of G with the given decomposition of variables and with fixed values equal to the corresponding components of x. The equilibrium $\bar{x}$ to this general subgame defines the mapping T satisfying

$$\bar{x} = T(x).$$

The iterative process using T as the successor function is called a *basic adjustment process.*

There are many interesting special cases of basic adjustment processes. Selecting

$$I_i = \{i\}$$

amounts to letting only the ith variable be free in f_i, and corresponds to simultaneous displacements. Setting

$$I_i = \{1, 2, \ldots, i\}$$

corresponds to *successive displacements*, where the variables are updated in order. And setting

$$I_i = \{1, 2, \ldots, n\}$$

corresponds to the original problem itself, and one iteration yields a solution to the complete system. Most standard iterative algorithms (Ref. 19) are basic adjustment processes. An interpretation of an iterative scheme of this type is that the individual i bases his adjustment on information that he has available. He may know other individuals' adjustments either because they are made earlier and can be observed, or because he works out his decision jointly with others. Essentially, any information pattern is conceivable.

Proposition 3.1. If h is a contraction, any basic adjustment process is a contraction.

Proof. Let x and y be given, and let $\bar{x}$ and $\bar{y}$ be the corresponding updated points. Let i be such that

$$|\bar{x}_i - \bar{y}_i| = \max_k |\bar{x}_k - \bar{y}_k|.$$

Without loss of generality (by reordering the variables if necessary), it can be assumed, for this i, that

$$I_i = 1, 2, \ldots, i.$$

It is clear that $\bar{x}_i$ maximizes

$$f_i(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_i, x_{i+1}, \ldots, x_n)$$

with *all* other variables held fixed. This follows from the definition of $\bar{x}$. Likewise, $\bar{y}_i$ maximizes $f_i(\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_i, y_{i+1}, \ldots, y_n)$. Therefore, by the contraction property of the response mapping, it follows that

$$|\bar{x}_i - \bar{y}_i| < \max_{\substack{k<i\\ j>i}} \{|\bar{x}_k - \bar{y}_k|, |x_j - y_j|\}.$$

However, by definition of i, it follows that

$$|\bar{x}_k - \bar{y}_k| \leq |\bar{x}_i - y_i|,$$

for all k. Thus,

$$|\bar{x}_i - \bar{y}_i| < \max_{j>i} |x_j - y_i| \leq \max_j |x_j - y_j|.$$

Therefore, the mapping is a contraction.

It is clear that under-relaxation can be used in conjunction with any basic adjustment process while preserving the contraction property. Likewise, convex combinations of basic adjustment processes form new processes that preserve the contraction property.

Definition 3.3. For $x \in \Omega$, and $0 < \alpha \leq 1$, let $G_\alpha(x)$ be the (closed) convex hull of points generated by all basic adjustment mappings, each with a relaxation factor between α and 1. The mapping G_α is a *general adjustment mapping*, and the corresponding process is a *general adjustment process*.

In view of the previous results and the above comments, it is clear that, if h is a contraction, any general adjustment mapping is a contraction.

Thus, the result motivates the final definition and the associated main theorem on stability.

Definition 3.4. A noncooperative game is *completely stable* if, in the complete game and in any general subgame, all general adjustment processes with $0<\alpha\leq 1$ converge.

Theorem 3.4. Any game with a contractive response mapping is completely stable.

4. Linear Case

In this section, it is assumed that the functions f_i consist of quadratic and linear terms. Accordingly, the functions w_i are linear (or affine). In this case, the system of relations satisfied by an equilibrium point reduces to an nth order linear system of the form

$$Ax = c$$

if boundary points

$$x_i = a_i \qquad \text{or} \qquad x_i = b_i$$

are tacitly excluded. Therefore, the iterative adjustment process can be considered as a method for solving a system of linear equations. In studying such methods for a linear system as above, it is generally sufficient to consider the homogeneous system

$$Ax = 0,$$

and hence, without loss of generality it can be assumed that

$$c = 0.$$

By the concavity assumption, the diagonal terms of the matrix A are all negative. Therefore, one may write

$$A = -D + B,$$

where D is diagonal with positive diagonal entries and B has zero diagonal. In the linear case, the method of simultaneous displacements takes the form

$$-Dx_{k+1} + Bx_k = 0$$

or, equivalently,

$$x_{k+1} = D^{-1}Bx_k.$$

The method is convergent for all initial conditions if the matrix $D^{-1}B$ has all of its eigenvalues of magnitude less than unity. This will be the case, of course, if A is quasidiagonally dominant, for then the response mapping is a contraction. A is quasidiagonally dominant if it can be made diagonally dominant by multiplying its columns by suitable positive constants. This result is well known and follows directly from the results of the previous section. Indeed, those results show that quasidiagonal dominance is sufficient for complete stability. The goal in this section, as mentioned earlier, is to show that quasidiagonal dominance is not only sufficient for complete stability, it is also *necessary*.

One case where necessity is known is where A is irreducible and nonnegative. The matrix A is *reducible* if there is an $n \times n$ permutation matrix P such that P^TAP takes the partitioned form

$$P^TAP = \begin{bmatrix} A_{11} & 0 \\ A_{12} & A_{22} \end{bmatrix}.$$

Otherwise, the matrix A is *irreducible*. For this case, the following result, derived from Frobenius–Perron theory, is applicable.

Proposition 4.1. Let $a_{ij} \geq 0$ for $i \neq j$, and suppose that A is irreducible. Then, for the method of simultaneous displacements to converge, it is necessary and sufficient that A be quasidiagonally dominant.

Proof. As before, one may write

$$A = -D + B,$$

where now $B \geq 0$. The iteration matrix $D^{-1}B$ is therefore nonnegative and irreducible. For this matrix to have all its eigenvalues less than unity in magnitude, it is necessary that there be an eigenvector $p, p > 0$, corresponding to a nonnegative eigenvalue of less than unity. That is,

$$D^{-1}Bp = \lambda p$$

with $0 \leq \lambda < 1$. It follows immediately that

$$D^{-1}Bp < p,$$

and thus

$$Ap = -Dp + Bp < 0.$$

This, however, implies quasidiagonal dominance.

The main result of this section is a generalization of the above which does not require nonnegativity. However, the full scope of the definition of general stability is employed.

Theorem 4.1. An irreducible linear noncooperative game is completely stable iff it is quasidiagonally dominant.

Proof. Write A in the form

$$A = -D + M - B,$$

where

$$D > 0, \qquad M \geq 0, \qquad B \geq 0, \qquad D \text{ is diagonal.}$$

M and B have zero diagonal; and, if an element of one is positive, the corresponding element in the other is zero.

Consider the general subgame defined with the free variables corresponding to the nonzero elements in $-D+M$. This general subgame must be stable with respect to simultaneous displacements, and therefore all eigenvalues of $D^{-1}M$ are less than 1. Thus, since $M \geq 0$, it follows, from the Frobenius–Perron theorem, that $(I - D^{-1}M)^{-1}$ exists and is nonnegative.

The basic adjustment mechanism corresponding to this same general subsystem is

$$(-D+M)x_{k+1} - Bx_k = 0$$

or, equivalently,

$$x_{k+1} = -(D-M)^{-1}Bx_k.$$

However,

$$(D-M)^{-1} = (I - D^{-1}M)^{-1}D^{-1},$$

which is nonnegative. Thus, $(D-M)^{-1}B$ is nonnegative.

Since this basic adjustment process must be stable, it follows that the nonnegative matrix $(D-M)^{-1}B$ has a dominant eigenvalue of less than unity. Thus, the expression

$$(D-M)^{-1}Bp \geq \sigma p \tag{6}$$

with $p \geq 0$, implies $\sigma < 1$.

Now, let μ and q be, respectively, the dominant eigenvalue and corresponding eigenvector of the matrix $D^{-1}(M+B)$. Since this matrix is nonnegative and irreducible, it follows that

$$\mu \geq 0, \qquad q > 0.$$

One easily verifies the series of algebraic manipulations:

$$D^{-1}(M+B)q = \mu q,$$

$$\{-I+D^{-1}(M+B)\}q = (\mu-1)q,$$

$$\{-D+M+B\}q = (\mu-1)Dq,$$

$$(D-M)\{-I+(D-M)^{-1}B\}q = (\mu-1)Dq,$$

$$(D-M)^{-1}Bq = q+(\mu-1)(D-M)^{-1}Dq.$$

Now, again, since $(D-M)^{-1}$ is nonnegative, one sees from the last line above that $\mu \geq 1$ implies that

$$(D-M)^{-1}Bq \geq q.$$

This, however, contradicts (6), and so it can be concluded that $\mu < 1$. Then, by the definition of q, it follows that

$$D^{-1}(M+B)q < q$$

or, equivalently,

$$Dq > (M+B)q.$$

Since $q > 0$, the above inequality shows that A is quasidiagonally dominant.

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