SIAM J. CONTROL AND OPTIMIZATION
Vol. 19, No. 5, September 1981

0363-0129/81/1905-0001 $01.00/0

ANALYSIS OF THE ASYMPTOTIC BEHAVIOR OF OPTIMAL CONTROL TRAJECTORIES: THE IMPLICIT PROGRAMMING PROBLEM*

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Abstract. The asymptotic behavior of the optimal trajectories of the infinite horizon control problem with discounting, is characterized by a static optimization problem. In the undiscounted case, the limit point of the optimal dynamic trajectory is the steady-state that minimizes the kernel of the objective functional. The corresponding static characterization of the limit point in the discounted case, called the implicit programming problem, is derived. The implicit programming problem is a mathematical programming problem with the special feature that part of the solution is contained in the definition of the problem. All results are achieved in the context of a sufficient maximum principle, which is shown to be equivalent to the other approaches taken in the literature to perform the dynamic analysis. The equivalence is based on convexity conditions assumed in the current dynamic theory. The class of problems that satisfy such convexity conditions is characterized in terms of a property of vector-valued mappings conceptually related to monotonicity.

1. Introduction. The objective of this paper is to characterize the asymptotic behavior of the solutions of the optimal control problem defined on an infinite horizon:

$$\text{minimize} \qquad \int_0^\infty L(x, u, t)\, dt \tag{1.1a}$$

$$\text{subject to} \qquad \dot{x}(t) = f(x(t), u(t), t), \tag{1.1b}$$

$$x(0) = x_0, \tag{1.1c}$$

$$(x(t), u(t)) \in X \times U \subseteq \mathbb{R}^n \times \mathbb{R}^m \quad \text{for each } t \in [0, \infty). \tag{1.1d}$$

The variable $x \in \mathbb{R}^n$ is the state variable, and the variable $u \in \mathbb{R}^m$ is the control variable. The set $U \subseteq \mathbb{R}^m$ may depend on $x(t)$ and t, explicitly. In that case, we shall write $U = U(x, t)$, where $U(x, t)$ is a set-valued mapping from $X \times [0, \infty)$ to $2^{\mathbb{R}^m}$, the set of all subsets of $\mathbb{R}^m$. L is a real-valued function and f is n-dimensional.

In this paper, we present a static characterization of the optimal steady-state trajectories of a subclass of problem (1.1), the optimal control problem with discounting. In this problem, a familiar model in mathematical economics, the kernel of the objective functional is $L(x, u, t) = e^{-\rho t} l(x, u)$ where ρ is the discount rate, and the system is autonomous, $f(x, u, t) = f(x, u)$.

It is known [26], [27] that the optimal trajectories of the discounted problem converge to a steady-state, under certain conditions. What is interesting about this property is that it follows from essentially static, geometric conditions about the data of the problem. We exploit this fact and characterize the optimal steady-state trajectory by a static optimization problem, the implicit programming problem. With this approach, we are able to determine the asymptotic properties of the optimal dynamic trajectories without having to solve the full dynamic problem itself.

It is often the case that precise knowledge of individual trajectories of a model is of less importance than the information that the optimal dynamic trajectory converges,

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* Received by the editors February 8, 1980. This research was supported in part by the National Science Foundation under grant NSF-ENG-76-18748.

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and converges to a particular point. Such limit points provide important information about the construction of approximate optimal trajectories. Moreover, the static characterization offers a convenient method for investigating the sensitivity of the optimal trajectory to various modelling assumptions; in particular, the sensitivity of the optimal steady-state to the discount rate is relatively easy to analyze using the implicit programming problem.

In the next section, we recall two approaches to the dynamic analysis of problem (1.1), given by the maximum principle, and the Hamiltonian dynamic system. In § 3 we present a third, equivalent characterization of the optimal dynamic trajectories that is given by sufficient conditions for dynamic optimality. The equivalence of these three approaches is a result of the basic convexity assumption invoked in the dynamic theory. This equivalence is discussed in § 4. The sufficient conditions provide a somewhat different perspective on the problem, and suggest a decomposition of the analysis into static and dynamic parts; the static aspect is interpreted as a supporting hyperplane result. In § 5 we give conditions under which the basic convexity assumption holds. In the last section, based on the decomposition perspective, we formulate the implicit programming problem and present a series of theorems that verify that the solutions to the implicit programming problem are the optimal steady-states of the optimal control problem with discounting.

2. Characterization of the optimal dynamic trajectory.

2.1. The maximum principle. The most familiar approach to the analysis of optimal control problems is based upon the necessary conditions for dynamic optimality that are expressed by the Pontryagin maximum principle [21]. The maximum principle has been extended by Halkin [12] to problems defined on an infinite horizon. In the process of that extension, Halkin proposed a relaxed concept of optimality, which is able to distinguish between trajectories even if the objective functional diverges. This extension of the maximum principle provides a set of necessary conditions that must be satisfied by a weakly overtaking-optimal trajectory. We include the following definitions for completeness, and then state the main result.

DEFINITION 2.1. A *trajectory* of problem (1.1) is a pair (x, u) such that:

(i) x is a continuous, piecewise continuously differentiable function from $[0, \infty)$ into $X \subseteq \mathbb{R}^n$;

(ii) u is a piecewise continuous function from $[0, \infty)$ into $U \subseteq \mathbb{R}^m$;

(iii) $\dot{x}(t) = f(x(t), u(t), t)$ for almost every $t \in [0, \infty)$;[1] and

(iv) $x(0) = x_0$.

DEFINITION 2.2. A trajectory (x^*, u^*) is said to be *weakly overtaking optimal* if for any feasible trajectory (x, u) and any $T \in [0, \infty)$ and any $\varepsilon > 0$ there exists a $t \geqq T$ such that

$$\int_0^t L(x^*, u^*, t)\,dt - \varepsilon \leqq \int_0^t L(x, u, t)\,dt,$$

or

$$\limsup_{t \to \infty} \int_0^t [L(x, u, t) - L(x^*, u^*, t)]\,dt \geqq 0. \tag{2.1}$$

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[1] Halkin defines the words "for almost every $t \in [0, \infty)$" to mean "for all $t \in [0, \infty)$ with the possible exception of a set I such that $I \cap [a, b]$ is finite for every closed bounded interval $[a, b] \subset [0, \infty)$" [12, p. 268n]. Similarly, piecewise continuity is qualified with respect to finite sets. Thus (i) implies that x is absolutely continuous on $[0, \infty)$. We shall employ Halkin's terminology. (See [12] for further details: alternate definitions are given in [17], [5], and [2].)

Other relaxed concepts of optimality have been proposed for the infinite horizon problem. In particular, weakly overtaking optimality is a modification of the concept of *overtaking optimality*, which was employed by Gale [10] and may be expressed as

$$\liminf_{t\to\infty} \int_0^t [L(x, u, t) - L(x^*, u^*, t)]\, dt \geqq 0. \tag{2.2}$$

A review of some of these concepts may be found in [15]. Brock and Haurie [5] gave conditions for the existence of overtaking- and weakly-overtaking optimal trajectories. We shall not be directly concerned with existence theories in this paper.

THEOREM 2.1. (Maximum principle). *Let U be a closed subset of $\mathbb{R}^m$. Let (f, L) be a continuous function from $\mathbb{R}^n \times U \times [0, \infty)$ into $\mathbb{R}^n \times \mathbb{R}$ whose first derivatives with respect to the first n arguments exist and are continuous over $\mathbb{R}^n \times U \times [0, \infty)$.*

If a trajectory (x^*, u^*) is weakly overtaking-optimal for problem (1.1) then there exist a nonnegative number p_0 and a continuous, piecewise continuously differentiable function p from $[0, \infty)$ into $\mathbb{R}^n$ such that:

(i) $\|(p_0, p(0))\| = 1$;

(ii) $\dot{p}(t) = -\partial/\partial\xi[H(\xi, u^*(t), t, p_0, p(t))]_{\xi=x^*(t)}$ *for almost every* $t \in [0, \infty)$;

(iii) $H(x^*(t), u^*(t), t, p_0, p(t) \geqq H(x^*(t), u, t, p_0, p(t)) \forall u \in U, \forall t \in [0, \infty)$;

where the Hamiltonian, H, is defined as

(iv) $H(x, u, t, p_0, p(t)) = -p_0 L(x, u, t) + \langle p(t), f(x, u, t)\rangle$.

Proof. See Halkin [12]. □

It is important to note that the maximum principle for the infinite horizon problem does not contain a transversality condition describing the behavior of the costate variable, $p(t)$, as t approaches infinity. In particular the boundary condition $\lim_{t\to\infty} p(t) = \theta$ is not generally satisfied.

It is also not proper to assume that $p_0 > 0$ (hence taken as 1), in general. This is the so-called normality assumption. Bliss [4] has given necessary and sufficient conditions for normality in problems in the calculus of variations; Berkovitz [3] has given a sufficient condition for normality in the control problem formulated on a finite horizon. Conditions for normality on an infinite horizon are not known at present. However, since the theory presented in this paper is essentially a sufficiency theory, our conditions will be written for the normal case.

2.2. Convexity theory and the Hamiltonian dynamic system. Another approach to the analysis of the optimal control problem has been studied by Rockafellar ([23], [25]–[27]). The main theme of this approach is the replacement of differentiability assumptions by convexity assumptions. The trajectories of the optimal control problem are then characterized by the Hamiltonian dynamic system. In his development of the application of convexity theory to the optimal control problem (1.1), Rockafellar [27] considers the problem of Lagrange, for an infinite time horizon,

$$\text{minimize} \quad \int_0^\infty L(t, x(t), \dot{x}(t))\, dt \tag{2.3a}$$

$$\text{subject to} \quad x(0) = x_0. \tag{2.3b}$$

The essential assumption that is made in the analysis of (2.3) is that $L(t, \cdot, \cdot)$ is a lower semicontinuous convex function on $\mathbb{R}^n \times \mathbb{R}^n$ with values in $(-\infty, +\infty]$, not identically $+\infty$; i.e., a lower semicontinuous *proper* convex function [25, p. 24].

The relationship between this problem and the optimal control problem (1.1) is effected by the formulation of the image function, $L^*(x, v, t)$, defined by

$$L^*(x, v, t) = \begin{cases} \inf\limits_{u} \{L(x, u, t) : v = f(x, u, t),\ u \in U(x, t)\} \\ +\infty \quad \text{if } x \notin X \text{ or } v \neq f(x, u, t) \forall u \in U(x, t). \end{cases} \tag{2.4}$$

The problem (2.3), in which the image $L^*(x, \dot{x}, t)$ replaces $L(t, x, \dot{x})$ is known as the deparametrized problem, in which the control parameter u has been eliminated. The effect of the control variable is felt as an infinite penalty, through the definition of $L^*(x, v, t)$ (2.4). Young [29] discusses the idea of control as a parameter in calculus of variations problems. Zachrisson [30] investigated the role of convexity theory in deparametrized problems. More recently, Goodman [11] analyzed a deparametrized formulation of the control problem.

The main results that can be achieved using this problem structure ([23], [25]) are that the optimal state-costate trajectories are solutions of the Hamiltonian dynamic system, defined by the subdifferential equations,

$$\dot{x}(t) \in \partial_p H^*(t, x(t), p(t)) \tag{2.5a}$$

and

$$\dot{p}(t) \in -\partial_x H^*(t, x(t), p(t)), \tag{2.5b}$$

with the Hamiltonian defined by the conjugacy formula

$$H^*(t, x, p) = \sup_{v} \{\langle v, p \rangle - L^*(t, x, v) : v \in \mathbb{R}^n\}. \tag{2.6}$$

The operator "∂_ξ" is the subdifferential operator. The subdifferential of a function at a point is the set of all subgradients of the function at that particular point. If the function happens to be differentiable at the point x, the subdifferential reduces to the gradient of the function at x [24, Thm. 25.1]. For completeness, we include

DEFINITION 2.3. A vector s is said to be a *subgradient* of a convex function h at a point x if

$$h(z) \geqq h(x) + \langle s, z - x \rangle \quad \forall z. \tag{2.7}$$

The relationship of the state-costate trajectories that satisfy the Hamiltonian dynamic system (2.5) and the state-control-costate trajectories that satisfy the necessary conditions of the maximum principle (Thm. 1.1) was discussed in [23, Example 12]. It was shown that if a state-costate trajectory is a solution to the Hamiltonian dynamic system, where l and f also satisfy the smoothness assumptions of the maximum principle, then there exists a control trajectory corresponding to the state-costate trajectory, such that all the conditions of the maximum principle are satisfied for the normal case ($p_0 = 1$). Further, in terms of the data of the control problem, the Hamiltonian H^* is equal to the optimal value Hamiltonian,

$$H^*(x, p, t) = \sup_{u} \{\langle p, f(x, u, t) \rangle - L(x, u, t)\}. \tag{2.8}$$

However, if the image function is indeed a lower semicontinuous proper convex function, there is yet another possible characterization of the optimal trajectories of the control problem. This third characterization is equivalent, in this case, to both the maximum principle and the Hamiltonian dynamic system, but is of a somewhat different nature, since it follows from sufficient conditions for dynamic optimality. It is discussed below.

3. A sufficient maximum principle and the support property. The next theorem is an extension of a result established by Peterson [20]. The sufficient maximum principle provides a context for all our subsequent developments. As we shall show, the more familiar characterizations of the optimal trajectories of problem (1.1) (given by the maximum principle or the Hamiltonian dynamic system), are actually equivalent to this sufficiency theorem, under the particular convexity assumptions that are generally invoked in the analysis of the optimal control problem.

THEOREM 3.1. (Sufficient maximum principle). *Let* (x^*, u^*) *be a trajectory (Definition* 1.1) *of problem* (1.1). *Let* p^* *be a continuous, piecewise continuously differentiable function from* $[0, \infty)$ *into* $\mathbb{R}^n$. *Define the Hamiltonian,* $H(x, u, t, p) = -L(x, u, t) + \langle p, f(x, u, t)\rangle$. *Suppose that*:

(i) $H(x^*(t), u^*(t), t, p^*(t)) + \langle \dot{p}^*(t), x^*(t)\rangle \geqq H(x, u, t, p^*(t)) + \langle \dot{p}^*(t), x\rangle \forall (x, u) \in X \times U$, *for almost every* $t \in [0, \infty)$;

(ii) $\lim_{t\to\infty} \langle p^*(t), x^*(t)\rangle$ *exists, and there holds*

$$-\infty < \lim_{t\to\infty} \langle p^*(t), x^*(t)\rangle \leqq \liminf_{t\to\infty} \langle p^*(t), x(t)\rangle < +\infty,$$

for any feasible state trajectory.

Then (x^*, u^*) *is overtaking-optimal for the optimal control problem* (1.1).

Proof. Since (x^*, u^*) is a trajectory, $\dot{x}^*(t) = f(x^*(t), u^*(t), t)$ holds for almost every $t \in [0, \infty)$. Let (x, u) be any other trajectory of problem (1.1). Then assumption (i) implies that, for any $T < +\infty$,

$$\int_0^T (-L(x^*(t), u^*(t), t) + d/dt[\langle p^*(t), x^*(t)\rangle])\, dt$$
$$\geqq \int_0^T (-L(x(t), u(t), t) + d/dt[\langle p^*(t), x(t)\rangle])\, dt.$$

Then

$$\int_0^T [L(x(t), u(t), t) - L(x^*(t), u^*(t), t)]\, dt \geqq \langle p^*(T), x(T) - x^*(T)\rangle.$$

Hence,

$$\liminf_{T\to\infty} \int_0^T [L(x, u, t) - L(x^*, u^*, t)]\, dt \geqq \liminf_{T\to\infty} \langle p^*(T), x(T) - x^*(T)\rangle$$
$$\geqq 0, \quad \text{by (ii).} \qquad \square$$

As Peterson [20] observed, no differentiability or continuity assumptions are invoked on L or f, directly, in Theorem 3.1. Moreover, condition (ii) of Definition 2.1 may be replaced by the simple inclusion, $u^*: [0, \infty) \to U$, since the continuity properties of u^* are irrelevant.

The assumption (i) of Theorem 3.1 occurs frequently in economic analysis, and is called the support property.

DEFINITION 3.1. A trajectory $(x^*(t), u^*(t))$ is said to be *supported* if there exists a continuous, piecewise continuously differentiable function $p^*: [0, \infty) \to \mathbb{R}^n$ such that

$$-L(x^*(t), u^*(t), t) + d/dt[\langle p^*(t), x^*(t)\rangle] \geqq -L(x, u, t) + \langle p^*(t), f(x, u, t)\rangle + \langle \dot{p}^*(t), x\rangle, \tag{3.1}$$
$$\forall (x, u) \in X \times U, \text{ for almost every } t \in [0, \infty).$$

The support property was defined by Gale [10] (he referred to a supported trajectory as "competitive"). More recently, Haurie [14] generalized the support property to suggest an approach to nonconvex problems of the form (1.1). Following Cass and Shell [7], the support functional

$$H(x, u, t, p^*(t)) + \langle \dot{p}^*(t), x \rangle = -L(x, u, t) + \langle p^*(t), f(x, u, t) \rangle + \langle \dot{p}^*(t), x \rangle, \tag{3.2}$$

may be interpreted as the profit rate given by the state control pair (x, u) at the "price" p^*, at time t. The sufficient maximum principle indicates the optimality of a "greedy" solution, one maximizing the profit rate at each instant and minimizing the asymptotic worth of the state variable, given by the inner product of the state and the supporting function, $\langle p^*(t), x^*(t) \rangle$.

It is evident that the concept of optimality provided by the sufficiency theorem is completely dependent upon the asymptotic behavior of the inner product $\langle p^*(T), x(T) - x^*(T) \rangle$. This suggests a decomposition of the analysis of the infinite horizon problem into separate components. One part of the analysis would determine conditions under which a trajectory is supported, and the other aspect would investigate the asymptotic properties of the inner product. We shall find this decomposition perspective useful in characterizing the optimal steady-states of the dynamic problem.

We shall now determine the source of the supporting function p^*. We show that the supporting function is the costate trajectory determined by the maximum principle, if a particular convexity assumption is satisfied. The convexity assumption we invoke is actually equivalent to the convexity assumption on the image function L^*. Moreover, under that convexity assumption, the maximum principle, the Hamiltonian dynamic system, and the support property are equivalent characterizations of the optimal dynamic trajectory.

4. The support theorem: equivalent characterizations of optimality. We relate the necessary conditions to the sufficient conditions through a convexity assumption. To motivate the assumption, we introduce some standard terminology. Suppose that the set of admissible controls, given that the system is in state x at time t, is described by the set $U(x, t)$. Then the *velocity set*, $F(x, t)$, is composed of all possible cost kernels and state velocities, as u ranges over the allowable set $U(x, t)$:

$$F(x, t) = \{(-L(x, u, t), f(x, u, t)) \colon u \in U(x, t)\}. \tag{4.1}$$

In his formulation of the optimal control problem, Filippov [9], in studying the system $\dot{x}(t) = f(x, u, t)$, where $u \in U(x, t)$, defined the set

$$R(x, t) = \{f(x, u, t) \colon u \in U(x, t)\} \tag{4.2}$$

and analyzed the contingent equation $\dot{x}(t) \in R(x, t)$ (which, by Filippov's lemma, is equivalent to the original dynamic system). Filippov made the important convexity assumption: $R(x, t)$ is convex for every pair (x, t). This assumption leads to several important existence results ([9], [28], [17]).

Baum [2] made assumptions about the set

$$Q(x, t) = \{(z^0, z) \colon z^0 \geqq L(x, u, t),\ z = f(x, u, t),\ u \in U(x, t)\}, \tag{4.3}$$

a set that is related to the velocity set $F(x, t)$. Baum required that $Q(x, t)$ be convex and closed for each pair (x, t) in order to achieve existence results for the infinite horizon problem.

For each state-time pair (x, t), $F(x, t)$ is a collection of points in $\mathbb{R} \times \mathbb{R}^n$ determined by all allowable controls, $u \in U(x, t)$. If, at time t, $(x^*(t), u^*(t))$ is on the optimal trajectory, then $(-L(x^*(t), u^*(t), t), f(x^*(t), u^*(t), t))$ is the optimal "velocity" vector of the joint cost-state dynamic system. Hence, any other velocity vector in $F(x^*(t), t)$ would be suboptimal. This essentially static property admits of a characterization in terms of supporting hyperplanes to convex sets, the convex set being $Q(x^*(t), t)$ and the hyperplane defined by the normal vector $(-1, p^*(t))$. Hence, Baum's convexity assumption expresses the maximum principle as a supporting hyperplane property of the costate variable, $p^*(t)$.

We will impose a stronger convexity assumption, extending Baum's assumption.

Assumption 4.1. The set

$$\Omega(t)=\{(x, v, \gamma): x \in X, v=f(x, u, t), \gamma \geqq L(x, u, t), u \in U(x, t)\}, \tag{4.4}$$

is convex and closed for each $t \in [0, \infty)$.

An immediate consequence of Assumption 4.1 is that Baum's convexity assumption follows. That is, the set

$$\begin{aligned}\Omega(x^*(t), t) &= \{(x^*(t), v, \gamma): \gamma \geqq L(x^*(t), u, t), v=f(x^*(t), u, t), u \in U(x^*(t), t)\}\\ &= \{(x^*(t), v, \gamma): (v, \gamma) \in Q(x^*(t), t)\}\end{aligned}$$

is also convex and closed, since it is the intersection of two closed, convex sets:

$$\Omega(x^*(t), t)=\Omega(t) \cap \{(x^*(t), v, \gamma): (v, \gamma) \in \mathbb{R}^n \times \mathbb{R}\}.$$

The main result that follows from this convexity assumption is that every trajectory that satisfies the conditions of the maximum principle, or is a solution of the Hamiltonian dynamic system, is actually supported. Just as the convexity of the set $Q(x, t)$ permits the maximum principle to be characterized by a supporting hyperplane, the convexity of the set $\Omega(t)$ permits the support property (Definition 3.1) to be characterized by a supporting hyperplane defined by the costate variable $p^*(t)$. We now state and prove the support theorem.

THEOREM 4.1 (Support theorem). *Let X be a subset of $\mathbb{R}^n$. Let $U(x, t)$ be a mapping* defined on $X \times [0, \infty)$ into $2^{\mathbb{R}^m}$. *Let (f, L) be a continuous function from the set*

$$D=\{(x, u, t): x \in X, t \in [0, \infty), u \in U(x, t)\},$$

into $\mathbb{R}^n \times \mathbb{R}$, whose first derivatives with respect to the first n arguments exist and are continuous over the set D.

Let (x^*, u^*) be a trajectory (Definition 2.1) *and suppose further that there exists a continuous, piecewise continuously differentiable function p^* from* $[0, \infty)$ *into $\mathbb{R}^n$ such that the triple (x^*, u^*, p^*) satisfies the conditions of the maximum principle* (Theorem 2.1) *for the normal case. Suppose, in addition, that $x^*(t) \in X^0$ (i.e., an interior point) for almost every $t \in [0, \infty)$, and that the set*

$$\Omega(t)=\{(x, v, \gamma): x \in X, v=f(x, u, t), \gamma \geqq L(x, u, t), u \in U(x, t)\}$$

is convex and closed $\forall t \in [0, \infty)$.

Then the trajectory (x^*, u^*) is supported by p^*.

Proof. Since $\Omega(t)$ is a closed, convex set, it follows that

$$\Omega(x^*(t), t)=\{(x^*(t), v, \gamma): v=f(x^*(t), u, t), \gamma \geqq L(x^*(t), u, t), u \in U(x^*(t), t)\}$$

is closed and convex for each $t \in [0, \infty)$. Consider the restriction of $\Omega(x^*(t), t)$ to $\mathbb{R}^n \times \mathbb{R}$, the set

$$\omega(x^*(t), t)=\{(v, \gamma): (x^*(t), v, \gamma) \in \Omega(x^*(t), t)\}.$$

Let $H(x, u, t, p^*(t)) = -L(x, u, t) + \langle p^*(t), f(x, u, t)\rangle$ and define the hyperplane in $\mathbb{R}^n \times \mathbb{R}$,

$$\begin{aligned}&\pi(x^*(t), u^*(t), t, p^*(t))\\ &\quad = \{(\eta, \gamma): \langle (p^*(t), -1), (\eta, \gamma)\rangle = H(x^*(t), u^*(t), t, p^*(t)), (\eta, \gamma) \in \mathbb{R}^n \times \mathbb{R}\}.\end{aligned}$$

By the maximum principle (Thm. 2.1 (iii), the maximization of the Hamiltonian), $(\eta^*(t), \gamma^*(t)) = (f(x^*(t), u^*(t), t), L(x^*(t), u^*(t), t))$ is a boundary point of the convex set $\omega(x^*(t), t)$ and $\pi(x^*(t), u^*(t), t, p^*(t))$ is a supporting hyperplane of $\omega(x^*(t), t)$ at $(\eta^*(t), \gamma^*(t))$.

Then, the point $(x^*(t), \eta^*(t), \gamma^*(t))$ is a boundary point of the convex set $\Omega(t)$ (since all ε-spheres centered at $(x^*(t), \eta^*(t), \gamma^*(t))$ contain points of the form $(x^*(t), \eta^*(t), \gamma)$, $\gamma < \gamma^*(t)$, and hence, not in $\Omega(t)$). By the support theorem for convex sets [19, Thm. 2, p. 133] (note that for finite dimensional space, the requirement that the supported set contain an interior point may be eliminated) there exists a closed hyperplane containing the point $(x^*(t), \eta^*(t), \gamma^*(t))$ such that $\Omega(t)$ is on one side of the hyperplane. Let the normal to the hyperplane be given by the functional $(-r(t), p^*(t), -1): \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$. Hence, the hyperplane

$$\begin{aligned}&\Pi(x^*(t), u^*(t), t, p^*(t))\\ &\quad = \{(\xi, \eta, \gamma): \langle (-r(t), p^*(t), -1), (\xi, \eta, \gamma)\rangle\\ &\qquad\qquad = \langle -r(t), x^*(t)\rangle + H(x^*(t), u^*(t), t, p^*(t)), (\xi, \eta, \gamma) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}\}\end{aligned}$$

contains $(x^*(t), \eta^*(t), \gamma^*(t))$ and supports the convex set $\Omega(t)$. Therefore, the functional $\langle (-r(t), p^*(t), -1), (\xi, \eta, \gamma)\rangle$ is maximized over the set $(\xi, \eta, \gamma) \in \Omega(t)$ at the point $(x^*(t), \eta^*(t), \gamma^*(t))$. Equivalently, we consider the maximization of the functional

$$-L(x, u, t) + \langle p^*(t), f(x, u, t)\rangle + \langle -r(t), x\rangle,$$

subject to the constraints $x \in \boldsymbol{X}$, $u \in U(x, t)$. The maximum is attained at $(x^*(t), u^*(t))$. Clearly, the maximization of the Hamiltonian (Thm. 2.1 (iii)) is a necessary condition for this maximization. Another necessary condition follows from the differentiability assumptions on the function (f, L) and the interior point assumption, $x^*(t) \in \boldsymbol{X}^0$. Thus the gradient, with respect to x, of the functional vanishes at $(x^*(t), u^*(t))$. We have

$$\partial/\partial\xi[-L(\xi, u^*(t), t) + \langle p^*(t), f(\xi, u^*(t), t)\rangle]_{\xi = x^*(t)} - r(t) = \theta.$$

By the costate differential equation (Thm. 2.1 (ii)), we conclude that $-r(t) = \dot{p}^*(t)$, for almost every $t \in [0, \infty)$. Therefore,

$$\begin{aligned}-L(x^*(t), u^*(t), t) + \frac{d}{dt}\langle p^*(t), x^*(t)\rangle \geqq -L(x, u, t) + \langle p^*(t), f(x, u, t)\rangle + \langle \dot{p}^*(t), x\rangle,&\\ \forall (x, u), \quad x \in \boldsymbol{X}, u \in U(x, t),&\\ \text{for almost every } t \in [0, \infty).&\end{aligned}$$

(A similar result was given in [14].) □

We will now show that the convexity assumption on the set $\Omega(t)$ is equivalent to assuming that the image function is a lower semi-continuous convex function. This follows from the fact that $\Omega(t)$ is the epigraph of the image function.

LEMMA 4.1. *The set* $\Omega(t) = \{(x, v, \gamma): x \in \boldsymbol{X}, v = f(x, u, t), \gamma \geqq L(x, u, t), u \in U(x, t)\}$ *is convex and closed if and only if the function*

$$L^*(x, v, t) = \begin{cases} \inf\limits_u\{L(x, u, t): v = f(x, u, t), u \in U(x, t)\} \\ +\infty \quad \text{if } x \notin \boldsymbol{X} \quad \text{or} \quad v \neq f(x, u, t) \quad \forall u \in U(x, t), \end{cases} \tag{4.5}$$

is convex and lower semi-continuous (hence, with effective domain the convex set

$$\operatorname{dom} L^*(t) = \{(x, v) : x \in X, \exists u \in U(x, t) \ni v = f(x, u, t)\}).$$

Proof. The epigraph of $L^*(x, v, t)$, for fixed $t \in [0, \infty)$, is the set

$$\begin{aligned}\operatorname{epi} L^*(t) &= \{(x, v, \gamma) : \gamma \geqq L^*(x, v, t), (x, v) \in \operatorname{dom} L^*(t)\}\\ &= \{(x, v, \gamma) : \gamma \geqq \inf_u \{L(x, u, t) : v = f(x, u, t), u \in U(x, t)\}\}\\ &= \{(x, v, \gamma) : \gamma \geqq L(x, u, t), v = f(x, u, t), u \in U(x, t)\}\\ &= \Omega(t).\end{aligned}$$

By definition, a function is convex if its epigraph is convex. The equivalence between lower semi-continuity of a function and closedness of the epigraph of the function is given by Rockafellar [24, Thm. 7.1.]. □

Lemma 4.1 implies the conclusion that every trajectory generated by the Hamiltonian dynamic system is supported. An indirect proof of this claim follows from Theorem 4.1 and the equivalence between the Hamiltonian dynamic system and the maximum principle, as noted in § 2. Directly, we observe that the Hamiltonian dynamic system and the Euler-Lagrange conditions for the image function L^*, the subdifferential equations $(\dot{p}^*(t), p^*(t)) \in \partial L^*(x^*(t), \dot{x}^*(t), t)$, for almost every t, are equivalent. The Euler–Lagrange conditions are themselves equivalent to the support property if L^* is convex. (The equivalence between the Hamiltonian dynamic system and the Euler–Lagrange conditions is based on the conjugacy properties of L^* and H^*, which we state explicitly in § 6 (see Lemma 6.1).)

Therefore, if Assumption 4.1 holds, we may characterize the optimal trajectories by the support property. The importance of this convexity assumption may be understood by observing the prominent place it occupies in the literature; the dynamic theory of Rockafellar ([23], [25]–[27]), the existence theory of Brock and Haurie [5], and the turnpike theory of Haurie [15] all are based on this essential assumption. It is significant to note that the difference between Assumption 4.1 and Baum's assumption, the convexity and closedness of $Q(x, t)$ for each (x, t), is that the latter is equivalent to assuming that $L^*(x, v, t)$ is convex as a function of v only. The joint variation in (x, v) is unspecified under Baum's assumption. Conditions under which Assumption 4.1 holds are presented in the immediately following section.

5. Convexity assumptions and the M-property.

This section presents conditions under which the set $\Omega(t)$ (4.4) is closed and convex; or equivalently, (Lemma 4.1) that the image function $L^*(x, v, t)$ (4.5) is convex and lower semi-continuous. To proceed, we require the following assumptions:

Assumption 5.1. The set $X \subseteq \mathbb{R}^n$ is convex.

Assumption 5.2. The mapping $U : X \times [0, \infty) \to 2^{\mathbb{R}^m}$ satisfies the convexity property: if $u_i \in U(x_i, t)$, $x_i \in X$, $i = 1, 2$, then, for each $t \in [0, \infty)$, $\lambda u_1 + (1 - \lambda) u_2 \in U(\lambda x_1 + (1 - \lambda) x_2, t), \forall \lambda \in [0, 1]$.

Observe that Assumption 5.2 is satisfied if $U(x, t) = U(t)$, for all $x \in X$, with $U(t)$ a convex set for each $t \in [0, \infty)$. Assumption 5.2 also holds if $U(x, t) = \{u : u \in \mathbb{R}^m, g(x, u, t) \leqq \theta\}$, where $g : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^s$ is a convex mapping (jointly in (x, u)) for each $t \in [0, \infty)$.

Assumption 5.3. The non-empty set $\Delta(t) = \{(x, v) : x \in X, v = f(x, u, t), u \in U(x, t)\}$ is convex and closed, for each $t \in [0, \infty)$.

Assumption 5.3 is stronger than Filippov's convexity assumption. Clearly, $\Delta(t)$ convex implies that $R(x, t)=\{f(x, u, t)\colon u \in U(x, t)\}$ is convex for each $(x, t) \in X \times [0, \infty)$.

This assumption may be interpreted as a condition on the inverse mapping, $f^{-1}(v; x, t)$, that assigns to each v in $\mathbb{R}^n$ the set of points ω in $\mathbb{R}^m$ such that $v = f(x, \omega, t)$. If $v \in R(x, t)$, then there exists at least one control, u, in $U(x, t)$ such that $v = f(x, u, t)$. That is, for $v \in R(x, t)$ the intersection $\{f^{-1}(v; x, t) \cap U(x, t)\} \neq \emptyset$. Assumption 5.3 requires that for all $\lambda \in [0, 1]$, for all $v_i \in R(x_i, t)$, $i = 1, 2$, the intersection of the sets f^{-1} $(\lambda(x_1, v_1, t) + (1-\lambda)(x_2, v_2, t))$ and $U(\lambda x_1 + (1-\lambda)x_2, t)$ is not empty.

Observe that the set $\Delta(t)$ is the effective domain of the function $L^*(x, v, t)$, defined by (4.5), for each $t \in [0, \infty)$. One might conjecture that since the domain is assumed to be convex, the convexity of $L^*(x, v, t)$ would follow from the convexity of the function $L(x, u, t)$, for each $t \in [0, \infty)$. This is not the case. In fact, more than convexity of $L(x, u, t)$ is required to assert the convexity of the function $L^*(x, v, t)$ or, equivalently, the convexity of the set $\Omega(t)$. The technical problem is the presence of the equality constraint in the definition of $\Omega(t)$ and the minimization problem that defines $L^*(x, v, t)$. Indeed if $\Omega(t)$ were defined by the inequality $v \geqq f(x, u, t)$ instead of the equality, convexity would follow from the assumption that (f, L) is a convex mapping for each $t \in [0, \infty)$. The actual definition of $\Omega(t)$ necessitates a further assumption.

Functions defined by equality-constrained minimization operations have been studied by Rockafellar [24]; he called them images, hence our use of the term to describe the function L^*. For a linear transformation A from $\mathbb{R}^n$ to $\mathbb{R}^m$, the image of the convex function h under A is defined by $(Ah)(y) = \inf \{h(x)\colon Ax = y\}$. The image (Ah) is easily shown to be convex [24, Thm. 5.7]. For given (x, t), the function $L^*(x, v, t)$ is then the image of $L(x, \cdot, t)\colon U(x, t) \to \mathbb{R}$ under $f(x, \cdot, t)\colon U(x, t) \to \mathbb{R}^n$.

For nonlinear mappings A, no general theorems relating to the convexity of the image (Ah) are known. We propose a condition to be satisfied by the function f that will aid in the characterization of images. The condition is called the M-property, since it is a property conceptually related to the monotonicity of mappings. As shown in Lemma 5.1, below, the M-property enables equality constraints to be treated as inequality constraints for certain nonlinear programming problems.

DEFINITION 5.1 A mapping $f\colon D \subseteq \mathbb{R}^s \to \mathbb{R}^n$ is said to satisfy the *M-property* on D if, for any v in the range of f and for any $x \in D$ such that $f(x) \geqq v$, there exists $y \in D$, $y \leqq x$, such that $f(y) = v$.

The M-property is concerned with positive cones. Let v belong to the range of f; i.e., $v \in f(D)$. At each element $y \in D$ that is in the level set $\{y\colon y \in D, f(y) = v\}$ erect a positive cone. The M-property is satisfied if all $x \in D$ such that $f(x) \geqq v$ are contained in the union of all positive cones with vertices in the level set.

A simple example of a mapping that satisfies the M-property is given by

$$f\colon \mathbb{R}^2 \to \mathbb{R}, \quad f(x_1, x_2) = x_1^2 + x_2^2, \quad D = \{(x_1, x_2)\colon x_i \geqq 0, i = 1, 2\}.$$

The level sets are quarter circles in the positive quadrant of the $x_1 - x_2$ plane, centered about (0, 0). The point $y = (0, 0)$ validates the M-property for $v = 0$. It is easy to see that the property holds for any $v > 0$.

The first result that can be proven using the M-property is an equivalence relationship between equality- and inequality-constrained nonlinear programming problems.

LEMMA 5.1 *Let* $l\colon D \subseteq \mathbb{R}^s \to \mathbb{R}$ *be nondecreasing on* D. *That is, if* $x_1, x_2 \in D$, $x_1 \leqq x_2$, *then* $l(x_1) \leqq l(x_2)$. *Let* $f\colon D \subseteq \mathbb{R}^s \to \mathbb{R}^n$ *satisfy the M-property on D. Suppose that the set*

$\{x: x \in D, f(x) = \theta\} \neq \emptyset$. *Then the following problems have equal optimal objective values*:

$$\begin{array}{lll} \min l(x), & \text{and} & \min l(x), \\ \text{s.t. } f(x) \geqq \theta, & & \text{s.t. } f(x) = \theta, \\ \quad x \in D, & & \quad x \in D, \end{array}$$

or

$$\min \{l(x): f(x) \geqq \theta, x \in D\} = \min \{l(x): f(x) = \theta, x \in D\}.$$

Proof. Since $f: D \to \mathbb{R}^n$ satisfies the M-property, for all $x \in D$ such that $f(x) \geqq \theta$, there exists $y \in D$, $y \leqq x$, $f(y) = \theta$. Since l is nondecreasing on D, $l(y) \leqq l(x)$. □

Using Lemma 5.1., we may now prove that the image $L^*(x, v, t)$ is convex and lower semicontinuous in (x, v) for each $t \in [0, \infty)$.

PROPOSITION 5.1. *Let Assumptions* 5.1, 5.2 *and* 5.3 *hold for* X, U, *and* f. *Let* $D = \{(x, u, t): x \in X, t \in [0, \infty), u \in U(x, t)\} \subseteq \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}$.

Suppose further that:

(i) $L: D \to \mathbb{R}$ *is convex and continuous with respect to* (x, u) *and non-decreasing with respect to u, for each* $t \in [0, \infty)$;

(ii) $f: D \to \mathbb{R}^n$ *is a continuous concave mapping with respect to* (x, u) *for each* $t \in [0, \infty)$; *and*

(iii) *for each* $(x, t) \in X \times [0, \infty)$ *the function* $f(x, \cdot, t): U(x, t) \to \mathbb{R}^n$ *satisfies the M-property on* $U(x, t)$.

Then $L^*(x, v, t)$ *is convex and lower semicontinuous in* (x, v) *for each* $t \in [0, \infty)$.

Proof. Fix $t \in [0, \infty)$. Let $(x_i, v_i) \in \Delta(t)$. That is, there exist $u_i \in U(x_i, t)$ such that $v_i = f(x_i, u_i, t)$, $i = 1, 2$. By definition,

$$\begin{aligned} &L^*(\lambda(x_1, v_1) + (1-\lambda)(x_2, v_2), t) \\ &\quad = \inf_u \{L(\lambda x_1 + (1-\lambda)x_2, u, t): \lambda v_1 + (1-\lambda)v_2 \\ &\qquad\qquad = f(\lambda x_1 + (1-\lambda)x_2, u, t), u \in U(\lambda x_1 + (1-\lambda)x_2, t)\}. \end{aligned}$$

By Assumption 5.3, the set

$$\{u: u \in U(\lambda x_1 + (1-\lambda)x_2, t), \lambda v_1 + (1-\lambda)v_2 = f(\lambda x_1 + (1-\lambda)x_2, u, t)\} \neq \emptyset.$$

By Assumption 5.1, $\lambda x_1 + (1-\lambda)x_2 \in X$.

Since the functions

$$L(\lambda x_1 + (1-\lambda)x_2, \cdot, t): U(\lambda x_1 + (1-\lambda)x_2, t) \to \mathbb{R},$$

and

$$f(\lambda x_1 + (1-\lambda)x_2, \cdot, t): U(\lambda x_1 + (1-\lambda)x_2, t) \to \mathbb{R}^n,$$

satisfy the assumptions of Lemma 5.1, and the feasible set is not empty, the equality constraints may be replaced by inequalities. Hence

$$\begin{aligned} &L^*(\lambda(x_1, v_1) + (1-\lambda)(x_2, v_2), t) \\ &\quad = \inf_u \{L(\lambda x_1 + (1-\lambda)x_2, u, t): \lambda v_1 + (1-\lambda)v_2 \\ &\qquad\qquad \leqq f(\lambda x_1 + (1-\lambda)x_2, u, t), u \in U(\lambda x_1 + (1-\lambda)x_2, t)\}. \end{aligned}$$

Let $u=\lambda\omega_1+(1-\lambda)\omega_2$, with $\omega_i\in U(x_i,t)$, $i=1,2$. By Assumption 5.2, $u\in U(\lambda x_1+(1-\lambda)x_2,t)$. However, restricting u to be of this form increases the value of the infimum. Moreover, by convexity of L and concavity of f, it follows that

$$\begin{aligned}L^*(\lambda(x_1,v_1)+(1-\lambda)(x_2,v_2),t)&\\ \leqq\lambda\inf_{\omega_1}\{L(x_1,\omega_1,t)\colon v_1\leqq f(x_1,\omega_1,t),\ \omega_1\in U(x_1,t)\}&\\ +(1-\lambda)\inf_{\omega_2}\{L(x_2,\omega_2,t)\colon v_2\leqq f(x_2,\omega_2,t),\ \omega_2\in U(x_2,t)\}.&\end{aligned}$$

By Lemma 5.1, we can replace the inequalities by equality constraints since the sets $\{\omega : v_i=f(x_i,\omega,t),\ \omega\in U(x_i,t)\}$ are not empty, $i=1,2$. Hence,

$$L^*(\lambda(x_1,v_1)+(1-\lambda)(x_2,v_2),t)\leqq\lambda L^*(x_1,v_1,t)+(1-\lambda)L^*(x_2,v_2,t).$$

The lower semicontinuity of L^* follows from Assumption 5.3 and the continuity of (f,L). □

Thus, Proposition 5.1 provides a set of conditions that are sufficient for the conclusion that the set $\Omega(t)$ is convex and closed for each $t\in[0,\infty)$, or equivalently, that the image function is convex and lower semicontinuous. That L^* is a proper convex function follows if L is never $\pm\infty$.

6. The static characterizations of the asymptotic behavior of the optimal trajectories of the discounted optimal control problem: the implicit programming problem.

6.1 The optimal control problem with discounting. We now apply the theory to a subclass of problems (1.1) that is of interest in mathematical economics, the so-called problem with discounting. Here a discount rate ρ is introduced and the kernel of the objective functional becomes $L(x,u,t)=e^{-\rho t}l(x,u)$. The dynamic system will be autonomous, $f(x,u,t)=f(x,u)$. The basic control problem (1.1) then becomes the discounted problem, with discount rate ρ:

(6.1a) minimize $$\int_0^\infty e^{-\rho t}\,l(x,u)\,dt$$

(6.1b) subject to $\dot{x}(t)=f(x,u)$,

(6.1c) $x(0)=x_0$,

(6.1d) $(x(t),u(t))\in X\times U(x(t))\subseteq\mathbb{R}^n\times\mathbb{R}^m$, for each $t\in[0,\infty)$.

We seek a characterization of the optimal steady-state trajectories of the discounted optimal control problem, pairs $(x^*,u^*)\equiv(x^0,u^0)$, for all t. We have seen that if the basic convexity property, given by Assumption 4.1, holds for the data of the problem, we may characterize the optimal trajectories of the problem by any one of three approaches. We shall describe the optimal steady-state trajectories in terms of the support property and formulate a static optimization problem, based on that description, that has as its solution the optimal steady-state. Then, we shall show that the dynamic theory that has been developed to analyze problem (6.1) may be interpreted as a means of establishing the boundary condition contained in Theorem 3.1, the sufficient conditions for dynamic optimality.

6.2 Static characterization of the optimal steady-state for the undiscounted problem. There is a complete body of literature analyzing problem (6.1) when $\rho=0$.

Some of the more recent references are Rockafellar [26], Haurie [15] and Brock and Haurie [5]; the classic reference is Ramsey [22]. It is known that, under sufficiently strict convexity conditions (including the convexity of the image L^*) the optimal state-costate trajectory converges to the saddlepoint (x^0, p^0) of the optimal value Hamiltonian [26, Thm. 1.2]. This saddlepoint may be characterized in terms of a static optimization problem, as noted by Brock and Haurie [5]. This result is given by the next theorem.

The theorem is motivated by the intuitively appealing idea that if the optimal trajectory converges toward a steady-state, or if a steady-state is an optimal trajectory for the dynamic problem, then that steady-state should minimize the kernel of the objective functional, $l(x, u)$. To estabish this result, as well as the other results of this section, we require the convexity assumption on $\Omega(t)$ (Assumption 4.1) or equivalently (Lemma 4.1).

Assumption 6.1. The image function, defined by

$$L^*(x, v) = \begin{cases} \inf\limits_u \{l(x, u)\colon v = f(x, u),\ u \in U(x)\} \\ +\infty \quad \text{if } x \notin X \quad \text{or} \quad v \neq f(x, u) \quad \forall u \in U(x), \end{cases} \tag{6.2}$$

is a lower semicontinuous proper convex function. (Recall that Proposition 5.1 provides a set of conditions under which Assumption 6.1 holds.)

THEOREM 6.1. *Suppose that Assumption* 6.1 *holds. Define the optimal value Hamiltonian*

$$H^*(x, p) = \sup_v \{\langle p, v\rangle - L^*(x, v)\colon v \in \mathbb{R}^n\}.$$

Let (x^0, p^0) *be a saddlepoint of* H^*. *Assume* $H^*(x^0, p^0)$ *is finite. Then* (x^0, u^0) *is a solution to the mathematical programming problem*

$$\text{(6.4a)} \qquad \text{minimize} \qquad l(x, u)$$

$$\text{(6.4b)} \qquad \text{subject to} \qquad f(x, u) = \theta, \quad x \in X,\ u \in U(x)$$

where $u^0 = \inf_u^{-1} \{l(x^0, u)\colon \theta = f(x^0, u),\ u \in U(x^0)\}$.

Proof. Apply the proof of Theorem 6.2, below, with $\rho = 0$ □

Theorem 6.1 provides a static characterization of the optimal steady state trajectory of the undiscounted optimal control problem. We relate the static problem (6.4) to the dynamic theory through the support property (Definition 3.1). For a steady-state trajectory, $(x^*(t), u^*(t), p^*(t)) \equiv (x^0, u^0, p^0)$, the support property becomes

$$\begin{aligned} &l(x^0, u^0) \leqq l(x, u) - \langle p^0, f(x, u)\rangle, \\ &\qquad \forall (x, u) \in D = \{(x, u)\colon x \in X,\ u \in U(x)\}. \end{aligned} \tag{6.5}$$

It is clear that a supported steady-state trajectory is necessarily a solution of the static problem (6.4). This observation suggests a similar characterization for the optimal steady-state trajectories of the discounted optimal control problem.

6.3 The implicit programming problem. The static characterization of the optimal steady-state trajectories of the discounted problem will be derived from the support property. The support property for the discounted problem is, for the trajectory

(x^*, u^*) and supporting function $p^*(t) = e^{-\rho t} q^*(t)$ (see § 6.4), the inequality

$$-e^{-\rho t} l(x^*(t), u^*(t)) + \frac{d}{dt}[\langle e^{-\rho t} q^*(t), x^*(t)\rangle]$$

$$\geqq -e^{-\rho t} l(x, u) + \langle e^{-\rho t} q^*(t), f(x, u)\rangle + \langle e^{-\rho t}(\dot{q}^*(t) - \rho q^*(t)), x\rangle \tag{6.6a}$$

$$\forall (x, u) \in D = \{(x, u) : x \in X, u \in U(x)\}, \text{ for almost every } t \in [0, \infty).$$

For a steady-state trajectory, $(x^*, u^*, q^*) \equiv (x^0, u^0, q^0)$ with $f(x^0, u^0) = \theta$, the support property becomes the inequality

$$l(x^0, u^0) \leqq l(x, u) - \langle q^0, f(x, u) - \rho(x - x^0)\rangle, \quad \forall (x, u) \in D. \tag{6.6b}$$

A necessary condition for (x^0, u^0) to be supported is that (x^0, u^0) is a solution to the well-defined nonlinear programming problem

(6.7a) minimize $l(x, \text{u})$

(6.7b) subject to $f(x, u) - \rho(x - x^0) = \theta$,

(6.7c) $x \in X, \quad u \in U(x)$.

Since x^0 is a fixed vector, the constraint (6.7b) is specified precisely and problem (6.7) is well defined. We now wish to consider the problem

(6.8a) minimize $l(x, u)$

(6.8b) subject to $f(x, u) - \rho(x - x^*) = \theta$,

(6.8c) $x \in X, \quad u \in U(x)$

where x^* is not fixed in advance. Indeed, x^*, as it appears in the constraint (6.8b), indicates the value of the x-component of the solution to the problem (6.8). In other words, the constraint is defined implicitly by the solution to the problem itself.

We call problem (6.8) an implicit programming problem, and we claim that it is a well-defined mathematical programming problem. Furthermore, we claim that under certain conditions, to be described below, the solution to the implicit programming problem is an optimal steady-state trajectory for the optimal control problem with discounting.

The most natural way to interpret the implicit programming problem is that it actually defines a mapping from $\mathbb{R}^n$ to $\mathbb{R}^n \times \mathbb{R}^m$. To highlight this interpretation, let us replace x^* in the constraint (6.8b) with a parameter $c \in \mathbb{R}^n$. As c varies over $\mathbb{R}^n$, a family of nonlinear programming problems is created. Moreover, for any c, the problem

(6.9a) minimize $l(x, u)$

(6.9b) subject to $f(x, u) - \rho(x - c) = \theta$,

(6.9c) $x \in X, \quad u \in U(x)$

defines a mapping that takes $c \in \mathbb{R}^n$ into the solution of problem (6.9), $(x^*(c), u^*(c)) \in D$.

The implicit programming problem (6.8) may then be written:

(6.10a) minimize $l(x^*(c), u^*(c))$

(6.10b) subject to $x^*(c) = c$,

(6.10c) $c \in X$.

where the minimization is taken over the fixed-points of the mapping $c \to x^*(c)$. The fact that this mapping is defined implicitly by the mathematical programming problem (6.9) suggests the terminology "implicit programming problem." One would expect the minimization defined by (6.10) to be over a discrete set (although there is a possibility of degeneracy in the data of the problem, in which case the fixed-points of the mapping form a continuum). Indeed, the discrete nature of the "feasible" set for (6.10) indicates that what is of primary importance in the analysis of the implicit programming problem is the set of local solutions; i.e., all the fixed-point of the implicit mapping defined by (6.9). We will designate such points as feasible points for the implicit programming problem (6.8).

It is clear that every feasible point of the implicit programming problem (i.e., a fixed-point of the implicit mapping) is a steady-state of the dynamic system $\dot{x} = f(x, u)$, since the term $\rho(x-c)$ in the constraint (6.9b) vanishes identically at the solution $x^*(c)$ whenever $x^*(c) = c$. Moreover, the solution to the implicit programming problem will not be the same as the solution of the undiscounted static problem (6.4); in general, the (globally) optimal value of the implicit programming problem will be higher than that of the undiscounted problem. This follows from the fact that only a subset of the steady-states correspond to fixed-points of the implicit mapping defined by (6.9), while the entire null space of f is feasible for (6.4). Thus, the optimal steady-state for the discounted problem is inferior compared with that for the undiscounted problem. This behavior is a direct result of the discounting of the objective; since later performance is valued less than earlier performance, the optimal trajectory exploits the dynamic possibilities in the structure of the problem at the beginning of the period to converge to what appears to be a suboptimal steady-state at the end. In fact, even if the initial condition were specified as the (global) solution to (6.4), the optimal trajectory would not remain at that point, but instead converge to a local solution of the implicit programming problem.

6.4 Determination of stationary points of the Hamiltonian dynamic system. We will now verify the claim that every local solution to the implicit programming problem is an optimal steady-state trajectory for the optimal control problem with discounting. To perform the analysis, we formulate the Hamiltonian dynamic system for the discounted problem, and express the steady-state trajectories in terms of the subdifferential equations of the dynamic system. We first show that the implicit programming problem characterizes the steady-states of the Hamiltonian dynamic system.

The Hamiltonian dynamic system is formulated subject ot the basic assumption that the image function L^* is a lower semicontinuous proper convex function (Assumption 6.1). The Hamiltonian is defined by:

$$H^*(t, x, p) = \sup_v \{\langle p, v\rangle - e^{-\rho t}L^*(x, v) \colon v \in \mathbb{R}^n\}. \tag{6.11}$$

It is convenient to introduce the change of costate variables, $p(t) = e^{-\rho t}q(t)$, which defines the current-value Hamiltonian, $H^*(x, q)$, such that $H^*(t, x, p) = e^{-\rho t}H^*(x, q)$, where

$$\begin{aligned} H^*(x, q) &= \sup_u \{\langle q, f(x, u)\rangle - l(x, u) \colon u \in U(x)\} \\ &= \sup_v \{\langle q, v\rangle - L^*(x, v) \colon v \in \mathbb{R}^n\}. \end{aligned} \tag{6.12}$$

The optimal state-costate trajectory of problem (6.1) is a solution to the Hamiltonian dynamic system, which due to the change of variables, becomes the autonomous system

$$(-\dot{q}(t)+\rho q(t),\ \dot{x}(t)) \in \partial H^*(x(t),\ q(t)), \tag{6.13}$$

the so-called *modified* Hamiltonian dynamic system. The steady-state trajectories of the modified Hamiltonian dynamic system are pairs (x^0, q^0) such that

$$(\rho q^0,\ \theta) \in \partial H^*(x^0,\ q^0). \tag{6.14}$$

We begin by proving the counterpart to Theorem 6.1, which indicates that every stationary point of the modified Hamiltonian dynamic system determines a feasible point of the implicit programming problem. The proof requires the following lemma, which relates the Hamiltonian dynamic system to the Euler-Lagrange Equations.

LEMMA 6.1 (Conjugate duality (partial conjugates)). *Let* $L(x, v): \mathbb{R}^n \times R^n \to (-\infty, +\infty]$ *be a lower semicontinuous proper convex function. Define*

$$H(x, p) = \sup_{v} \{\langle p, v\rangle - L(x, v):\ v \in \mathbb{R}^n\}.$$

Then $H(x, p): \mathbb{R}^n \times \mathbb{R}^n \to [-\infty, +\infty]$ *is concave in x and convex in p, and the following conditions on* $(x, p) \in \mathbb{R}^n \times \mathbb{R}^n$ *and* $(x^*, v^*) \in \mathbb{R}^n \times \mathbb{R}^n$ *are equivalent*:

(i) $-x^* \in \partial_x H(x, p), \qquad v^* \in \partial_p H(x, p);$
(ii) $x^* \in \partial_x L(x, v^*), \qquad p \in \partial_v L(x, v^*).$

Proof. These results are all given by Rockafellar [24, see Thms. 33.1 and 37.5]. □

THEOREM 6.2. *Suppose that Assumption* 6.1 *holds. Let* $(x^0, q^0) \in X \times \mathbb{R}^n$ *be a stationary point of the modified Hamiltonian dynamic system.*

Then (x^0, u^0) *is a solution to the mathematical programming problem* (6.9), *with* $c = x^0$,

$$\begin{aligned} &\text{minimize} && l(x, u)\\ &\text{subject to} && f(x, u) - \rho(x - x^0) = \theta,\\ & && x \in X, \quad u \in U(x), \end{aligned}$$

where $u^0 = \min_u{}^{-1}\{l(x^0, u):\ \theta = f(x^0, u),\ u \in U(x^0)\}$.

Proof. The stationary point condition (6.14), is equivalent to the subdifferential condition on L^* (Lemma 6.1), $(-\rho q^0, q^0) \in \partial L^*(x^0, \theta)$. By definition of a subgradient of a convex function (Definition 2.2), it follows that

$$L^*(x^0 + \xi, v) \geqq L^*(x^0, \theta) + \langle q^0, v - \rho\xi\rangle, \quad \forall (\xi, v) \in \mathbb{R}^n \times \mathbb{R}^n.$$

Thus, for $v = \rho\xi$, we have

$$\begin{aligned} &L^*(x^0 + \xi, \rho\xi) \geqq L^*(x^0, \theta) = l(x^0, u^0), \quad \text{where}\\ &u^0 = \min_u{}^{-1}\{l(x^0, u):\ \theta = f(x^0, u),\ u \in U(x^0)\}. \end{aligned}$$

By definition of $L^*(x^0 + \xi, \rho\xi)$, letting $\xi = x - x^0$,

$$\begin{aligned} l(x^0, u^0) &\leqq \inf_u \{l(x, u):\ \rho(x - x^0) = f(x, u),\ u \in U(x),\ x \in \mathbb{R}^n\}\\ &= \min_{(x,u)} \{l(x, u):\ \rho(x - x^0) = f(x, u),\ u \in U(x),\ x \in X\}. \end{aligned}$$ □

We are far more interested in a converse result, which would indicate how the implicit programming problem may be used as an analytic tool in conjunction with the established dynamic theory. Our objective is to pose sufficient conditions on the solution of the implicit programming problem to ensure that an optimal steady-state has been determined. We first show that every local solution of the implicit programming problem (i.e., a fixed-point of the implicit mapping $x^*(c)$) determines the state component of a stationary point of the modified Hamiltonian dynamic system.

PROPOSITION 6.1. *Let (x^*, u^*) be a solution to the mathematical programming problem* (6.9) *with $c = x^*$, hence feasible for* (6.8). *Suppose that Assumption* 6.1 *holds. Suppose further that the point (x^*, θ) is the interior of the effective domain of L^*. Then there exists a stationary point of the modified Hamiltonian dynamic system at (x^*, q^*), for some $q^* \in \mathbb{R}^n$.*

Proof. Since (x^*, u^*) is a solution to the mathematical programming problem (6.9) with $c = x^*$ we have

$$L^*(x^*, \theta) = \inf_u \{l(x^*, u)\colon \theta = f(x^*, u),\ u \in U(x^*),\ x^* \in X\} = l(x^*, u^*).$$

By assumption, $L^*(x, v)$ is a proper convex function, hence, if (x^*, θ) is in the interior of the effective domain of L^*, then the subdifferential of L^* at (x^*, θ) is not empty [24, Thm. 23.4]. Hence, for some $(\zeta, q^*) \in \mathbb{R}^n \times \mathbb{R}^n$,

$$L^*(\xi + x^*, v) \geqq L^*(x^*, \theta) + \langle q^*, v\rangle + \langle \zeta, \xi\rangle \quad \forall(\xi, v) \in \mathbb{R}^n \times \mathbb{R}^n.$$

Now let $v = \rho\xi$. We have

$$L^*(\xi + x^*, \rho\xi) \geqq L^*(x^*, \theta) + \langle \rho q^* + \zeta, \xi\rangle \quad \forall \xi \in \mathbb{R}^n.$$

However, since (x^*, u^*) is the solution to problem (6.9), we also have

$$\begin{aligned} l(x^*, u^*) &= L^*(x^*, \theta) \\ &= \min_{(x,u)} \{l(x, u) : \rho(x - x^*) = f(x, u),\ u \in U(x),\ x \in X\} \\ &\leqq L^*(x, \rho(x - x^*)) \quad \forall x \in X. \end{aligned}$$

Letting $\xi = x - x^*$, this implies $L^*(\xi + x^*, \rho\xi) - L^*(x^*, \theta) \geqq 0, \forall \xi \in \mathbb{R}^n$. Hence, it follows that $\rho q^* + \zeta = \theta$, which implies $(-\rho q^*, q^*) \in \partial L^*(x^*, \theta)$. By Lemma 6.1, we conclude that the subdifferential condition holds:

$$(\rho q^*, \theta) \in \partial H^*(x^*, q^*). \qquad \square$$

We will now characterize the costate component of the stationary point of the modified Hamiltonian dynamic system in terms of the Lagrange multiplier of the implicit programming problem. Recall the following results from the theory of nonlinear programming.

We consider the nonlinear programming problem with equality constraints:

(6.15a) minimize $l(x)$

(6.15b) subject to $f(x) = \theta$,

(6.15c) $x \in D \subseteq \mathbb{R}^s$.

DEFINITION 6.1. Let $f: \mathbb{R}^s \to \mathbb{R}^n$ be continuously differentiable. A point x^0 satisfying the constraints $f(x^0) = \theta$ is said to be a *regular point* of the constraints if the gradient vectors $\nabla_x f_1(x^0), \cdots, \nabla_x f_n(x^0)$ are linearly independent.

Definition 6.1 is equivalent to the fact that the $n \times s$ Jacobian matrix $[\nabla_x f(x)]$ is full rank at x^0.

The idea of a regular point is essential for characterizing the solution of problem (6.15), as indicated in the next theorem.

THEOREM 6.3. (First-order necessary conditions for a minimum; Lagrange multipliers). *Let* (l, f) *be* $\mathbb{C}^1$ *functions. Let* x^0 *be a solution to* (6.15), *that is a regular point of the constraints. Suppose further that* $x^0 \in D^0$, *an interior point of the feasible set. Then there exists* $\lambda^0 \in \mathbb{R}^n$ *such that* $\nabla_x l(x^0) - \langle \lambda^0, \nabla_x f(x^0)\rangle = \theta$.

Proof. See [18, Chpt. 10]. □

We shall define the Lagrangian of the implicit programming problem as follows: let

$$L_\rho(x, u, \lambda; c) = \begin{cases} l(x, u) - \langle \lambda, f(x, u) - \rho(x - c)\rangle, \\ +\infty \quad \text{if } x \notin X \quad \text{or} \quad u \notin U(x), \end{cases} \tag{6.16}$$

where $c \in \mathbb{R}^n$ is a fixed vector. This is the Lagrangian of a member of the class of problems (6.9).

With this definition of the Lagrangian, we may express the optimal value Hamiltonian as

$$H^*(x, q) = \begin{cases} \sup_u \{-L_\rho(x, u, q; c) : u \in U(x)\} + \rho\langle q, x - c\rangle, \\ -\infty \quad \text{if } x \notin X. \end{cases} \tag{6.17}$$

We now relate the saddlepoints of the Lagrangian to the saddlepoints of the (perturbed) optimal value Hamiltonian,

$$H_\rho^*(x, q; c) = H^*(x, q) - \rho\langle q, x - c\rangle. \tag{6.18}$$

LEMMA 6.2. *If* $L_\rho(x, u, q; x^0)$, *the Lagrangian* (6.16) *with* $c = x^0$, *possesses a saddlepoint* (x^0, u^0, q^0), *with* $x^0 \in X$ *and* $u^0 \in U(x^0)$, *then* (x^0, q^0) *is a saddlepoint of* $H_\rho^*(x, q; x^0)$ *where* u^0 *furnishes the supremum in the definition of* $H^*(x^0, q^0)$.

Further, if $H^*(x, q)$ *is a concave-convex function, then* (x^0, q^0) *is a stationary point of the modified Hamiltonian dynamic system.*

Proof. The saddlepoint condition on L_ρ is equivalent to the inequalities

$$-L_\rho(x, u, q^0; x^0) \leqq -L_\rho(x^0, u^0, q^0; x^0) \leqq -L_\rho(x^0, u^0, q; x^0) \quad \forall (x, u, q).$$

By definition, for $x^0 \in X$,

$$\begin{aligned} H_\rho^*(x^0, q; x^0) &= \sup_u \{-L_\rho(x^0, u, q; x^0) : u \in U(x^0)\} \\ &\geqq -L_\rho(x^0, u^0, q; x^0) \quad \forall q. \end{aligned}$$

Further, $-L_\rho(x, u, q^0; x^0) \leqq -L_\rho(x^0, u^0, q^0; x^0)$ implies that

$$\begin{aligned} H_\rho^*(x, q^0; x^0) &= \sup_u \{-L_\rho(x, u, q^0; x^0) : u \in U(x)\} \\ &\leqq -L_\rho(x^0, u^0, q^0; x^0) \quad \forall x. \end{aligned}$$

Hence,

$$H_\rho^*(x, q^0; x^0) \leqq -L_\rho(x^0, u^0, q^0, x^0) \leqq H_\rho^*(x^0, q; x^0) \quad \forall (x, q).$$

Thus,

$$H_\rho^*(x^0, q^0; x^0) = -L_\rho(x^0, u^0, q^0; x^0).$$

Therefore u^0 furnishes the supremum in the definition of $H_\rho^*(x^0, q^0; x^0)$, and (x^0, q^0) is a saddlepoint of H_ρ^*.

If $H^*(x, q)$ is a concave-convex function, it follows that $H_\rho^*(x, q; x^0)$ is also concave-convex. Hence, the saddlepoint condition on $H_\rho^*(x, q : x^0)$ is equivalent to the subgradient condition

$$(\theta, \theta) \in \partial H_\rho^*(x^0, q^0; x^0) \quad \text{or} \quad (\rho q^0, \theta) \in \partial H^*(x^0, q^0). \qquad \square$$

The search for stationary points of the modified Hamiltonian dynamic system becomes, by Lemma 6.2, a search for saddlepoints of the Lagrangian $L_\rho(x, u, q; c)$, with the special property that the state-component of the saddlepoint, x^0, is equal to the parameter c that defines the Lagrangian. This bears out the observation of Cass and Shell that the optimal steady-state is "something like a saddlepoint for the modified current value Hamiltonian $H^*(x, q) - \rho\langle q, x\rangle$" [7, p. 54]. The implicit programming formulation suggests consideration of the perturbed Hamiltonian $H_\rho^*(x, q; c)$; the saddlepoint behavior of this function is clearly a restatement of the support property. The main result of Lemma 6.2 is that every stationary point of the modified Hamiltonian dynamic system determines a stationary trajectory that is supported. Moreover, a necessary condition for such a trajectory to exist is that (x^0, u^0) is a solution to problem (6.9), with $c = x^0$, hence (x^0, u^0) is feasible for the implicit programming problem.

The next result provides sufficient conditions under which a feasible point of the implicit programming problem completely characterizes the stationary point of the modified Hamiltonian dynamic system. The result is a corollary to Proposition 6.1 and Lemma 6.2.

COROLLARY 6.1. *Let (x^*, u^*, λ^*) be a solution to the mathematical programming problem* (6.9) *with $c = x^*$. Assume that f and l are $\mathbb{C}^1$ functions. Suppose (x^*, u^*) is a regular point of the constraints $f(x, u) - \rho(x - x^*) = \theta$, and $x^* \in X^0$, $u^* \in [U(x^*)]^0$. Suppose further that Assumption* 6.1 *holds and that the point (x^*, θ) is in the interior of the effective domain of L^*. Then (x^*, λ^*) is a stationary point of the modified Hamiltonian dynamic system.*

Proof. The assumptions that L^* is a proper convex function and $(x^*, \theta) \in [\text{dom } L^*]^0$ imply, as in the proof of Proposition 6.1, that for some $q^* \in \mathbb{R}^n$ there holds

$$L^*(\xi + x^*, v) \geqq L^*(x^*, \theta) + \langle q^*, v - \rho\xi\rangle, \quad \forall(\xi, v) \in \mathbb{R}^n \times \mathbb{R}^n.$$

By definition of L^*, letting $\xi = x - x^*$, this is equivalent to

$$\begin{aligned} l(x^*, u^*) &\leqq \inf_u \{l(x, u) : v = f(x, u), u \in U(x)\} - \langle q^*, v - \rho(x - x^*)\rangle, \quad \forall(x, v) \\ &\leqq \inf_u \{l(x, u) - \langle q^*, f(x, u) - \rho(x - x^*)\rangle, u \in U(x)\}, \quad \forall x. \end{aligned}$$

Now let $x = x^*$. The infimum is then attained at $u = u^* \in [U(x^*)]^0$. Since u^* is an interior point, and (x^*, u^*) is a regular point of the constraints $f(x, u) - \rho(x - x^*) = \theta$, the necessary conditions hold,

$$\nabla_u[l(x^*, u) - \langle q^*, f(x^*, u)\rangle]_{u = u^*} = \theta,$$

and imply that $q^* = \lambda^*$, by Theorem 6.3. It follows that

$$l(x^*, u^*) \leqq l(x, u) - \langle \lambda^*, f(x, u) - \rho(x - x^*)\rangle \quad \forall x \in X, \quad u \in U(x).$$

Then the Langrangian $L_\rho(x, u, q; x^*)$ possesses a saddlepoint at (x^*, u^*, λ^*). Lemma 6.1 indicates that the optimal value Hamiltonian, $H^*(x, q)$, is a concave-convex function, given the assumptions on L^*. Hence, by Lemma 6.2, (x^*, λ^*) is a stationary point of the modified Hamiltonian dynamic system. □

Thus, we have established conditions under which a local solution to the implicit programming problem, with Lagrange multiplier, completely determines a stationary point of the modified Hamiltonian dynamic system. To assert the dynamic optimality of a local solution to the implicit programming problem, we must apply the sufficient maximum principle. This requires an investigation of the relationship of the solutions of the implicit programming problem to the dynamic theory.

To this point, our analysis has presented new results of a purely static nature. What we seek to demonstrate below is the extent to which the static approach can substitute for the dynamic analysis, for the purpose of characterizing the complete dynamic trajectory. The novel aspects of these subsequent results are contained in their relationship to the formulation of the implicit programming problem itself, and to the decomposition perspective suggested by the sufficient maximum principle. However, this portion of the theory is derived from the results of the current dynamic theory; we are, in this last section, indicating how implicit programming complements the dynamic analysis, rather than presenting any new results of a dynamic nature.

6.5 Application of the dynamic theory. The most complete dynamic analysis of the discounted problem is due to Rockafellar [27]. The theory developed rests on the following curvature assumption.

Assumption 6.2 (*curvature assumption*). We assume that for certain values $\alpha > 0$ and $\beta > 0$ the Hamiltonian H^* is locally α-concave and β-convex near the stationary point of the modified Hamiltonian dynamic system, (x^0, q^0), or in other words, that there exists a convex neighborhood $S \times T$ of (x^0, q^0) in $\mathbb{R}^n \times \mathbb{R}^n$ such that $H^*(x, q)$ is (finite and) α-concave in $x \in S$ for each $q \in T$ and β-convex in $q \in T$ for each $x \in S$. Moreover, the discount rate $\rho > 0$ is small enough so that $\rho^2 < 4\alpha\beta$.

The notion of α-convexity is a measure of strict convexity.

DEFINITION 6.2. A finite function h on a convex set $C \subseteq \mathbb{R}^n$ is said to be *α-convex*, where $\alpha \in \mathbb{R}$, if for all $x \in C$, $x' \in C$ and $\lambda \in [0, 1]$ it is true that

$$h((1-\lambda)x + \lambda x') \leqq (1-\lambda)h(x) + \lambda h(x') - \tfrac{1}{2}\alpha\lambda(1-\lambda)\|x - x'\|^2. \tag{6.19a}$$

α-concavity is defined by replacing the inequality above with

$$h((1-\lambda)x + \lambda x') \geqq (1-\lambda)h(x) + \lambda h(x') + \tfrac{1}{2}\alpha\lambda(1-\lambda)\|x - x'\|^2. \tag{6.19b}$$

The optimality of a steady state trajectory is based on the following convergence lemma. The convergence properties of the optimal dynamic trajectories are established subject to the curvature assumption (see [27]). It is this convergence property that indicates the important role that the optimal steady-state trajectories play in the dynamic theory.

LEMMA 6.3. *Suppose $x : [0, \infty) \rightarrow X \subseteq \mathbb{R}^n$ is a continuous, piecewise continuously differentiable function such that the objective functional converges, where L^* satisfies Assumption* 6.1:

$$\int_0^\infty e^{-\rho t} L^*(x, \dot{x})\, dt < +\infty.$$

Let (x^0, q^0) be a stationary point of the modified Hamiltonian dynamic system such that Assumption 6.2 *holds. Suppose further that*

$$\liminf_{t \rightarrow \infty} e^{-\rho t}\langle q^0, x(t)\rangle > -\infty.$$

Then, if Assumptions 6.1 *and* 6.2 *hold*, $\lim_{t \to \infty} e^{-\rho t}(x(t) - x^0) = \theta$.

Proof. See [27, Propositions 1 and 2]. □

We may now apply the sufficient maximum principle to claim that a local solution to the implicit programming problem is an optimal steady-state trajectory.

THEOREM 6.4. *Let* (x^*, u^*, λ^*) *be a local solution to the implicit programming problem. Assume that* f *and* l *are* $\mathbb{C}^1$ *functions. Suppose* (x^*, u^*) *is a regular point of the constraints* $f(x, u) - \rho(x - x^*) = \theta$, *and* $x^* \in X^0$, $u^* \in [U(x^*)]^0$. *Suppose that Assumptions* 6.1 *and* 6.2 *hold, the latter in a neighborhood of* (x^*, λ^*). *Then the stationary trajectory* (x^*, u^*) *is optimal in the class of trajectories with initial condition* $x(0) = x^*$ *such that* $e^{-\rho t} x(t)$ *remains bounded as* $t \to \infty$.

Proof. It follows, from Corollary 6.1, that (x^*, u^*) is a stationary trajectory that is supported by λ^*. (The interior point condition on (x^*, θ) follows from the curvature assumption, (see [27, Proposition 1]).) Lemma 6.3 indicates that the boundary condition

$$\lim_{t \to \infty} e^{-\rho t} \langle \lambda^*, (x(t) - x^*) \rangle \geqq 0,$$

is satisfied for all state trajectories x such that $e^{-\rho t} x(t)$ remains bounded as $t \to \infty$, that also provide finite cost sums. Since the support property and the boundary condition are satisfied, the optimality of the stationary trajectory follows from Theorem 3.1. □

Theorem 6.4 indicates conditions for a local solution of the implicit programming problem to be an optimal stationary trajectory. Moreover, the proof of the theorem is based on the decomposition perspective that is suggested by the sufficient maximum principle. The steady-state that is characterized by the implicit programming problem is a supported trajectory, because of the convexity assumption on L^* (Assumption 6.1). The optimality results from the boundary condition describing the asymptotic behavior of other trajectories. The boundary condition is established by the dynamic theory, as indicated by Lemma 6.3. In fact, the curvature assumption is actually a strengthening of the basic convexity assumption. By Lemma 6.1, the convexity of L^* implies that the Hamiltonian is concave-convex; Assumption 6.2 strengthens that property sufficiently to allow the boundary condition to be established.

If we view the theory from this perspective, there is another point to be mentioned. The curvature assumption is sufficiently powerful that we are able to conclude that it is actually the limit of the inner product that satisfies the nonnegativity condition required in the sufficiency theorem. Yet the relaxed concepts of optimality (e.g., weakly overtaking) require weaker asymptotic conditions. This suggests that one direction in which to proceed would be to weaken the curvature assumption and aim the dynamic theory towards establishing the boundary condition of Theorem 3.1.

There are two other kinds of theorems that can be proven about the solutions to the implicit programming problem. The first is a uniqueness theorem that indicates sufficient conditions for (at least) the state-component of the implicit programming problem to be unique. Corollary 6.1 indicates that every solution of the implicit programming problem determines a stationary point of the modified Hamiltonian dynamic system. Hence, conditions that are sufficient for the stationary point to be unique also imply that the solution to the implicit programming problem is unique (if the control that defines the optimal value Hamiltonian is unique). It is straightforward to establish that if the curvature assumption holds globally, then the stationary point is unique. (This result follows from an argument similar to that of [27, Proposition 5].) It is not true, however, that the stationary point is unique if the curvature assumption happens to hold at a particular point (locally). Further, even if the Hamiltonian H^* is globally strictly concave-convex, there may be multiple stationary points if the inequality $\rho^2 < 4\alpha\beta$ is violated (see [16, Example 1]).

The other kind of theorem indicates how the implicit programming problem can be used to analyze the asymptotic behavior of other optimal trajectories, with initial conditions sufficiently close to the steady-state. This is essentially a stability question, particularly when it is viewed in the context of the Hamiltonian dynamic system. The dynamic theory has established that, if Assumption 6.2 holds, the stationary point of the Hamiltonian dynamic system behaves like a saddlepoint of a differential equation. That is, there exist, locally, stable and unstable manifolds, which intersect only at the stationary point, that are comprised of the trajectories that converge to the stationary point as t approaches plus or minus infinity, respectively [27, Thms. 1 and 1′]. Then, for any initial condition on the manifold, the optimal trajectory remains in the manifold and converges to the stationary point [27, Thm. 2]. It is this result that indicates the important role played by the optimal steady-state trajectories; they are limit points of other optimal trajectories. Moreover, if Assumption 6.2 holds globally, then the solution of the implicit programming problem is unique, the stationary point of the modified Hamiltonian dynamic system is unique, and all optimal trajectories converge to this stationary point. In any event, if the curvature assumption holds in a neighborhood of a local solution to the implicit programming problem, the dynamic theory indicates that the solution is the limit point of other optimal trajectories, as well as an optimal steady-state trajectory.

Since the curvature assumption is a static property of the Hamiltonian, additional conditions may be imposed on the solution of the implicit programming problem that are sufficient to conclude that the curvature assumption holds. One way of establishing the curvature assumption would be to investigate the Hessian matrices of the Hamiltonian. This is the approach taken in the next theorem. We add smoothness conditions to the functions l and f, and a regularity condition on the mapping U. We also add two special assumptions, the local duality assumption [18, Ch. 12], that guarantees that the Lagrangian is locally convex at the solution of the implicit programming problem, and, a local controllability assumption that takes the form of a rank condition of the u-Jacobian of the function f. This latter assumption implies that the dimension of the control space is at least as great as the dimension of the state space. This condition is met in the calculus of variations, where control is identified with the derivative of the arc; in general control problems, it amounts to a restriction. However, in economic models, such a condition is generally satisfied. The main result of the theorem is that the Hamiltonian is strictly concave-convex in the neighborhood of a local solution to the implicit programming problem.

THEOREM 6.5. *Let* (x^*, u^*, λ^*) *be a solution to the implicit programming problem* (6.8), *such that* (x^*, u^*) *is a regular point of the constraints* $f(x, u) - \rho(x - x^*) = \theta$, *and* $x^* \in X^0$, $u^* \in [U(x^*)]^0$. *Let Assumption* 6.1 *hold and suppose further that*:

(i) *the functions f and l possess continuous second derivatives with respect to all arguments*;

(ii) *the mapping* $U : X \to 2^{\mathbb{R}^n}$ *is lower semi-continuous*;

(iii) *the Hessian of the Lagrangian*

$$\nabla^2_{(x,u)} L_\rho(x, u, \lambda^*; x^*) = \nabla^2_{(x,u)}(l(x, u) - \langle \lambda^*, f(x, u) \rangle)$$

evaluated at (x^*, u^*) *is positive-definite on all of* $\mathbb{R}^n \times \mathbb{R}^n$; *and*

(iv) *the matrix* $[\nabla_u f(x, u)]_{(x^*, u^*)}$ *is an* $n \times m$ *matrix of rank* n (hence $m \geqq n$).

Then the concave-convex optimal value Hamiltonian $H^*(x, q)$ *is strictly concave-convex in an open convex neighborhood,* $S \times T$, *of the stationary point* (x^*, λ^*) *of the modified Hamiltonian dynamic system.*

In addition, if $H^*(x, \lambda^*)$ *is* α*-concave for* $x \in S$ *and if* $H^*(x^*, q)$ *is* β*-convex for* $q \in T$, *such that* $\rho^2 < 4\alpha\beta$, *then there exists an open neighborhood* $S_+ \subset S$, $x^* \in S_+$, *such that for any initial condition* $x(0) \in S_+$, *the optimal state-control-costate trajectory converges to* (x^*, u^*, λ^*).

Outline of proof. Defining $u^*(x, q)$ as the minimizer in the definition of the optimal value Hamiltonian (6.12), the first-order necessary conditions for the implicit programming problem contain a system of m equations in $(n + m + n)$ variables that defines $u^*(x, q)$ implicitly. The local duality assumption (iii) indicates that the Jacobian of this system is nonsingular at (x^*, u^*, λ^*), so that the implicit function theorem applies to $u^*(x, q)$ in a neighborhood of the solution (x^*, λ^*). Calculating the Hessians of the optimal value Hamiltonian, $H^*(x, q) = \langle q, f(x, u^*(x, q))\rangle - l(x, u^*(x, q))$, it is easy to see that they are definite. The lower semi-continuity assumption on the mapping U is required to assert that the implicit function $u^*(x, q)$ actually belongs to the set $U(x)$ for x in a neighborhood S of x^*. The assumption on the relative sizes of the state and control spaces is required to express the Hessian of the Hamiltonian, with respect to q, as an inner product of full-rank matrices, hence definite. The last statement of the theorem is precisely Theorem 2, [27]. For further details, see [8]. □

6.6 Summary. We have shown that, under the basic convexity assumption describing the image function L^* (Assumption 6.1):

(i) every stationary point of the modified Hamiltonian dynamic system is a feasible point of the implicit programming problem (Theorem 6.2);

(ii) conversely, every feasible point of the implicit programming problem determines a stationary point of the modified Hamiltonian dynamic system (Proposition 6.1);

(iii) every stationary point of the modified Hamiltonian dynamic system determines a stationary trajectory that is supported (Lemma 6.2);

(iv) if the solution to the implicit programming problem is a regular point, then the stationary point of the Hamiltonian dynamic system is determined by the Lagrange multiplier (Corollary 6.1);

and, complementing the dynamic theory,

(v) if the curvature assumption holds in a neighborhood of the solution of the implicit programming problem, then the solution of the implicit programming problem is an optimal steady-state trajectory in the class of dynamic trajectories such that $e^{-\rho t}x(t)$ is bounded as $t \to \infty$ (Theorem 6.4);

(vi) the solution to the implicit programming problem is the limit point of other optimal trajectories, under additional (local) assumptions designed to establish the curvature assumption (Theorem 6.5).

The implicit programming problem also remedies the lack of a transversality condition in the maximum principle for the infinite horizon problem. The Lagrange multiplier of the implicit programming problem determines the boundary condition at infinity of the costate trajectory, and the familiar two-point boundary-value problem determines the optimal trajectory.

7. Conclusions. We have formulated a static optimization problem, the implicit programming problem, that characterizes the optimal steady-states of the optimal control problem with discounting; all results hold for the undiscounted case as well. This characterization does not require the solution of the full dynamic problem to determine the asymptotic behavior of the optimal dynamic trajectories. The formulation of the problem is based on an application of sufficient conditions for dynamic optimality, which have been shown to be equivalent to the more familiar approaches found in the literature. The sufficient conditions suggest a decomposition of the analysis of optimal control problems defined on an infinite horizon. The implicit programming problem responds to one aspect of that decomposition, the support property. The current dynamic theory may be interpreted as an approach to the other aspect of this decomposition, the boundary condition. This perspective suggests a possible direction in which to proceed, that of investigating ways in which to weaken the assumptions of the current dynamic theory.

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