CANONICAL FORMS FOR LINEAR MULTIVARIABLE SYSTEMS

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Reprinted from IEEE TRANSACTIONS
ON *AUTOMATIC CONTROL*
Volume AC-12, Number 3, June, 1967
pp. 290-293

Canonical Forms for Linear Multivariable Systems

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Abstract—**In this paper a class of well-known canonical forms for single-input or single-output controllable and observable systems are extended to multivariable systems. It is shown that, unlike the single-variable case, the canonical forms are generally not unique, but that the structure of the canonical form can be controlled to some extent by the designer.**

A major result of the paper is that a multi-input system can be transformed to a set of coupled single-input subsystems.

I. INTRODUCTION

The development of the phase-variable canonical form for single-input linear controllable systems has been an active area of research [1]–[8]. Partly this is because the phase-variable form has proved to be an extremely convenient starting point for certain control design problems and partly it is because canonical forms are mathematically intriguing in their own right.

Unlike the single-variable case, the corresponding canonical forms for multivariable systems are not unique. This lack of uniqueness not only tends to make their derivation more difficult but also forces the design engineer, faced with a practical application, to determine the best form from the several possibilities.

One particular derivation of a multivariable canonical form is given in a previous paper [9]. A more complete development along those lines can be found in Tuel [10]. This paper contains a new, unified approach to multivariable canonical forms which includes the form in the previous paper [9] as an important special case. The derivation given here, however, is more general and notationally simpler since computations are expressed in terms of matrix algebra whenever possible.

Some early derivations of the single-input phase-variable form made heavy use of the Jordan form of the system [2]–[4]. Other derivations are more direct and require relatively little computational effort [5]–[8]. When restricted to the single-input case, the development in this paper closely parallels Wonham and Johnson [5].

II. SELECTION OF INDEPENDENT VECTORS

Consider a system governed by the set of first-order differential equations:

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

where

$x(t)$ is an $n \times 1$ state vector
A is an $n \times n$ matrix
$u(t)$ is an $r \times 1$ input vector
B is an $n \times r$ input matrix.

The fundamental assumption imposed on the system is that of system controllability; i.e., it is assumed that the $n\times(n\cdot r)$ *controllability matrix*

$$\Gamma = [B, AB, A^2B, \cdots, A^{n-1}B] \tag{2}$$

has rank n. In addition, it is generally assumed in this short paper that the r columns of B are linearly independent.

The *controllability index* ν_c of the system (1) is defined as the smallest positive integer for which the matrix

$$\Gamma_\nu = [B, AB, A^2B, \cdots, A^{\nu-1}B] \tag{3}$$

has rank n. Generally, for multivariable controllable systems $\nu_c \leq n$.

Canonical forms for the system (1) are constructed by transforming the state vector to a new coordinate system in which the system equations take a particular form. The transformation employed to affect the coordinate change is essentially always constructed from independent columns of the controllability matrix (2).

The first step in the development of a canonical form of the class discussed in this paper is the selection of n linearly independent vectors from the $n\cdot r$ columns of this controllability matrix (2). It will be required that the selection procedure be so devised that the n chosen linearly independent vectors comprise the columns of a matrix P of the form

$$P = [b_1, Ab_1, \cdots, A^{p_1-1}b_1, b_2, Ab_2, \cdots, A^{p_2-1}b_2, b_3, \cdots, A^{p_r-1}b_r]. \tag{4}$$

The essential restriction, then, is that no vector of the form A^kb_j is selected unless all lower powers of A times b_j are also selected.

As will be shown below, it is always possible to make a selection of the required form; but in general, it is not unique. The real difficulty is in determining which of the many possible P matrices leads to the best canonical form.

The selection of the vectors comprising the P matrix is straightforward (but still somewhat arbitrary) if it is done according to the following procedure (special cases of this type of selection have been employed previously [9], [11], [12]):

1) Select one of the columns of B (without loss of generality, it may be assumed that b_1 is selected).
2) Select either another column of B (say b_2) or the vector Ab_1. If the selected vector is linearly independent of b_1, retain it, otherwise omit it from the selection.
3) At any stage of the process, select the new vector to be of the form A^ib_k where all lower powers of A times b_k have already been retained. If the new vector is linearly independent of all previously selected vectors, retain it, otherwise omit if from the selection.
4) The selection process terminates when n linearly independent vectors are found. Arrange the n vectors in their proper order to form the matrix P.

It is shown in Appendix I that this process does not terminate before n independent vectors have been selected.

It may happen that, as a result of the selection procedure, not all columns of the original B matrix occur in the P matrix. In this case, the corresponding input components play no special role in the associated canonical forms and will appear in an arbitrary fashion in the final result. The other input components enter the canonical system in a special way. For notational simplicity, we shall assume that each column of B appears in P.

Although there is a certain amount of freedom in the selection process, there are two specific plans for selection that have special interest. In the first plan, one starts with the vector b_1 and then proceeds to Ab_1, $A^2b_1, \cdots$, etc., until either $A^{n-1}b_1$ is obtained, in which case the system is controllable from the first input alone, or until a dependency arises. If more independent vectors are required, one then selects b_2, Ab_2, A^2b_2, etc., until a dependency arises. The procedure continues in this manner through the b_k's until n linearly independent vectors are obtained. The tendency is to develop a few long chains in this case. The P matrix (4) obtained in this fashion has the property that $A^{p_k}b_k$ is linearly dependent on vectors of the form A^ib_i with $i \leq k$. In other words, the dependence is always with

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Manuscript received April 11, 1966; revised October 14, 1966, and January 27, 1967. This research was supported by Joint Services Contract Nonr 225 (83).

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vectors to the left of a given column of the P matrix. A canonical form based on this plan is given in Anderson and Luenberger [12].

In the second plan, the vectors are examined in the order:

$$b_1, b_2, \cdots, b_r, Ab_1, Ab_2, \cdots, \cdots, A^{n-1}b_r. \tag{5}$$

By definition of the controllability index ν_c, this procedure will terminate before $A^{\nu_c}b_1$ is examined. Thus, in this case, $p_i \leq \nu_c$ for each i. The tendency is to produce several chains of nearly equal length.

Obviously, these are not the only possible selection plans and even these two are not unique since the outcome depends heavily on the a priori ordering of the columns b_i.

III. CANONICAL FORMS

A change of coordinates from state vector x to y defined by $y = Sx$ transforms the system (1) from

$$\dot{x} = Ax + Bu \tag{1}$$

to

$$\dot{y} = SAS^{-1}y + SBu. \tag{6}$$

Appropriate choices of S lead to canonical forms of the system (1).

In this paper, two basic canonical forms are developed from the matrix P (4) constructed in Section II. Of course, there are variations possible within each of the two basic forms since there are variations possible in the choice of P. Each choice of P, however, leads to two basically different canonical forms.

The first canonical form is produced by setting $S = P^{-1}$. Simple matrix bookkeeping verifies that the system is then transformed to the form

$$\dot{y} = \tilde{A}y + \tilde{B}u \tag{7}$$

with

$$\tilde{A} = \begin{bmatrix} 0 & 0\cdots & x & & & x & & x \\ 1 & 0\cdots & x & & & x & & x \\ 0 & 1\cdots & x & & & x & & \vdots \\ \vdots & & \vdots & & & \vdots & & \vdots \\ 0\cdots & 1 & x & & & & & \\ & & x & 0 & 0\cdots & x & & \\ & & x & 1 & 0\cdots & x & & \\ & & \vdots & \vdots & & \vdots & & \\ & & & 0\cdots & 1 & x & & \\ & & & & & x & & \vdots \\ & & & & & x & & x \\ & & & & & \vdots & & x \\ & & & & & & 0\cdots & x \\ & & & & & & 1\cdots & x \\ & & & & & & & \vdots \\ & & & & & & 0 \quad 1 & x \end{bmatrix} \tag{8}$$

$$\tilde{B} = \begin{bmatrix} 1 & 0\cdots & 0 \\ 0 & 0 & \vdots \\ \vdots & \vdots & \\ & 1 & \\ & 0 & \\ & \vdots & \\ & & 1 \\ & & 0 \\ & & \vdots \\ 0 & & 0 \end{bmatrix} \tag{9}$$

The system may be considered as composed of fundamental companion matrices located in blocks along the diagonal. The x's in the matrix represent possible nonzero elements and, except for the indicated 1's, these occur only in the columns corresponding to the right-hand edge of a fundamental companion matrix.

Different choices of P lead to different sizes and number of companion matrices as well as different values for the nonzero elements. If P were chosen according to the first special plan of the last section, the x's in a given column of $\tilde{A}$ would be zero below the companion matrix corresponding to the column. Each of the companion matrices can be considered to represent a subsystem coupled to other subsystems. For the special choice of P mentioned above, the coupling between two subsystems is in one direction only.

The second type of canonical form is more useful than the first but is somewhat more difficult to derive.

Again, start with a P matrix of the form (4). Write P^{-1} in terms of its row vectors

$$P^{-1} = \begin{bmatrix} e_{11} \\ e_{12} \\ \vdots \\ e_{1p_1} \\ e_{21} \\ \vdots \\ e_{rp_r} \end{bmatrix}. \tag{10}$$

Actually, only the r's of these rows play a direct role in the canonical form; these are the last rows of each of the r groups of rows, i.e., the (row) vectors e_{ip_i}, $i = 1, 2, \cdots, r$. For simplicity of notation, these vectors are now labeled

$$e_i = e_{ip_i}. \tag{11}$$

The vectors $e_1, e_2, \cdots, e_r$ are used to construct the transformation matrix

$$S = \begin{bmatrix} e_1 \\ e_1A \\ e_1A^2 \\ \vdots \\ e_1A^{p_1-1} \\ e_2 \\ e_2A \\ \vdots \\ e_rA^{p_r-1} \end{bmatrix}. \tag{12}$$

It is shown in Appendix II that S defined by (12) is nonsingular.

It is again a simple matter of bookkeeping to verify that the transformation S defined by (12) reduces the system (1) to the form

$$\dot{y} = \tilde{A}y + \tilde{B}u \tag{13}$$

where now

$$\tilde{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & & & & & & & & & & & \\ 0 & 0 & 1 & 0 & & & & & & & & & & & \\ \vdots & & \ddots & & & & & & & & & & & & \\ & & & 1 & & & & & & & & & & & \\ x & x & x\cdots & x & x & x & x\cdots & x & x & \cdots & & & & & x \\ & & & & 0 & 1 & 0 & 0 & & & & & & & \\ & & & & 0 & 0 & 1 & 0 & & & & & & & \\ & & & & \vdots & & \ddots & & \cdot & & & & & & \\ & & & & & & & 1 & & & & & & & \\ x & x & x\cdots & x & x & x & x\cdots & x & x & \cdots & & & & & x \\ x & x & & & & \cdots & & & & & x & x & x & x\cdots & x \\ & & & & & & & & & & & & 0 & 1\cdots & 0 \\ & & & & & & & & & & & & \vdots & \ddots & \\ & & & & & & & & & & & & & & 1 \\ x & x & & & & \cdots & & & & & x & x & x & \cdots & x \end{bmatrix} \tag{14}$$

$$\tilde{B} = \left[\begin{array}{ccc} 0 & & \\ 0 & \times & \\ \vdots & & \\ 1 & & \\ & 0 & \\ & \vdots & \\ & 1 & \\ & & 0 \\ & & 0 \\ & \times & \vdots \\ & & 1 \end{array}\right] \tag{15}$$

where at least half of the elements not shown in (15) are zero.

More can be said about the exact structure of $\tilde{B}$ but instead a special case will be examined below. The main point here is that $\tilde{A}$ takes the canonical form for any P matrix of the form (4).

IV. A Special Canonical Form

If the P matrix is constructed according to the second special selection plan of Section II, the canonical form (14), (15) becomes particularly nice. The $\tilde{A}$ matrix will, of course, appear as in (14) but $\tilde{B}$ reduces to

$$\tilde{B} = \left[\begin{array}{cccc} 0 & 0 & & 0 \\ 0 & & & 0 \\ \vdots & & & \vdots \\ 1 & x & x \cdots & x \\ 0 & 0 & & 0 \\ 0 & 0 & & 0 \\ & & & \vdots \\ & & & 0 \\ & 1 & x \cdots & x \\ & & & \vdots \\ & & & 0 \\ & & & \vdots \\ 0 & \cdots & 0 & 1 \end{array}\right] \tag{16}$$

which is most easily verified by (tedious) inspection rather than by algebraic manipulation.

It follows that

$$\tilde{B} = \hat{B}C \tag{17}$$

where

$$\hat{B} = \left[\begin{array}{ccc} 0 & 0 \cdots & \\ 0 & 0 & \\ \vdots & \vdots & \\ 1 & 0 & \\ 0 & & \\ 0 & & \vdots \\ \vdots & 1 & \\ \vdots & 0 & \\ & \vdots & 0 \\ 0 & & 1 \end{array}\right] \tag{18}$$

and C is an $r \times r$ upper-triangular matrix with 1's on the main diagonal.

Since C is invertible, a new but equivalent set of system inputs can be defined as

$$v = Cu \tag{19}$$

With respect to these inputs, the system equations take the particularly nice form

$$\dot{y} = \tilde{A}y + \hat{B}v. \tag{20}$$

Theorem 1

Suppose the system (1) is controllable with controllability index ν_c. Then there is a nonsingular transformation of the state vector and a nonsingular transformation of the input vector which reduce the system to a coupled set of r single-input subsystems. Each subsystem is of order ν_c or less and is in the standard single-input phase-variable form. Furthermore, all additional coupling enters a subsystem at its input.

V. Conclusion

It has been shown that the standard phase-variable canonical form for single-input systems can be extended to multi-input systems but that the extended version is not unique. Probably the most interesting form is the one described by Theorem 1 and illustrated in Fig. 1.

The canonical forms developed in this paper for multi-input systems have direct analogs as canonical forms for multi-output systems. The details of their development from the results given here are straightforward.

Canonical forms for multivariable systems are useful as a starting point for deriving certain other general results for multivariable systems or for initiating design considerations. For example, it can be shown that by linear state-vector feedback the poles of a multivariable system can be placed arbitrarily. It is hoped that the canonical forms given here will prove useful in further research on multivariable systems.

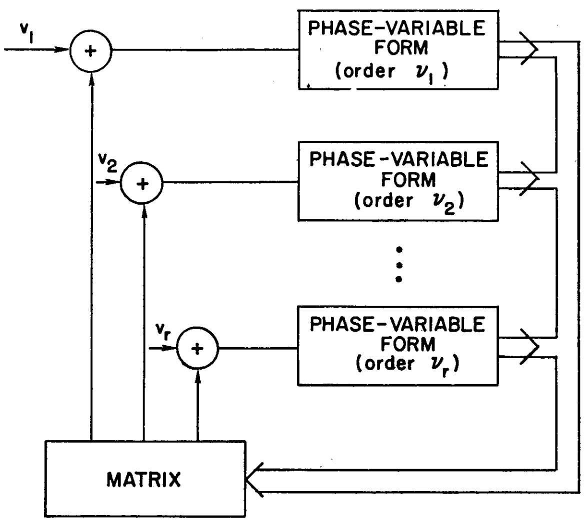

Fig. 1. Canonical form for multiple-input system.

Appendix I

Lemma 1

Unless n vectors have already been selected (and retained) in the process 1) to 4) above, there is a vector of the form $A^j b_k$, where all lower powers of A times b_k have been retained, which is linearly independent of all previously selected vectors.

Proof: Suppose that the selected vectors are:

$$b_1, Ab_1, \cdots, A^{q_1}b_1, b_2, Ab_2, \cdots, A^{q_2}b_2, b_3, \cdots, A^{q_r}b_r \tag{21}$$

and that each of the vectors

$$A^{q_1+1}b_1, A^{q_2+1}b_2, \cdots, A^{q_r+1}b_r$$

are linearly dependent on the selected vectors, so that the process terminates. It then follows by the induction argument sketched below that all other vectors in the controllability matrix (2) are linearly dependent on the selected vectors (21). This in turn implies that either the rank of the controllability matrix is less than n or there are n independent vectors in the selection (21).

A sketch of the induction proof is as follows:

The vector $\boldsymbol{A}^{q_1+2}\boldsymbol{b}_1$ is $\boldsymbol{A}\cdot\boldsymbol{A}^{q_1+1}\boldsymbol{b}_1$, so since $\boldsymbol{A}^{q_1+1}\boldsymbol{b}_1$ is a linear combination of the selected vectors, $\boldsymbol{A}^{q_1+2}\boldsymbol{b}_1$ is the same linear combination of $\boldsymbol{A}$ times the selected vectors. However, by hypothesis, $\boldsymbol{A}$ times any selected vector is also a linear combination of selected vectors, thus $\boldsymbol{A}^{q_1+2}\boldsymbol{b}_1$ is a linear combination of selected vectors. Proceeding in this fashion, one proves that all remaining vectors in the controllability matrix are dependent on the selected vectors.

This completes the proof.

Appendix II

It is shown here that the matrix S defined by (12) is nonsingular.

To this end it is sufficient to show that the rows of S are linearly independent; or equivalently, that any null linear combination of the rows must be the linear combination consisting of zeros.

Suppose there are constants a_{ij} such that

$$\sum_{i=1}^{r}\sum_{j=1}^{p_i} a_{ij}\mathbf{e}_i {\boldsymbol{A}^{j-1}}' = 0. \tag{22}$$

Taking the inner product of both sides of this equation with $\boldsymbol{b}_k$ produces

$$a_{kp_k} = 0 \tag{23}$$

since by definition of the $\mathbf{e}_i$'s each term in the inner product is zero except the one involving $\mathbf{e}_k\boldsymbol{A}^{p_k-1}\boldsymbol{b}_k$ which is unity.

In view of (23), (22) can be written equivalently as

$$\sum_{i=1}^{r}\sum_{j=1}^{p_i-1} a_{ij}\mathbf{e}_i\boldsymbol{A}^{j-1} = 0.$$

Taking the inner product of both sides of this with $\boldsymbol{A}\boldsymbol{b}_k$ produces

$$a_{k,p_k-1} = 0.$$

Continuing in this manner, by induction, it is proved that each $a_{ij}=0$ which completes the proof.

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