Control Problems with Kinks

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***Abstract*—An important class of optimal control problems, arising frequently in an economic framework, is characterized as having a cost functional that is continuous but has discontinuous partial derivatives with respect to the state variables. Such problems are said to have kinks. Along a kink the classical adjoint equation breaks down, and it is impossible to define a gradient. In this paper it is shown that the gradient can be replaced by a more general definition of the direction of steepest descent but that the adjoint equation must in general be replaced by an adjoint optimal control problem. This yields a complete set of necessary conditions for problems of this type. The results derived are then combined with the theory of penalty functions to convert a problem having state constraints to one without such constraints.**

I. Introduction

THE VERY heart of modern optimal control theory, both from the viewpoint of describing optimal controls in terms of necessary conditions and from the viewpoint of constructing optimal controls numerically, is the adjoint differential equation. For a majority of problems the adjoint equation leads directly to the appropriate gradient of the cost functional with respect to control. Setting this gradient equal to zero yields necessary conditions, while evaluation of this gradient at a nonoptimal point often forms the basis for modification of the control according to a numerical descent procedure.

Some problems of importance, however, are not smooth enough to possess gradients. If the nonsmoothness is with respect to the state vector, a situation common in problems found in an economic setting, gradients do not exist, the value of the adjoint equation itself is lost, and the dynamic nature of perturbation behavior is destroyed. A new approach is required for problems of this type.

To make the discussion specific, consider the system described by

$$\dot{x}(t) = f(x, u), \quad x(t_0) = x_0 \tag{1}$$

where $x(t)$ is an $n \times 1$ state vector and $u(t)$ is an $m \times 1$ control vector. The function f is n dimensional. Consider also the cost functional

$$J = \int_{t_0}^{t_1} l(x, u)\, dt \tag{2}$$

where t_0 and t_1 are fixed. The functional J can be regarded as a function only of u because once u is selected, x can be determined from (1).

If the functions f and l have continuous derivatives with respect to x and u, then, as is well known, the gradient of J with respect to u is the function

$$\lambda'(t) f_u + l_u \tag{3}$$

where λ satisfies the adjoint differential equation

$$-\dot{\lambda}(t) = [f_x]'\lambda(t) + l_x, \quad \lambda(t_1) = 0. \tag{4}$$

If any of the partial derivatives fails to exist for a given x, u, then, in general, J will not possess a gradient at that point.

Nonexistence of the partials with respect to u is not, however, a serious matter. The adjoint equation (4) can still be evaluated and retains its position of central importance for such problems. The difficulties associated with the nonexistence of partials with respect to u can be circumvented in most cases by consideration of two-sided derivatives or by maximizing the associated Hamiltonian. These techniques and their supporting theory have been available for some time, and today nonexistence of partials with respect to u is not considered a major hindrance.

Nonexistence of partial derivatives with respect to x is more serious. The adjoint equation breaks down and cannot be easily repaired by consideration of two-sided derivatives or other simple measures. The fundamental dynamic property of the perturbation equations, at the root of the classical theory, breaks down and must be replaced by new machinery.

In this paper problems in which l_x does not exist are treated. (The difficulties encountered when f_x does not exist appear to be substantially greater.) Such problems arise naturally in many economic or physical systems where the cost functional has a kink, such as in the function $|x|$ at $x = 0$. Objective functions of this form also arise in the implementation of a particular penalty function scheme discussed in Section IV of this paper.

The approach taken in this paper treats the kink exactly, with no approximation. An alternative procedure that comes to mind is to approximate the kink by a tight but smooth arc and hence seemingly bypass the difficulty. This approach is unsatisfactory since the limiting conditions when the arc radius approaches zero are difficult to obtain.

Example 1

As an illustration of the kind of difficulty that is of concern, consider the problem

$$\dot{x}(t) = u(t) - d(t), \quad x(0) = 0$$

$$J = \int_0^T \{|x(t)| + \tfrac{1}{2}u(t)^2\}\, dt$$

$$u(t) \geq 0.$$

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This problem can be interpreted as a production scheduling problem in which $x(t)$ represents inventory on hand (with negative inventory representing back orders); $d(t)$ is the demand rate at time t (it is assumed known); and $u(t)$, the control, is the production rate. There is a unit cost for storing inventory and a unit cost, for loss of goodwill, in carrying back orders. The cost of production is quadratic.

A simple trial solution to this problem is $u(t) = d(t)$. This yields $x(t) = 0$, which lies on the kink in the objective functional. At this point the usual adjoint equation cannot be calculated since l_x does not exist. Furthermore, it is not at all clear what value could be substituted for l_x to obtain a meaningful result.

Example 2

As a somewhat more dramatic illustration of the difficulties implied by kinks, consider the following problem, which can be regarded as a discrete-time analog of the production problem but with goodwill and production costs only:

$$x(k+1) = x(k) + r(k) - d(k), \quad x(0) = 0$$
$$J = \sum_{k=0}^{N-1} \tfrac{1}{2} r(k)^2 - N \times \min\{0, x(k+1)\}.$$

To be specific, suppose $d(k) = k$, and again, consider the proposed solution $r(k) = d(k)$, $k = 0, 1, \cdots, N$.

Again the adjoint equation breaks down. More surprising, however, is that from a simple local perturbation analysis the solution appears optimal. If, for any k, $r(k)$ is increased, then certainly the cost increases. On the other hand, a decrease by an amount $a > 0$ yields an immediate saving of $ak - (a^2/2)$ in the $\frac{1}{2}r(k)^2$ term but a long-term goodwill loss of $Na(N-k) \geq Na$. Hence the losses exceed the savings and therefore either an increase or decrease of $r(k)$ leads to higher cost. In the classical control situation this condition would imply local optimality. However, in this problem with kinks the implication is not valid; in fact, the proposed solution is not locally optimal.

In the next section control problems characterized by nonsmooth cost functionals, fixed final time, and free final endpoint are analyzed. In this simple framework the basic technique for handling kinks is developed.

In Section III the approach is generalized to problems characterized by free terminal time and terminal state constraints. The direction of steepest descent in the tangent space of the constraints is found.

Finally, in Section IV, an interesting application of the technique developed in the earlier sections is explored. It is shown that a control problem with state constraints can be exactly reduced, via a penalty function, to a single optimal control problem without state constraints but with a nonsmooth cost functional.

II. Direction of Steepest Descent with Fixed Final Time and No Final Constraints

In this section the basic control problem discussed in the previous section is analyzed and it is shown how the direction of steepest descent can be determined. In the next section the method is extended to more complex problems. To begin the analysis the nature of the local behavior of l at a kink is made explicit.

Definition: A functional L on X has a directional Gateau differential $\delta^+ L(x_0; h)$ at x_0 with increment h if

$$\lim_{\alpha \to 0^+} \frac{L(x_0 + \alpha h) - L(x_0)}{\alpha} = \delta^+ L(x_0; h)$$

exists. If $\delta^+ L(x_0; h)$ exists for all $h \in X$, then L is said to be directionally Gateau differentiable at x_0.

Let

$$J(x, u) = \int_{t_0}^{t_1} l(x, u)\, dt. \tag{5}$$

It is assumed that $\delta^+ J(x, u; h, v)$ exists for all $(x, u) \in X \times U$ and $(h, v) \in X \times U$ and is of the form

$$\delta^+ J(x, u; h, v) = \int_{t_0}^{t_1} \delta^+ l(x, u; h, 0)\, dt + \int_{t_0}^{t_1} l_u v(t)\, dt \tag{6}$$

where

$$\delta^+ l(x, u; h, 0) = l_x^* h + \sum_{i=1}^{p} |y_i^* h| \tag{7}$$

and l_x^*, y_i^*, $i = 1, 2, \cdots, p$, are linear functions (i.e., n vectors) on E^n. In particular, if l has a partial derivative with respect to x, then $l_x^* = l_x$, $y_i^* = 0$.

Examples

1) $l(x, u) = |x| + \frac{1}{2}u^2$, with x and u scalars,

$$\delta^+ l(x, u; h, 0) = \begin{cases} h, & x > 0 \\ |h|, & x = 0 \\ -h, & x < 0. \end{cases}$$

2) $l(x, u) = u^2 - \min[x, a]$, with x, u, and a scalars,

$$\delta^+ l(x, u; h, 0) = \begin{cases} 0 \cdot h, & x > a \\ -\frac{1}{2}h + \frac{1}{2}|h|, & x = a \\ -h, & x < a. \end{cases}$$

In order to compute the direction of steepest descent of J at (x, u) satisfying (1), one computes the variation of J, $\delta^+ J(x, u; h, v)$, as given in (6), with (h, v) satisfying the equation of variation along (x, u):

$$\dot{h}(t) = f_x h(t) + f_u v(t), \quad h(t_0) = 0. \tag{8}$$

For this purpose first rewrite (6) in the form

$$\delta^+ J(x, u; h, v) = \int_{t_0}^{t_1} \left\{ l_u v + l_x^* h + \sum_{i=1}^{p} \max_{|s_i(t)| \leq 1} s_i(t) y_i^* h \right\} dt.$$

Or, interchanging maximization and integration, since the maximization is pointwise,

$$\delta^+ J(x, u; h, v) = \max_{|s(t)| \leq 1} \int_{t_0}^{t_1} \{ l_u v + (l_x^* + s(t)Y) h(t) \}\, dt \tag{9}$$

where $s = (s_1, s_2, \cdots, s_p)$, $Y = (y_1^*, y_2^*, \cdots, y_p^*)$, and $|s(t)| = \max |s_i(t)|$. Introduction of the functions s_i transforms the integrand to a linear expression.

Under the variational constraint (8), which determines h once v is specified, the expression for the variation of the objective can now be written in the form

$$\delta^+ J(x, u; h, v) = \max_{|s(t)| \leq 1} \int_{t_0}^{t_1} (l_u + \lambda' f_u) v(t)\, dt \qquad (10)$$

where

$$-\dot{\lambda}(t) = [f_x]' \lambda(t) + l_x^* + s(t)Y, \quad \lambda(t_1) = 0. \qquad (11)$$

The last equation is the familiar adjoint equation, except that it contains the unknown functions s_i, $i = 1, 2, \cdots, p$.

Defining a quadratic integral norm on the variation in control

$$\|v\| = \left\{ \int_{t_0}^{t_1} v'(t) v(t)\, dt \right\}^{1/2}$$

the direction of steepest descent for J at the point (x, u) is the function v that solves

$$\min_{\|v\| \leq 1} \delta^+ J(x, u; h, v)$$

subject to (8). Explicitly, this is

$$\min_{\|v\| \leq 1} \max_{|s(t)| \leq 1} \int_{t_0}^{t_1} (l_u + \lambda' f_u) v(t)\, dt \qquad (12)$$

subject to the modified adjoint equation (11).

The next step of the analysis, representing perhaps the first real progress, is to interchange the order of minimization and maximization in (12). The detailed justification for this step for continuous-time problems is beyond the mathematical scope of this paper. The essential observations in that justification are that the variation in the objective is a linear function of both v and s, and the constraint sets $\|v\| \leq 1$ and $|s(t)| \leq 1$ are convex and compact in an appropriate (weak-star) topology [1]. In the discrete-time case, using a standard min–max theorem [2], validity of the interchange follows from linearity with respect to v and s and convexity and boundedness of the constraint sets.

It has been determined that the direction of steepest descent is the function v solving

$$\max_{|s(t)| \leq 1} \min_{\|v\| \leq 1} \int_{t_0}^{t_1} (l_u + \lambda' f_u) v(t)\, dt \qquad (13)$$

subject to (11). This leads to the next critical step of the analysis. The inner minimization of (13) is carried out explicitly. The minimum with respect to v is attained by

$$\bar{v}(t) = -c(l_u + \lambda' f_u)$$

where $c > 0$ is chosen to normalize $\bar{v}$.

Determination of the direction of steepest descent then reduces to solving

$$\min_{|s(t)| \leq 1} \int_{t_0}^{t_1} [l_u + \lambda' f_u]^2\, dt \qquad (14)$$

subject to (11). The steepest descent direction, without regard for its magnitude, is then

$$v(t) = [-l_u - \lambda' f_u]'. \qquad (15)$$

There are a number of observations concerning this final result that seem to be worth emphasizing. First, note that if throughout $[t_0, t_1]$ the current trajectory is not on a kink, then $Y \equiv 0$, $l_x^* = l_x$, and hence the solution of (11), (14) reduces to the integration of (11) alone. In other words, the direction of steepest descent given by the method previously deduced reduces to a standard adjoint equation determination of the negative gradient.

Second, note that if the current trajectory is on a kink for a subinterval of $[t_0, t_1]$, then solving (11), (14) is equivalent to solving an optimal control problem on this subinterval. The optimal control problem that must be solved is characterized by linear dynamics, a quadratic cost functional, and a magnitude constraint on the control $s(t)$. It is perhaps appropriate to refer to this as the *adjoint optimal control problem.*

Finally, as a result of the preceding calculations, a set of necessary conditions satisfied by an optimal trajectory is obtained. The minimum variation due to changes in v must be zero, and hence the necessary conditions can be stated as follows.

Theorem 1

If a control history u and its corresponding trajectory x are optimal, there is a function s, $|s(t)| \leq 1$, such that

$$l_u + \lambda' f_u = 0 \qquad (16)$$

where

$$-\dot{\lambda}(t) = [f_x]' \lambda(t) + l_x^* + s(t)Y, \quad \lambda(t_1) = 0. \qquad (11)$$

Example (Solution to the Production Problem): Suppose in Example 1) that $d(t) \equiv 1$. As a candidate for solution consider

$$u^0(t) = \begin{cases} 1, & 0 \leq t \leq T-1 \\ T-t, & T-1 \leq t \leq T. \end{cases}$$

This solution yields

$$x^0(t) = \begin{cases} 0, & 0 \leq t \leq T-1 \\ (T-1)t - \frac{1}{2}t^2 - \frac{1}{2}(T-1)^2, & T-1 \leq t \leq T. \end{cases}$$

Denoting by $\chi[a, b]$ the characteristic function of the interval $[a, b]$, i.e.,

$$\chi[a, b](t) = \begin{cases} 1, & t \in [a, b] \\ 0, & t \notin [a, b] \end{cases}$$

the associated adjoint equation becomes

$$-\dot{\lambda}(t) = s(t)\chi[0, T-1] - \chi[T-1, T], \quad \lambda(T) = 0$$

and we seek s, $|s(t)| \leq 1$, such that

$$\lambda(t) + u^0(t) = 0.$$

The solution $s(t) \equiv 0$ satisfies these requirements and verifies that the proposed solution u^0 satisfies the necessary condition for optimality.

III. Direction of Steepest Descent in a Constraint Tangent Space

In analyzing optimal control problems having terminal constraints, it is common to calculate the projection of the gradient onto the tangent space associated with the terminal constraints. Setting this projected gradient to zero gives necessary conditions on the problem. It is the purpose of this section to develop a similar approach for problems having kinks. Unlike the classical case where gradients exist, the direction of steepest descent in a subspace cannot, in general, be found by projecting the direction of steepest descent for the whole space onto that subspace. Thus the problem posed and solved in this section is nontrivial. It can, however, be solved by suitably extending the method developed in the previous section.

Consider the problem of finding $(x^0, u^0, \bar{t}_1)$ to minimize

$$J(x, u, t_1) = g(x(t_1)) + \int_{t_0}^{t_1} l(x, u)\, dt \tag{17}$$

subject to

$$\dot{x}(t) = f(x, u), \quad x(t_0) = x_0 \tag{1}$$

$$\psi(x(t_1)) = 0. \tag{18}$$

Here x_0 and t_0 are given, while t_1 is free. The function ψ is assumed to have q components. All functions except l are continuously differentiable. It is assumed that (1) uniquely defines

$$x(t) = \varphi(u, t) \tag{19}$$

for every piecewise continuous u. Through the use of (19) the cost functional and the terminal constraint may be regarded as dependent on u alone. Thus (18) can be written as

$$\Phi(u, t_1) = 0$$

where Φ is the composite mapping $\psi\varphi$. It is further assumed that ψ and φ are differentiable mappings, in which case Φ is differentiable with

$$\Phi_u = \psi_x \varphi_u(u, t_1)$$

$$\Phi_t = \psi_x f(x, u).$$

Given (x, u, t_1) satisfying (1) and (18), it is desired to find the direction of steepest descent in the space tangent to $\psi(x(t_1)) = \Phi(u, t_1) = 0$, i.e., to solve

$$\min_{VT} \delta^+ J(u, t_1; v, \tau) \tag{20}$$

subject to

$$[\Phi_u v + \Phi_t \tau]_{t_1} = 0 \tag{21}$$

where

$$VT = \{(v, \tau) : \|v\|^2 + |\tau|^2 \leq 1\}.$$

Equation (21) can be written as

$$\int_{t_0}^{t_1} \Lambda' f_u v\, dt + \psi_x f(x, u)\Big|_{t_1} \tau = 0 \tag{22}$$

where

$$-\dot{\Lambda}(t) = [f_x]' \Lambda(t), \quad \Lambda(t_1) = \psi_x'\Big|_{t_1}. \tag{23}$$

Problem (20), (21) is then explicitly

$$\min_{VT} \max_{|s(t)| \leq 1} \int_{t_0}^{t_1} (l_u + \lambda' f_u) v(t)\, dt + [g_x f(x, u) + l(x, u)]_{t_1} \tau \tag{24}$$

subject to (22), where

$$-\dot{\lambda}(t) = [f_x]' \lambda(t) + l_x^{*\prime} + (sY), \quad \lambda(t_1) = g_x]_{t_1}. \tag{25}$$

Again, as in the previous section, the minimization and maximization can be interchanged in (24). Then also, as before, an explicit solution to the inner operation of minimization is performed, i.e.,

$$\min_{VT} \int_{t_0}^{t_1} (l_u + \lambda' f_u) v\, dt + [g_x f(x, u) + l(x, u)]_{t_1} \tau \tag{26}$$

subject to (21).

The solution to this inner minimization problem can then be found immediately by introduction of a Lagrange multiplier vector ν having q components. The solution is

$$v = -[l_u + (\lambda - \Lambda\nu)' f_u]' \tag{27}$$

$$\tau = -[(g_x - \nu' \psi_x) f + l]_{t_1} \tag{28}$$

where ν is selected so that this solution satisfies (22). Thus substituting into (22) and solving there obtains

$$\nu = Q^{-1} \xi \tag{29}$$

where

$$Q = \int_{t_0}^{t_1} \Lambda' f_u f_u' \Lambda\, dt + [\psi_x f f' \psi_x']_{t_1} \tag{30}$$

$$\xi = \int_{t_0}^{t_1} \Lambda' f_u (l_u + \lambda' f_u)'\, dt + [\psi_x f (g_x f + l)]_{t_1}. \tag{31}$$

It can be noted that the matrix Q is positive definite (and hence invertible) if the linearized system equation (8) is completely controllable.

Substituting the results of this inner minimization into the problem (25), we deduce that the direction of steepest descent in the tangent space of the terminal constraint is obtained by solving the adjoint optimal control problem for problems with terminal constraints:

$$\min_{|s(t)| \leq 1} \int_{t_0}^{t_1} |l_u + (\lambda - \Lambda\nu)' f_u|^2\, dt + \|[g_x - \nu' \psi_x] f(x, u) + l(x, u)]_{t_1}'\|^2 \tag{32}$$

subject to (25), and where ν is regarded as a function of λ through (29)–(31).

This result generalizes that of the previous section and is subject to similar interpretations. Note that if l is continuously differentiable, the adjoint optimal control problem reduces to a simple evaluation, and the steepest descent direction reduces to the projected gradient.

Finally, as with the unconstrained problem, the formulas previously derived for the direction of steepest descent directly yield necessary conditions for optimality. For an optimal solution the minimum in (32) is zero. Thus Theorem 2 is the appropriate statement.

Theorem 2

If a control history u, final time t_1, and corresponding trajectory x are optimal for the problem (17), (1), (18), then there exists a q vector v and a function $s, |s(t)| \leq 1$, such that

$$l_u + (\lambda - \Lambda v)' f_u = 0 \tag{33}$$

$$(g_x - v'\psi_x) f(x, u) + l(x, u)|_{t_1} = 0 \tag{34}$$

$$\psi(x(t_1)) = 0 \tag{18}$$

where Λ and λ are determined from (25) and (23), respectively.

If the optimal solution does not lie on a kink, then s does not enter (25) and the preceding necessary conditions reduce to the standard conditions for constrained optimal control.

IV. An Exact Penalty Function Solution to Constrained Optimal Control Problems

The penalty function method is widely recognized as an effective technique for handling constraints in all types of optimization problems, including optimal control problems. The method is based on approximating the original constrained problem by an unconstrained problem that incorporates a penalty term in its cost functional. The penalty term consists of a large positive constant times a measure of constraint violation. In the usual implementation of the scheme, however, the true solution to the constrained problem cannot be obtained with a finite penalty constant. This has led to the ploy of solving a series of penalty function minimization problems for an increasing sequence of penalty constants. In this section it is shown how a constrained optimal control problem can be converted into a single penalty function problem with a cost functional that has kinks.

Consider the constrained optimal control problem

$$\min \int_{t_0}^{t_1} l(x, u)\, dt \tag{35}$$

subject to

$$\dot{x}(t) = f(x, u), \quad x(t_0) = x_0 \tag{1}$$

$$G(x(t), u(t)) \leq 0, \quad t_0 \leq t \leq t_1 \tag{36}$$

where G is p-dimensional. For simplicity, assume that all functions are continuously differentiable. The constraint (36) is quite general and includes various state-constrained problems. Other constraints such as integral or terminal constraints can be treated in a manner similar to that developed for (36).

Now under certain standard regularity conditions, the constraint (36) can be handled by introducing an appropriate Lagrange multiplier–vector function $\mu(t) \geq 0$, so that the original problem is equivalent to

$$\min \int_{t_0}^{t_1} [l(x, u) + \mu'(t) G(x, u)]\, dt \tag{37}$$

subject to (1).

If the minimum value of the cost functional in either (35) or (37) is denoted by σ_0, then

$$\sigma_0 \leq \min \int_{t_0}^{t_1} \left[l(x, u) + K \sum_{i=1}^{p} \max(0, G_i(x, u)) \right] dt \tag{38}$$

subject to (1), where

$$K \geq K_0 = \max_{j,t} \mu_j(t). \tag{39}$$

Hence, defining

$$P(x, u) = \sum_{i=1}^{p} \max(0, G_i(x, u)) \tag{40}$$

we have

$$\sigma_0 \leq \min \int_{t_0}^{t_1} [l(x, u) + KP(x, u)]\, dt \tag{41}$$

for $K \geq K_0$, where the minimization is subject to the system equations (1).

On the other hand, consider the number σ_K derived from the following:

$$\sigma_K = \min \int_{t_0}^{t_1} [l(x, u) + KP(x, u)]\, dt \tag{42}$$

subject to (1) for an arbitrary positive K. Clearly, this minimum is no greater than that which would be obtained if $P(x, u)$ were forced to be identically zero. Thus

$$\sigma_K < \sigma_0 \tag{43}$$

for all $K \geq 0$. The inequalities (41) and (43) together imply

$$\sigma_K = \sigma_0 \tag{44}$$

for $K \geq K_0$.

This result forms the basis of our penalty function method. For any $K \geq K_0$, the solution to the unconstrained optimal control problem (42) is precisely the solution to the original constrained problem. The new unconstrained problem, however, is characterized by the fact that the integrand of the objective functional has kinks.

V. Conclusions

Optimal control problems with kinks are important both because they are natural models of a fairly large class of economic allocation problems and because they serve as unconstrained equivalents to complex state-constrained problems. Although the classical optimal control theory cannot be directly applied to these problems, it has been shown that the standard methods can be suitably extended to such problems by consideration of an adjoint optimal control problem.

There is obviously the possibility that a computational scheme can be developed based on the direction of steepest descent derived in this paper. The method would be a more or less straightforward extension of gradient descent methods, or, in the case of problems with terminal constraints, of various gradient projection methods. Such a method would suffer, however, from the need to repeatedly solve the adjoint optimal control problem. The method has been tried, nevertheless, and on the small examples tested appears to work satisfactorily. It is expected, however, that the theory given here will be more applicable to the attainment of analytical solutions than computational ones.

An area of investigation, in which some progress has already been made, is the development of a maximum principle for problems with kinks. This result should have extensive application to economic problems. The study of problems with kinks is in its earliest stages and it seems that there is still much to be learned about their behavior.

Finally, it should be noted that there is a connection between the theory of this paper and that of the notions of duality and subdifferentiability [6]. In this characterization the direction of steepest descent would be identified with a special hyperplane supporting the cost functional.

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