

CONTROL OF LINEAR DYNAMIC MARKET SYSTEMS*

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This paper introduces and studies a new type of dynamic system that characterizes the behavior of a dynamic market in dynamic equilibrium. The system consists of a physical production process operated by a large collection of producers, a group of consumers with a demand function, and a market that provides the dynamic equilibrium as the producers attempt to maximize profit. The paper introduces these dynamic equilibrium systems and discusses some of their basic properties. The concepts of reachability and observability are considered and it is shown that there are extensions of these concepts to equilibrium systems.

1. Introduction

This paper is concerned with the study of control theory in the context of a market. The control problem corresponds to a producer, or group of producers, seeking to operate a dynamic production facility so as to maximize total profit. Control action consists, at least in part, of market participation: purchase and sale of goods.

The market aspect of the problem arises because the prices in the market, although apparently fixed from the perspective of the supplier, are in fact determined to be an equilibrium between the suppliers as a group and the consumers. The basic system, therefore, is one that determines a time sequence of prices that comprise an equilibrium in this dynamic context. The sequence of prices is such that the response to it by producers and consumers will clear markets in each time period.

This paper focuses on some aspects of this equilibrium system that are analogous to the basic structural issues in modern linear control theory. The first issue is whether initial and final conditions can be specified arbitrarily, leading to results that generalize the standard conditions of controllability or reachability. It is shown how this depends not only on the inherent reachability of the dynamic system of production, but also on the demand structure. The second issue is whether the system state can be inferred from observa-

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tions. This is a somewhat complex issue, since one must account for observations of prices as well as state variables. Overall, this paper is not intended to represent a new model of economic phenomena, for the economic content is quite standard. Rather, it is intended as an exploration of structural issues of dynamic economic models, so that their general properties can be better understood.

2. Basic formulations

As motivation for the structures considered in this paper, imagine a commodity, or group of commodities, that is processed and eventually sold for consumption. During processing, commodities move from stage to stage or are stored at a given stage for later processing or sale. This type of activity is represented by a system of the form

$$x(k+1) = A(k)x(k) - B(k)u(k) + a(k). \quad (2.1)$$

In this system $x(k)$ is an n -dimensional state vector, $u(k)$ is an m -dimensional market vector, and $a(k)$ is an n -dimensional vector of known inputs. The matrices $A(k)$ and $B(k)$ are of the appropriate consistent dimensions. These matrices determine the system structure.

The model of (2.1) is not the most general possible. Certainly many production processes are non-linear. However, linearity very appropriately describes a large class of processes where material is moved from stage to stage. Within the linear realm (2.1) is very general, and goes well beyond the study of particular systems. In this sense the model attempts to establish a general framework, similar in intent to that of Takayama and Judge (1971), but much more in the spirit of modern control theory.

Another reason that linearity is important is that, with respect to the market, (2.1) really represents the aggregate of many producers with identical processes. Because of linearity the individual processes can be summed to obtain an aggregate of the same form.

External to the production system is a market for the products produced by it. The quantities supplied to the market are the components of the vector $u(k)$. The value of the vector $u(k)$ is determined by market equilibrium between supply and demand. Accordingly, associated with the market is an m -dimensional vector of prices $p(k)$. From the demand side, these prices determine the value of the market vector through an m -dimensional linear demand function,

$$v(k) = e(k) - E(k)p(k),$$

where $v(k)$ has the same dimension as $u(k)$.

The (identical) producers seek to maximize profit over the time intervals $k = 0, \dots, N$ by selecting $u(k)$. The selection of $u(k)$ corresponds to their participation in the market; that is, $u(k)$ is the amount they sell.

2.1. Basic linear reward structure

To complete the specification of the model, we must specify the reward structure of the producers. In this paper we assume a linear structure; that is, the producers' costs and revenues are linear in the system variables. This leads to the overall market consisting of

Supply: The producers operate the system

$$x(k+1) = A(k)x(k) - B(k)u(k) + a(k), \quad (2.1)$$

[by selection of $u(k)$] to maximize

$$J = \sum_{k=0}^N \{ p(k)'u(k) - c(k)'x(k) \}, \quad (2.2)$$

subject to the initial conditions

$$x(0) = x_0, \quad (2.3)$$

and to the terminal conditions

$$Gx(N+1) = g, \quad (2.4)$$

where G is a $q \times n$ matrix with $q \leq n$ and g is a q -vector.

Demand:

$$v(k) = e(k) - E(k)p(k). \quad (2.5)$$

Equilibrium: The price sequence $p(k)$ is determined so that for each k

$$u(k) = v(k). \quad (2.6)$$

The first and most complex part of the model is the production problem. Each producer seeks to maximize the objective function (2.2). In this objective, the term $c(k)'x(k)$ represents one-period storage costs. A producer's revenue term $p(k)'u(k)$ is obtained through sales (or purchases) of $u(k)$ at prices $p(k)$. The price vector sequence $p(k)$, $k = 0, \dots, N$, is considered to be fixed for producers; that is, a producer maximizes without regard for possible price

changes. This assumption is based on two economic assumptions. First, each producer is considered to be a *price-taker*. His action, as an individual, has negligible effect on the market. Second, the model is based on *rational expectations*, or in this deterministic framework, *perfect foresight*. At a given period k the producer can observe current prices $p(k)$, but future prices cannot be directly observed. It is assumed, however, that producers are able to forecast future prices perfectly (perhaps by solving the equilibrium problem posed here). Thus producers forecast prices that are consistent with their optimal behavior.

The objective function of the producer represents total profit (exclusive of fixed costs). It can be easily generalized to include discounting.

The initial condition (2.3) represents the initial state of the production process. It is the sum of all individual producers' initial conditions. We assume that the individual initial conditions are identical (to within a scale factor) and the final condition requirements are apportioned by the same scale factors. Then the optimal solutions for producers also will be proportional, and the collection of producers can be viewed as a single producer.

Demand is assumed to be governed by a standard period-by-period linear demand function as described by (2.5). Consumers do not store goods in anticipation of future price changes. The system represents a dynamic competitive equilibrium. Each agent acts independently of others, but the price sequence clears the market.

To be realistic the state variables of the system must be non-negative. Explicit non-negativity constraints can be easily adjoined to the structure (2.1)–(2.6) with no significant change in concept. However, in order to investigate the linear structure of the system in complete generality, it is necessary to neglect the non-negativity constraints. This, of course, entails no loss of generality if the linear system we consider represents a perturbation of a system with positive prices.

A final point concerning this model is that the production portion need not represent the actions of totally integrated production operations. That is, in addition to the assumption that no one producer can directly influence price, no one producer need be responsible for the complete set of production operations. It is quite possible for production to be decentralized, with one producer buying intermediate products from another at market clearing prices. These intermediate markets are suppressed in this framework but they are consistent with it. This is a standard decentralization result. [See Arrow and Hurwicz (1960).]

The framework above is illustrated by the following example.

Example 1 (Wholesaler Problem)

As a specific example, which will be used later as well, consider a simple wholesaler problem. A commodity (such as livestock) is produced over the

period $0 \leq k \leq N$ according to the fixed sequence $\{a(k)\}$. There is an initial inventory $x(0)$. Successive inventories are governed by the equations

$$x(k+1) = x(k) - u(k) + a(k), \quad (2.7a)$$

where $u(k)$ is the amount sold in period k . Given a price sequence $\{p(k)\}$, the wholesaler selects the sequence of $u(k)$'s so as to maximize the net profit

$$J = \sum_{k=0}^N \{ p(k)u(k) - cx(k) \}, \quad (2.7b)$$

where c is the unit holding cost. In this model the original purchase price to the wholesaler of the commodity is assumed to be exogenous.

Demand follows the simple rule

$$v(k) = e - p(k). \quad (2.7c)$$

One seeks the sequence $\{p(k)\}$ that yields equilibrium.

3. Necessary conditions

We consider the basic optimal control problem:

$$x(k+1) = A(k)x(k) - B(x)u(k) + a(k), \quad (3.1a)$$

$$J = \sum_{k=0}^N \{ p(k)'u(k) - c(k)'x(k) \}, \quad (3.1b)$$

$$x(0) = x_0, \quad (3.1c)$$

$$Gx(N+1) = g. \quad (3.1d)$$

This problem is singular, since it is a completely linear problem. Therefore, there must be a proper relationship among the parameters in order that a solution exist and be finite. These relationships can be found by writing the standard optimality conditions for the problem. These conditions can be written by introducing a sequence of adjoint variables [Luenberger (1979)] $\{\lambda(k)\}$ satisfying the adjoint equation

$$\lambda(k) = A(k)'\lambda(k+1) - c(k), \quad k = 0, \dots, N, \quad (3.2a)$$

and the terminal condition

$$\lambda(N+1) = G'z, \quad (3.2b)$$

where z is a q -dimensional vector (which must be determined). In addition, the Hamiltonian

$$H = \lambda(k+1)'A(k)x(k) - \lambda(k+1)'B(k)u(k) \\ + p(k)'u(k) - c(k)'x(k)$$

must be a maximum with respect to the control variables for each k , $0 \leq k \leq N$. It follows readily that the necessary conditions for finite optimality in the producers' problem are

$$x(k+1) = A(k)x(k) - B(k)u(k) + a(k), \quad (3.3a)$$

$$\lambda(k) = A(k)'\lambda(k+1) - c(k), \quad (3.3b)$$

$$p(k) = B(k)'\lambda(k+1), \quad (3.3c)$$

$$x(0) = x_0, \quad Gx(N+1) = g, \quad (3.3d)$$

$$\lambda(N+1) = G'z. \quad (3.3e)$$

In addition, we must have

$$u(k) = v(k), \quad (3.4)$$

$$v(k) = e(k) - E(k)p(k). \quad (3.5)$$

Combining, the complete system of equations can be written as

$$x(k+1) = A(k)x(k) + B(k)E(k)B(k)'\lambda(k+1) \\ - B(k)e(k) + a(k), \quad (3.6a)$$

$$\lambda(k) = A(k)'\lambda(k+1) - c(k), \quad (3.6b)$$

$$p(k) = B(k)'\lambda(k+1), \quad (3.6c)$$

$$x(0) = x_0, \quad Gx(N+1) = g, \quad (3.6d)$$

$$\lambda(N+1) = G'z. \quad (3.6e)$$

Eq. (3.6c) can be solved separately after the others. The remaining equations constitute a two-point boundary value problem (or *descriptor variable* system [Luenberger (1977)], especially if A is singular) defining the solution to the dynamic equilibrium problem.

It is important to emphasize the significance of the fact that $E(k)$ is *not* generally symmetric, nor positive semi-definite. First, from an economic

viewpoint, the imposition of such assumptions would be highly restrictive because they are not often satisfied by actual demand functions. Second, from a theoretical viewpoint, if such assumptions were imposed, the problem could be transformed to an equivalent quadratic optimal control problem and solved accordingly, thus avoiding the necessity for a more careful study of (3.6).

4. Terminal conditions and reachability

To make the equilibrium model practical, it must be recognized that the process will in fact continue beyond period $N + 1$. There are two ways of implicitly accounting for this extension of periods: one is by imposing a terminal constraint $Gx(N + 1) = g$; and the other is to impose a terminal cost, say $d'Gx(N + 1)$, in addition to the other costs. The two methods are related and can be considered dual. We focus on the terminal constraint method here for it provides a natural way to consider a parameterized family of solutions, and it is a good preliminary to the issues of inference and observability considered in the next section.

Consider the case where the terminal constraint is $x(N + 1) = 0$. It will not always be possible to guide the system, with appropriate inputs, in such a way as to drive $x(N + 1)$ to zero – unless, of course, the system is controllable and $N \geq n$. Moreover, even if the system state can be brought to zero with suitable controls, it is not necessarily true that it can be brought to zero using inputs that are in economic equilibrium.

To illustrate the issue, consider the case where $E(k) \equiv 0$, which reduces the demand equation to $v(k) = e(k)$, corresponding to constant (totally inelastic) demand. The solution to the overall equilibrium system will also have $u(k) = e(k)$. The controls are therefore fixed in this case and hence the terminal point is uniquely determined from the initial conditions. We cannot specify an arbitrary final point; for even if the production system is completely reachable, the requirement of meeting demand leaves no flexibility. In general, even if $E(k)$ is not zero, the equilibrium requirement essentially imposes a constraint on the allowable input patterns. With this additional constraint, complete reachability of the system is not generally sufficient to be able to achieve arbitrary terminal conditions, so a more complete analysis is required. We find, however, that the correct condition is a mathematically natural extension of complete reachability, but must incorporate the demand matrix. We formalize the notions, for the general situation, by the following definitions.

Definition 1. The system (3.6) is *equilibrium reachable* if for any x_0 and any g there is a solution. The system (3.6) is *completely equilibrium reachable* if it is *equilibrium reachable* for $G = I$.

The condition for equilibrium reachability depends on the matrix C of the following definition.

Definition 2. Corresponding to the system (3.6) the *Terminality Matrix* is

$$C = \sum_{k=0}^N \Phi(N+1, k+1)B(k)E(k)B(k)'\Phi(N+1, k+1)'. \quad (4.1)$$

The required condition for equilibrium reachability, as spelled out in the following theorem, is a generalization of standard conditions for reachability.

Theorem 1. The system (3.6) has a solution for any value of g (that is, it is equilibrium reachable) if and only if the matrix GCG' is non-singular.

Proof. The solution to the adjoint equation (3.6b) can be written

$$\lambda(k) = \Phi(N+1, k)\lambda(N+1) + \bar{c},$$

where \bar{c} is a constant. Likewise, the solution to (3.6a) can be written

$$\begin{aligned} x(N+1) = & \Phi(N+1, 0)x(0) + \sum_{k=0}^N \Phi(N+1, k+1) \\ & \times B(k)E(k)B(k)'\lambda(k+1) + \bar{a}, \end{aligned}$$

where \bar{a} is a constant. Combining these we obtain

$$\begin{aligned} x(N+1) = & \sum_{k=0}^N \Phi(N+1, k+1)B(k)E(k) \\ & \times B(k)'\Phi(N+1, k+1)'\lambda(N+1) + \bar{b}, \end{aligned}$$

where \bar{b} is a constant. Multiplication by G and use of the boundary requirement $\lambda(N+1) = G'v$ yields

$$Gx(N+1) = GCG'v + G\bar{b}.$$

For this to take arbitrary values g , it is clearly necessary and sufficient that GCG' be non-singular. ■

In economic terms this theorem shows that the demand matrix must be related to the system matrix in a way that avoids overspecifying the control; there must be enough flexibility both in the demand matrix and in the producer's system for the equilibrium itself to be flexible.

The result itself can be deduced intuitively for the case $G = I$ in the following way. All final conditions must be of the form

$$\sum_{k=0}^N \Phi(N+1, k+1)B(k)u(k),$$

plus a constant. However, all controls must be equal to demands, so $u(k) = E(k)p(k)$, plus a constant. This shows explicitly how demand drives the system. Now the price itself must be aligned with the adjoint variables in order to obtain a finite optimal solution. From (3.6b) and (3.6c) it follows that $p(k) = B(k)'\Phi(N+1, k+1)'$ plus a constant. This together with the above comments about $u(k)$ lead to the Theorem 1. Both the demand and optimality of supply are wrapped up in the result.

The special case of $G = I$, corresponding to the terminal condition that $x(N+1)$ be specified entirely, is analogous to the problem posed by complete reachability of the system

$$x(k+1) = A(k)x(k) + B(k)u(k). \quad (4.2)$$

The requirement of Theorem 1 is that C be non-singular. The relation of this to complete reachability is stated in the following theorem.

Theorem 2. If C is non-singular, then the system (4.2) is completely reachable. If the demand matrices $E(k)$ are symmetric and positive definite, then complete reachability of (4.2) implies that C is positive definite.

Proof. Assume C is non-singular; then let

$$u(k) = E(k)B(k)'\Phi(N+1, k+1)C^{-1}y,$$

where y is arbitrary. Then, for (4.2),

$$\begin{aligned} x(N+1) &= \Phi(N+1, 0)x(0) + \sum_{k=0}^N \Phi(N+1, k+1) \\ &\quad \times B(k)E(k)B(k)'\Phi(N+1, k+1)C^{-1}y \\ &= \Phi(N+1, 0)x(0) + y, \end{aligned}$$

which can be made arbitrary, and hence the system is completely reachable.

Suppose now that (4.2) is completely reachable and that, for each k , $E(k)$ is symmetric and positive definite. Complete reachability implies that the set of columns $\Phi(N+1, k+1)B(k)$ span the whole space. Now suppose there is a y

such that $y'Cy = 0$. Since the $E(k)$'s are positive definite this implies that

$$y'\Phi(N+1, k+1)B(k)E(k)B(k)'\Phi(N+1, k+1)y = 0,$$

since each of these terms is always non-negative and their sum is $y'Cy$. Again by the positive definiteness of $E(k)$ it follows that, for each k ,

$$y'\Phi(N+1, k+1)B(k) = 0.$$

However, by the spanning condition this implies that $y = 0$. Thus C is positive definite. ■

From the above we see that when the $E(k)$'s are symmetric and positive definite then complete reachability is necessary and sufficient to achieve arbitrary terminal conditions on the state. For $E(k)$ matrices of other structure, the situation is more complex. However, this generalization of the well-known conditions of complete reachability is mathematically and economically quite natural.

It is worth pointing out the special form of the above results in the time-invariant case. If A , B and E do not depend on k , then the Terminality Matrix is

$$C = \sum_{k=0}^N A^k B E B' (A')^k. \quad (4.3)$$

If E is positive definite and symmetric we know that non-singularity of C is equivalent to the condition that the matrix

$$[B, AB, \dots, A^{n-1}B]$$

be of full rank. However, there seems to be no analogous form for general E ; instead, the full form (4.3) must be used.

5. Inference and observability

The next issue that we consider is that of inferring the state of a dynamic market from observations. This turns out to be somewhat more complicated than one might first suppose, and for simplicity we restrict attention in this section to the time-invariant case.

Assume that the initial state is unknown, but that the required final condition $Gx(N+1) = g$ is known. Partial observations are available on both the state vector and the price vector. Specifically, assume that we have the

observations

$$y(k) = Hx(k), \quad (5.1a)$$

$$z(k) = Fp(k) \quad \text{for } k = 0, \dots, N. \quad (5.1b)$$

In practice there may also be observations on market level [that is, on the demand $v(k)$]; however, these observations can be converted to observations on price by use of the demand function.

Ignoring constant terms, the observations generate the sequences

$$Hx(0), HAx(0) + HBEB'A^N G'v, \dots, \quad (5.2a)$$

$$FB'A^N G'v, FB'A^{N-1} G'v, \dots, \quad (5.2b)$$

from which it is clear that the observations may be regarded as observations of linear functionals of the unknown composite vector $(x(0), v)$. Once $(x(0), v)$ is known, then the system (3.6) can be solved. After $q+1$ observations, the known information can be written in matrix form as

$$\begin{bmatrix} Gx(N+1) \\ y(0) \\ y(1) \\ \vdots \\ y(q) \\ z(0) \\ z(1) \\ \vdots \\ z(q) \end{bmatrix} = \begin{bmatrix} GA^{N+1} & GC_N G' \\ H & HC_1 G' \\ HA & HC_2 G' \\ \vdots & \vdots \\ HA^q & HC_q G' \\ & FB'A^N G' \\ & FB'A^{N-1} G' \\ \vdots & \vdots \\ & FB'A^{N-q} G' \end{bmatrix} \begin{bmatrix} x(0) \\ v \end{bmatrix}, \quad (5.3)$$

where again constant terms have been ignored. The general expression for C_i can be found by noting that

$$x(i) = A^i x(0) + \sum_{k=0}^{i-1} A^{i-1-k} BEB' \lambda(k+1) + \text{constant},$$

and

$$\lambda(k+1) = (A')^{N-k} G'v + \text{constant}.$$

This yields

$$x(i) = A^i x(0) + \sum_{k=0}^{i-1} A^{i-1-k} BEB' (A')^{N-k} G'v. \quad (5.4)$$

Therefore,

$$C_i = \sum_{k=0}^{i-1} A^{i-1-k} B E B' (A')^{N-k}. \quad (5.5)$$

We denote the large matrix on the right-hand side of (5.3) by M_q . The first (block) row of M_q is not due to an observation. Instead it represents the prior knowledge of the terminal condition.

We are led immediately to the following general result whose proof is elementary.

Theorem 3. The complete trajectory of the system (3.6) can be inferred after at most $q + 1$ observations if and only if M_q is of rank $n + m$.

The general condition developed above is rather complex. It is useful to consider some special cases:

Case 1. $x(N)$ free, measurements of state only [i.e., $G = 0$, $F = 0$]. In this case $\dim(v) = 0$. Trajectory inference after n steps is equivalent to complete observability of the pair (H, A) .

Case 2. $x(N)$ free, measurement of price only [i.e., $G = 0$, $H = 0$]. In this case the trajectory can never be determined.

Case 3. $x(N + 1) = \bar{x}$ is specified, A^{-1} exists, measurements of price only [i.e., $G = I$, A^{-1} exists, $H = 0$, $F = I$]. In this case trajectory inference after n steps is equivalent to complete reachability of the pair (A, B) .

Case 4 (general). In the general case the pair (H, A) must be controllable for the nullspace of GA^N . However, other conditions are required.

Example 2

Consider again Example 1. It is possible to determine the complete trajectory from a single price measurement. This follows from the above with $G = 1$, $H = 0$ and $F = 1$. The intuitive explanation is that the price trajectory can be shown to be linear in k with coefficient c . A single price measurement determines the whole price trajectory. Total demand can then be found, and hence the initial stock that is just sufficient to supply that demand can be calculated.

6. Conclusions

This paper has initiated the systematic study of linear dynamic markets, in which the dynamic equations represent both physical production processes and conditions for dynamic economic equilibrium. The system is generally a two-point boundary system since it is derived from the standard system and adjoint equations associated with the optimization problem of producers that is inherent in an economic equilibrium. The dynamic equilibrium is postulated to be a result of free competition, and the agents achieve it indirectly by each acting optimally. In this way, the optimization is an integral part of the system description rather than a process imposed later by the analyst.

Standard concepts of reachability and observability have analogs in this new type of dynamic system, and they are explored in the paper. This led to new definitions that extend the results of standard dynamic systems.

Future work will be devoted to models derived from a linear-quadratic framework and other extensions.

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