APPROXIMATION OF LINEAR CONSTANT SYSTEMS

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Abstract

In this paper two problems are considered: the problem of modeling a given constant linear system by a constant linear system of fixed lower order, and the problem of finding a filter of fixed order to estimate a time invariant random process from a related time invariant random process. A quadratic criterion is used to select the optimum system in both cases.

It is shown that the filtering problem reduces to the problem of modeling the corresponding Wiener filter. The modeling problem is found to be analogous to the geometric problem of finding the point on a nonlinear submanifold of a space nearest to a given point in the whole space. Necessary conditions for a solution are developed and stated in terms of standard Wiener Filter theory notation. Numerical solution of the equations embodying the necessary conditions is considered and several examples are presented.

Introduction

It is often desirable to replace a complex system by a simpler system either because the simpler system is easier to analyze or because it is cheaper to build. If the system is linear and constant-coefficient, then a natural way to simplify it is to model it by a constant, linear system of lower dynamic order.

Such problems were first treated by Phillips[1], who considered a modified Wiener filter problem in which the filter was constrained to be of a given order. Mantey[2] investigated the problem of realizing as closely as possible a given input-output relation with a linear, constant system of fixed order. In this paper, both problems are considered; they will be referred to as the Phillip's filter problem and the modeling problem respectively.

A criterion of optimality must be selected for the modeling problem (a mean square criterion is assumed in the Phillip's filter problem). Mantey chose to use a criterion which lead to a simple procedure for determining the optimum model. In considering an adaptive control problem, Kalman[3] used the same criterion and derived the same procedures as Mantey. For this paper, the optimum model is defined to be the system (of order less than a fixed order less than the order of the given system) that minimizes the mean square error between the output of the given system and the output of the model, when both systems have the same random input. The filter problem is equivalent to the modeling problem when this criterion is used.

In principle, a set of necessary conditions that must be satisfied by the optimal approximating system can be derived by differentiating the objective function with respect to the unknown parameters. This approach was originally pursued by Phillips who obtained in this way a fairly complicated set of nonlinear equations. Phillips parameterized all transfer functions as the ratio of two polynomials and thus his necessary conditions were expressed in terms of the coefficients of these polynomials. In this paper the system transfer functions are represented in partial fraction form and a set of necessary conditions obtained that take an extremely simple form when expressed in terms of the $[\ ]_+$ operator commonly used in Wiener filter theory. Thus the necessary conditions, although still nonlinear, are easily interpreted in familiar terms. It is shown that numerical solutions to the approximation problem can be obtained by straight-forward numerical solution of the set of necessary equations.

Problem Statement

In this paper only constant, linear systems with a single input and a single output are considered; therefore, unless stated otherwise, the term "system" refers to such systems. A system is called nth order if its transfer function is rational with denominator of order n and numerator of order less than n. The transfer function of such a system is referred to as an nth order transfer function.

The modeling problem, illustrated in Figure 1, is:

Linear-Constant Coefficient--Modeling Problem

Given:

1. A linear, constant coefficient, asymptotically stable system S of order n.

2. A stationary random process X(t) with mean zero, and rational power spectral density $\Phi_x(s)$, [$\Phi_x(s)$ is the bilateral Laplace transform of $E\{X(\tau)\,X(\tau+t)\}$, with $E\{\ \}$ denoting expectation*] bounded on the $j\omega$ axis.

Find:

The linear, constant coefficient, system $\hat{S}$ of order m or less where $m<n$ that best models the given system in the sense that, if X(t) is used as input to both systems then in steady state

$$J = E\{[Y(t) - \hat{Y}(t)]^2\} \tag{1}$$

is minimized, where Y(t) is the output of S and $\hat{Y}(t)$ the output of $\hat{S}$.

The Phillips filter problem is illustrated in Figure 2:

Given:

1. A stationary random process X(t) with mean zero, bounded variance, and rational power spectral density $\Phi_x(s)$, bounded on the $j\omega$ axis.

*Note that since X(t) is stationary $E\{X(\tau)X(\tau+t)\}$ is independent of τ.

2. A stationary random process $Y(t)$ with mean zero, bounded variance, and power spectral density $\Phi_{\tilde{y}}(s)$, bounded on the $j\omega$ axis.

3. The cross power spectral density $\Phi_{x\tilde{y}}(s)$, between $X(t)$ and $\tilde{Y}(t)$, where $\Phi_{x\tilde{y}}(s)$ is the bilateral Laplace transform $E\{X(\tau)\ \tilde{Y}\ (\tau+t)\}$.

Find:

The linear, constant-coefficient, system $\hat{S}$ of order m or less for which

$$\tilde{J} \triangleq E\{[\tilde{Y}(t) - \hat{Y}(t)]^2\} \qquad (2)$$

is minimized.

Several remarks are in order at this point. $\hat{S}$ must be asymptotically stable since otherwise (1) or (2) will be unbounded; hence the optimum model or filter, if one exists, is asymptotically stable. Only constant systems are treated because only for such systems is dynamic order a good measure of complexity. This is not true for time varying systems; for example, a second order system with complicated time variation is not necessarily simpler than a fourth order system with simple time variation. These remarks apply with even greater force to nonlinear systems. Finally, it should be noted that the requirement of optimality depends upon the characteristics of the input to the system because it is unreasonable to expect the model to approximate the system well for all inputs.

Consider (1) and (2) in greater detail. When the right sides of these two equations are expanded:

$$J = E\{Y^2(t)\} - 2E\{Y(t)\hat{Y}(t)\} + E\{\hat{Y}^2(t)\}\ , \qquad (3)$$

$$\tilde{J} = E\{\tilde{Y}^2(t)\} - 2E\{\tilde{Y}(t)\hat{Y}(t)\} + E\{\hat{Y}^2(t)\}\ . \qquad (4)$$

Suppose that the given system S is the best Wiener filter for estimating $\tilde{Y}(t)$ from $X(t)$. The optimum Wiener filter satisfies the wiener-Hopf equation[4] which states that

$$E\{Y(t)\ X(\tau)\} = E\{\tilde{Y}(t)\ X(\tau)\} \qquad \tau < t\ . \qquad (5)$$

But $\tilde{Y}(t)$ is just the result of a linear operation on the $X(\tau)$'s for $\tau < t$. Hence

$$E\{Y(t)\ \hat{Y}(t)\} = E\{\tilde{Y}(t)\ \hat{Y}(t)\}\ . \qquad (6)$$

Substitution of (6) and (3) into (4) yields

$$\tilde{J} = E\{\tilde{Y}^2(t) - 2E\{Y(t)\ \hat{Y}(t)\} = E\{\hat{Y}^2(t)\}$$

$$= E\{\tilde{Y}^2(t)\} - E\{Y^2(t)\} + J\ . \qquad (7)$$

Since $E\{\tilde{Y}^2(t)\}$ and $E\{Y^2(t)\}$ are quantities fixed by the problem statement, the $\hat{S}$ which minimizes J also minimizes $\tilde{J}$; therefore, the optimum Phillips filter is the optimum model of the optimum Wiener filter if the random process $X(t)$ used as input in determining the optimum model is the same as the process which is available for measurement in the filter problem. This result is illustrated in Figure 3. Because it can be reduced to a modeling problem, the Phillip's filter problem will not be analyzed further.

Consider Figure 1 again. J can be expressed in terms of the transfer functions $T(s)$ and $\hat{T}(s)$ of S and $\hat{S}$ respectively:

$$J = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \left|T(s)-\hat{T}(s)\right|^2 \Phi_x(s)\ d\ s\ . \qquad (8)$$

Thus, the modeling problem consists of finding the $T(s)$ corresponding to a system of order m which minimizes the integral given above.

Geometric Interpretation

It is convenient to introduce at this point the inner product space[5] V of rational transfer functions $T(s)$ for which

$$\frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \left|T(s)\right|^2 \Phi_x(s)\ d\ s = \left|\left|T(s)\right|\right|^2 < \infty\ . \qquad (9)$$

The space V becomes a vector space if addition is defined by

$$T_{1+2}(s) = T_1(s) + T_2(s)\ , \qquad (10)$$

and scalar multiplication is defined by

$$T_{\alpha\cdot 1}(s) = \alpha T_1(s)\ .$$

The inner product of two elements $T_1(s)$ and $T_2(s)$ in this space is defined as

$$[T_1(s), T_2(s)] = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} T_1(s)T_2^*(s)\Phi_x(s)\ d\ s\ . \qquad (11)$$

The norm of an element $T_1(s)$ in V is*

$$\left|\left|T_1(s)\right|\right| = \sqrt{[T_1(s), T_1(s)]}\ . \qquad (12)$$

Since $\Phi_x(s)$ is bounded on the imaginary axis, it is easily seen that all finite order, asymptotically stable transfer functions are in V, therefore, all transfer functions relevant to the modeling problem discussed in this paper are elements of the inner product space V.

The criterion of optimality for the modeling problem may be expressed as

$$J = \left|\left|T(s) - \hat{T}(s)\right|\right|^2\ . \qquad (13)$$

Hence the modeling problem consists of picking the transfer function of order m or less which is closest to the transfer function of the given system. If the set of all transfer functions of order m or less were a subspace of the space of all rational transfer functions, the modeling problem could be solved by projecting the original transfer function T onto the subspace. A subspace has two properties: The sum of any two vectors in the subspace is also in the subspace, and the product of any scalar with any vector in the subspace is in the subspace. Unfortunately, the sum of two transfer functions of order m is not in general a transfer function of order m; therefore, this set is

*Since $\Phi(s)$ and $T_1(s)$ are rational $\left|\left|T_1(s)\right|\right|=0$ implies $T_1(s)\equiv 0$; hence (12) gives a valid norm.

not a subspace* and the modeling problem cannot be solved by projection. This implies that the modeling problem is not a linear problem, a point which will become clear in the next section.

Necessary Conditions

To solve the problem posed in the Introduction, the following procedure is followed: The transfer function of a general mth order system is written in terms of 2 m parameters; J is then a function of these parameters. Necessary conditions for minimum J are found by differentiating with respect to the parameters.

For ease of presentation, T(s) is assumed to have no multiple poles. To derive the necessary conditions when multiple poles occur requires a considerable increase in notation with little corresponding increase in insight. The results, however, will be presented in complete generality.

Any transfer function T(s) of order m without multiple poles can be expanded in a partial fraction expansion

$$\hat{T}(s) = \sum_{i=1}^{M} \frac{\hat{r}_i}{s-\hat{p}_i} \tag{14}$$

where the $\hat{p}_i$ are the system poles and the $\hat{r}_i$ are the corresponding residues. Thus

$$J = J(\hat{r}_i, \hat{r}_2, \ldots, \hat{r}_m, \hat{p}_1, \hat{p}_2, \ldots, \hat{p}_m) . \tag{15}$$

Consider the problem of taking the derivative of J with respect to any real parameter α of T(s). From (8)

$$\frac{\partial J}{\partial \alpha} = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} [T(s)-\hat{T}(s)]\, \frac{\partial \hat{T}^*(s)}{\partial \alpha} \Phi_x(s)ds$$

$$+ \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \frac{\partial \hat{T}(s)}{\partial \alpha}\, [T(s)-\hat{T}(s)]^* \Phi_x(s)ds . \tag{16}$$

Note that for $s = j\omega$,

$$T^*(s) = T(-s) , \tag{17}$$

$$\hat{T}^*(s) = \hat{T}(-s) \tag{18}$$

Hence, if the dummy variable is changed from s to -s, the second integral in (16) becomes identical to the first; hence

$$\frac{\partial J}{\partial \alpha} = \frac{2}{2\pi j}\int_{-j\infty}^{j\infty} [T(s)-\hat{T}(s)]\, \frac{\partial \hat{T}^*(s)}{\partial \alpha} \Phi_x(s)ds . \tag{19}$$

Earlier it was noted that the sum of two transfer functions of order m or less is not in general a transfer function of order m or less. However, if the two transfer functions have the same poles, then their sum is also a transfer function of order m or less. Thus the set of all transfer functions of order m or less with the same poles forms a subspace, and the problem of finding the optimum residues (i.e., the values of $\hat{r}_i$ which minimize J) when the poles are specified is a linear problem which can be solved by projection.

If $\hat{p}_i$ is real then $\hat{r}_i$ is real also, and from (14)

$$\frac{\partial \hat{T}(s)}{\partial \hat{r}_i} = \frac{1}{s-\hat{p}_i} , \tag{20}$$

hence by (19)

$$\frac{\partial J}{\partial \hat{r}_i} = \frac{2}{2\pi j}\int_{-j\infty}^{j\infty} [T(s)-\hat{T}(s)]\frac{\partial \hat{T}^*(s)}{\partial \alpha} \Phi_x(s)ds$$

$$= 2[T(s)-\hat{T}(s), \frac{1}{s-\hat{p}_i}] . \tag{21}$$

A necessary condition that $\hat{r}_i$ minimize J is that $\frac{\partial J}{\partial \hat{r}_i}$ vanish

$$[T(s)-\hat{T}(s), \frac{1}{s-\hat{p}_i}] = 0 , \tag{22}$$

If $\hat{p}_i$ is not real then there exists a $\hat{p}_j = \hat{p}_i^*$ with $\hat{r}_j = \hat{r}_i^*$. Letting $\hat{r}_i = \alpha + j\beta$ then necessary conditions for $\hat{r}_i$ to be optimum are

$$\frac{\partial J}{\partial \alpha} = 2[T(s)-\hat{T}(s), \frac{1}{s-\hat{p}_i} + \frac{1}{s-\hat{p}_i^*}] = 0 ;$$

$$\frac{\partial J}{\partial \beta} = 2[T(s)-\hat{T}(s), \frac{j}{s-\hat{p}_i} - \frac{j}{s-\hat{p}_i^*}] = 0 . \tag{23}$$

These two equations imply that (22) must also hold if $\hat{p}_i$ is complex.

Thus if the poles of the model are specified, the optimum residue corresponding to these poles must be such that

$$\boxed{[T(s)-\hat{T}(s), \frac{1}{s-\hat{p}_i}] = 0 \quad i=1,2, \ldots, m .} \tag{24}$$

Furthermore, because J is quadratic in the $\hat{r}_i$ satisfaction of these conditions is also sufficient.

Note that any transfer function $T_1(s)$ with the given poles can be written

$$T_1(s) = \sum_{i=1}^{m} \frac{\hat{a}_i}{s-\hat{p}_i} \tag{25}$$

hence by (24)

$$[T(s)-\hat{T}(s), T_1(s)] = 0 \tag{26}$$

for all mth order systems $T_1(s)$ with the given poles. Equation (26) states that the difference between the given system and model is perpendicular to all systems of order m or less with the given poles.

If we let $T_1(s)$ in (24) be T(s) then

$$J = ||T(s) - \hat{T}(s)||^2$$

$$= [T(s)-\hat{T}(s), T(s)-\hat{T}(s)]$$

$$= [T(s), T(s)]-[\hat{T}(s), T(s)]$$

$$= ||T(s)||^2 - [\hat{T}(s), \hat{T}(s)] - [\hat{T}(s), T(s)-\hat{T}(s)]$$

$$J = ||T(s)||^2 - ||\hat{T}(s)||^2 . \tag{27}$$

*The set of all transfer functions of order m or less is a 2 m dimensional nonlinear submanifold of the infinite dimensional space of all rational transfer functions.

Thus for a given set of model poles the minimum J is just the difference between the norm of the given system and the norm of the optimum model. This result is a direct consequence of the fact that $\hat{T}(s)$ is a projection of T(s). Since this relation holds for any given set of model poles it will hold for the optimum model poles.

Now return to the general problem, in which the poles of the model are not specified.

From (14) if $\hat{p}_i$ is real

$$\frac{\partial \hat{T}(s)}{\partial \hat{p}_i} = \frac{\hat{r}_i}{(s-\hat{p}_i)^2} \quad ; \tag{28}$$

hence by (20)

$$\frac{\partial J}{\partial \hat{p}_i} = 2\,\hat{r}_i \left[T(s) - \hat{T}(s),\ \frac{1}{(s-\hat{p}_i)^2} \right] . \tag{29}$$

In exactly the same way as was done when $\hat{r}_i$ was optimized it can be shown that (29) also holds when $\hat{p}_i$ is complex.

If $\hat{r}_i$ is zero then the model is of order less than m. Since the problem of finding a model of order m or less can be reduced to the problems of finding the best ℓth order model for all $\ell \le m$, the case $\hat{r}_i = 0$ will not be considered further. Thus in addition to satisfying (24), the $\hat{r}_i$ and $\hat{p}_i$ must be chosen so that

$$\boxed{\left[T(s)-\hat{T}(s),\ \frac{1}{(s-\hat{p}_i)^2} \right] = 0 \qquad i-1, \ldots, m} \ . \tag{30}$$

Because J is a non-quadratic function of $\hat{p}_i$ these conditions are not sufficient.

Consider the set A of all transfer functions which are linear combinations of

$$\frac{1}{s-\hat{p}_i} \qquad i = 1, \ldots, m \ , \tag{31}$$

and

$$\frac{1}{(s-\hat{p}_i)^2} \qquad i = 1, \ldots, m \ . \tag{32}$$

This set is clearly a subspace.

Now consider a pole near $\hat{p}_i$ given by $\hat{p}_i + \beta$:

$$\frac{1}{s-\hat{p}_i-\beta} \approx \frac{1}{s-\hat{p}_i} + \frac{\beta}{(s-\hat{p}_i)^2} \ , \tag{33}$$

and*

$$\left\| \frac{1}{s-\hat{p}_i-\beta} - \frac{1}{s-\hat{p}_i} - \frac{\beta}{(s-\hat{p}_i)^2} \right\| = o \left\| \frac{1}{s-\hat{p}_i-\beta} - \frac{1}{s-\hat{p}_i} \right\| \ . \tag{34}$$

Thus any transfer function $T_1(s)$ close to $\hat{T}(s)$ can be approximated by a transfer function $T_2(s)$ from the set A such that

$$\left\| T_2(s) - T_1(s) \right\| = o\left(\left\| T_1(s) - \hat{T}(s) \right\| \right) \ . \tag{35}$$

By analogy with ordinary space, the set A is just the tangent to the set of all transfer functions of order m at the transfer function of the optimum model. Hence the necessary conditions (24) and (30) may be restated in geometric terms as requiring that the difference between the transfer functions of the given system and model be perpendicular to the tangent to the submanifold of mth order transfer functions at transfer function of the model.

Now consider briefly what happens if the model has multiple poles. Suppose the model has a pole of order ℓ at p, then

$$\hat{T}(s) = \frac{c_1}{s-\hat{p}} + \frac{c_2}{(s-\hat{p})^2} + \ldots \frac{c_\ell}{(s-\hat{p})^\ell} + \ldots \tag{36}$$

Proceeding in the same manner as before, it is easy to show [6] that necessary and sufficient conditions on optimum c_i's for fixed p are

$$\left[T(s)-\hat{T}(s),\ \frac{1}{(s-\hat{p})^i} \right] = 0 \qquad i = 1, \ldots, \ell \ . \tag{37}$$

Similarly, a necessary condition that $\hat{p}$ be optimum is

$$\left[T(s)-\hat{T}(s),\ \frac{1}{(s-\hat{p})^{\ell+1}} \right] = 0 \ . \tag{38}$$

The $[\ ]_+$ Operator

In this discussion the necessary conditions just derived are restated in terms of Wiener Filter theory notation[7].

For a time function f(t) defined for all $-\infty < t < \infty$ with bilateral Laplace transform F(s), $f_+(t)$ is defined as

$$f_+(t) = f(t)\ u(t) \tag{39}$$

where u(t) is the unit step function. The notation $[F(s)]_+$ is used for the Laplace transform of $f_+(t)$. Application of the convolution rule to (39) yields for Re $[s] > 0$

$$[F(s)]_+ = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} F(s') \frac{1}{s-s'}\, ds' \ . \tag{40}$$

If F(s) is a rational function of s, $[F(s)]_+$ can be found by writing F(s) in a partial fraction expansion and dropping all terms corresponding to poles with positive real parts.

Consider first the necessary conditions given by (24); because the poles occur in conjugate pairs, $\hat{p}_i$ may be replaced by $\hat{p}_i^*$. If the inner product is written out as an integral, (24) becomes for p_i^* :

$$\frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \Phi_x(s') T(s') \left(\frac{1}{s'-\hat{p}_i^*} \right)^* ds'$$

$$= \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \Phi_x(s') \hat{T}(s') \left(\frac{1}{s'-\hat{p}_i^*} \right)^* ds' \tag{41}$$

or from (40) since along the $j\omega$ axis $s'^* = -s$ and since p_i has negative real part

*The statement y = o(x) implies $\lim_{x \to 0} \frac{y}{x} = 0$

$$[\Phi_x(-\hat{p}_i)\ T(-\hat{p}_i)]_+ = [\Phi_x(-\hat{p}_i)\hat{T}(-\hat{p}_i)]_+$$

$$i = 1, \ldots, m\ . \qquad (42)$$

Now turn to the necessary conditions given in (30). If the inner product is expressed in integral form, (30) becomes

$$\frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \Phi_x(s')T(s')\left[\frac{1}{(s'-\hat{p}_i^*)^2}\right]^* ds'$$

$$= \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \Phi_x(s')\hat{T}(s')\left[\frac{1}{(s'-\hat{p}_i^*)^2}\right]^* ds'\ . \qquad (43)$$

Differentiation of (40) yields

$$\frac{d}{ds}[F(s)]_+ = \frac{-1}{2\pi j}\int_{-j\infty}^{j\infty} F(s')\frac{1}{(s-s')^2}\, ds'\ . \qquad (44)$$

therefore, the necessary conditions can be written

$$[\Phi_x(-\hat{p}_i)T(-\hat{p}_i)]_+^{(j)} = [\Phi_x(-\hat{p}_i)\hat{T}(-\hat{p}_i)]_+^{(j)}$$

$$i = 1, \ldots, m \qquad (45)$$

$$j = 0, 1$$

where
$$F(a)^{(j)} \triangleq \frac{d^jF(s)}{ds^j}\Big|\ s = a \qquad (46)$$

for any function F(s).

In a similar manner we find that at an ℓth order pole p of T(s)

$$[\Phi_x(-\hat{p})T(-\hat{p})]_+^{(j)} = [\Phi_x(-\hat{p})\hat{T}(-\hat{p})]_+^{(j)}$$

$$j = 1, \ldots, \ell+1\ . \qquad (47)$$

These equations constitute a complete set of necessary conditions for the optimal approximating transfer function T(s). Although, the equations are nonlinear with respect to the optimal pole locations, they are expressed in a simple compact form.

A case of special interest is the case in which X(t) is white noise. In this case

$$\Phi_x(s) = 1 \qquad (48)$$

and

$$J = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}|T(s)-\hat{T}(s)|^2\, ds$$

$$= \int_0^\infty [w(t)-\hat{w}(t)]^2\, dt \qquad (49)$$

where w(t) and $\hat{w}(t)$ are the impulse responses of S and $\hat{S}$ respectively.

If $\Phi(s) = 1$ then

$$[\Phi(s)T(s)]_+ = [T(s)]_+ = T(s) \qquad (50)$$

hence in the white noise case the necessary conditions take the particularly simple compact form:

$$T(-\hat{p})^{(j)} = \hat{T}(-\hat{p})^{(j)} \qquad j = 1, \ldots, \ell+1\ . \qquad (51)$$

at each pole p of $\hat{T}(s)$ of order ℓ.

Numerical Solution and Examples

This section contains methods for finding a $\hat{T}(s)$ which satisfies the necessary conditions, and presents examples. The method to be used proceeds as follows: Equations (24), which are linear in the $\hat{r}_i$ are solved for the $\hat{r}_i$ in terms of the $\hat{p}_i$ and the solution substituted into (30) to give m nonlinear equations in the m unknowns $\hat{p}_i$. These equations are then solved numerically.

Substitution of (14) into (24) and (30) yields

$$[T(s), \frac{1}{s-\hat{p}_i}] = \sum_{j=1}^{m}\hat{r}_j\ (\frac{1}{s-\hat{p}_j}, \frac{1}{s-\hat{p}_i})$$

$$i = 1, \ldots, m, \qquad (52)$$

and

$$[T(s), \frac{1}{(s-\hat{p}_i)^2}] = \sum_{j=1}^{m}\hat{r}_j[\frac{1}{s-\hat{p}_j}, \frac{1}{(s-\hat{p}_i)^2}]$$

$$i = 1, \ldots, m\ . \qquad (53)$$

If the following definitions are made

$\underline{\hat{p}}$ = the vector with components $\hat{p}_i$,

$\underline{\hat{r}}$ = the vector with components $\hat{r}_i$,

$\underline{g}(\underline{\hat{p}})$ = the vector with components $[T(s), \frac{1}{s-\hat{p}_i}]$,

$\underline{g}'(\underline{\hat{p}})$ = the vector with components $[T(s), \frac{1}{(s-\hat{p}_i)^2}]$,

$M(\underline{\hat{p}})$ = the matrix with ijth element $(\frac{1}{s-\hat{p}_j}, \frac{1}{s-\hat{p}_i})$,

$N(p)$ = the matrix with ijth element

$$[\frac{1}{s-\hat{p}_j}\quad \frac{1}{(s-\hat{p}_i)^2}]\ ; \qquad (54)$$

then (52) and (53) become

$$\underline{g}(\underline{\hat{p}}) = M(\underline{\hat{p}})\ \underline{\hat{r}}\ , \qquad (55)$$

and

$$\underline{g}'(\underline{\hat{p}}) = N(\underline{\hat{p}})\underline{\hat{r}} \qquad (56)$$

Equation (55) can immediately be solved for $\underline{r}$ since $M(\underline{p})$ is invertible (for proof see reference 6).

$$\underline{\hat{r}} = M^{-1}(\underline{\hat{p}})\ \underline{g}(\underline{\hat{p}})\ . \qquad (57)$$

If this result is substituted into (55) then

$$g'(\underline{\hat{p}}) = N(\underline{\hat{p}})\ M^{-1}(\underline{\hat{p}})\ g(\underline{\hat{p}}) \qquad (58)$$

Equation (58) is a set of m non-linear algebraic equations in the m unknowns $\hat{p}_i$, which can be solved by a variety of numerical methods. Newton's method, for instance, has been used successfully on several examples [6]. An alternative iterative method of finding $\underline{\hat{r}}$ and $\underline{\hat{p}}$ may be obtained by modification of the work of McBride, et al[8], which is closely related to Mantey's work. By use of Mantey's method[2] or prony's method[9] good models may be obtained for initiating the iterative procedures for finding the optimum model. Experience indicates that the greatest numerical difficulties (for example multiple relative optima) occur when the optimum model for the given m is a poor model.

To illustrate how the numerical solution proceeds, some examples are presented. For simplicity, the case of approximating a second order system by a first order system will be considered; the random process X(t) is assumed white.

A general second order system (with no multiple poles) has transfer function

$$T(s) = \frac{r_1}{s-p_1} + \frac{r_2}{s-p_2} \quad . \tag{59}$$

From (11), since X is white,

$$\left(\frac{1}{s-a}, \frac{1}{s-b}\right) = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \left(\frac{1}{s-a}\right)\left(\frac{-1}{s+b}\right) ds = \frac{-1}{a+b} \quad . \tag{60}$$

Therefore, (since $\hat{S}$ is one dimensional)

$$\begin{aligned} \underline{g}(\hat{\underline{p}}) &= \left[T(s), \frac{1}{s-\hat{p}}\right] \\ &= r_1\left(\frac{1}{s-p_1}, \frac{1}{s-\hat{p}}\right) + r_2\left(\frac{1}{s-p_2}, \frac{1}{s-\hat{p}}\right) \\ &= -\frac{r_1}{\hat{p}+p_1} - \frac{r_2}{\hat{p}+p_2} \quad . \end{aligned} \tag{61}$$

and

$$\underline{M}(\underline{p}) = \left(\frac{1}{s-\hat{p}}, \frac{1}{s-\hat{p}}\right), = -\frac{1}{2\hat{p}} \quad . \tag{62}$$

Substitution of (61) and (62) into (55) yields

$$-\frac{r_1}{\hat{p}+p_1} - \frac{r_2}{\hat{p}+p_2} = -\frac{1}{2\hat{p}}\hat{r} \tag{63}$$

or

$$\hat{r} = \frac{2\hat{p}r_1}{\hat{p}+p_1} + \frac{2\hat{p}r_2}{\hat{p}+p_2} \quad . \tag{64}$$

Use of this result in (27) results in

$$\begin{aligned} J &= \|T(s)\|^2 - \|\hat{T}(s)\|^2 \\ &= \frac{r_1^2}{2p_1} + 2\frac{r_1 r_2}{p_1+p_2} + \frac{r_2^2}{2p_2} - 2\hat{p}\left(\frac{r_1}{p+p_1} + \frac{r^2}{\hat{p}+p_2}\right)^2 \quad . \end{aligned} \tag{65}$$

In Figure 4, $J/\|T(s)\|^2$ is plotted for several second order T(s).

The optimum value of $\hat{p}$ may be found graphically from Figure 4, or it may be found by numerical solution of (58). From (11)

$$\left[\frac{1}{s-a}, \frac{1}{(s-b)^2}\right] = \left[\frac{1}{a+b}\right]^2 \quad ; \tag{66}$$

thus from (54)

$$g'(\hat{\underline{p}}) = -\frac{r_1}{(\hat{p}+p_1)^2} + \frac{r_2}{(\hat{p}+p_2)^2} \quad , \tag{67}$$

$$\underline{N}(\hat{\underline{p}}) = \frac{1}{(2\hat{p})^2} \quad . \tag{68}$$

Therefore (58) becomes

$$\begin{aligned} \frac{r_1}{(\hat{p}+p_1)^2} + \frac{r_2}{(\hat{p}+p_2)^2} &= \frac{1}{(2\hat{p})^2}(-2\hat{p})\left(-\frac{r_1}{\hat{p}+p_1} - \frac{r_2}{\hat{p}+p_2}\right) \\ &= \frac{1}{2\hat{p}}\left(\frac{r_1}{\hat{p}+p_1} + \frac{r_2}{\hat{p}+p_2}\right) \quad , \end{aligned}$$

or

$$r_1\frac{\hat{p}-p_1}{(\hat{p}+p_1)^2} + r_2\frac{\hat{p}-p_2}{(\hat{p}+p_2)^2} = 0 \quad . \tag{69}$$

For the first example in Figure 4, (69) is

$$\frac{1}{9.9}\frac{(\hat{p}-0.1)}{(\hat{p}+0.1)^2} - \frac{1}{9.9}\frac{(\hat{p}-10)}{(\hat{p}+10)^2} = 0 \quad , \tag{70}$$

or

$$\begin{aligned} &(\hat{p}-0.1)(\hat{p}+10)^2 - (\hat{p}+0.1)^2(\hat{p}-10) = \\ &\hat{p}^3 + 19.9\hat{p}^2 + 98\hat{p} - 10 - \hat{p}^3 + 9.8\hat{p}^2 + 1.99\hat{p} + 0.1 = \\ &29.7(\hat{p}^2 + 3.36\hat{p} - 0.334) = 0 \quad . \end{aligned} \tag{71}$$

The roots of (71) are

$$\hat{p} = +0.1, -3.46 \quad . \tag{72}$$

The second root is discarded since it is negative and hence corresponds to an unstable system. A glance at Figure 4 will confirm this solution.

For examples presented in Figure 4 the best models are listed in Table I.

Table I Optimum Models

T(s)	T(s)	$J/\|T(s)\|^2$
$\frac{1}{(s+0.1)(s+10)}$	$\frac{-1}{s+0.1}$	0.02
$\frac{s+0.9}{(s+0.1)(s+10)}$	$\frac{0.9}{s+7}$	0.31
$\frac{s+0.2}{(s+0.1)(s+10)}$	$\frac{1}{s+10}$	0.25
$\frac{s-0.9}{(s+0.1)(s+10)}$	$\frac{0.9}{s+13.3}$	0.50

As a second example, consider the system with transfer function

$$T(s) = \frac{s+4}{(s+1)(s+3)(s+5)(s+10)} \tag{73}$$

Because of the zero at -4 and because -10 is fairly far from -3 it appears that a good model might be made by using a system with poles at -1 and -3. In fact these are the poles of the optimum second order model, which is given by

$$\hat{T}(s) = \frac{0.0042\ (s-19)}{(s+1)(s+3)} \tag{74}$$

The relative root mean square error for this model is

$$\sqrt{\bar{J}}/\|T(s)\| = 0.016 \tag{75}$$

Note that T(s) is approximately

$$\hat{T}(s) \approx T_1(s) = \frac{0.08}{(s+1)(s+3)} \tag{76}$$

but the relative root means square error for this model is

$$\sqrt{\bar{J}}/\|T(s)\| = 0.14 \tag{77}$$

The impulse responses corresponding to T(s), $\hat{T}(s)$, and $T_1(s)$ are shown in Figure 5.

Conclusions

The problem of modeling a given system by a system of lower dynamic order can be applied in order to simplify systems prior to analysis or to simplify designs. A special but important instance of this is the derivation of the Phillips filter by approximating the Wiener filter.

The approximation problem considered in this paper leads to a set of nonlinear algebraic equations, but when formulated in terms of the unknown pole locations these equations take a particularly simple form (at least conceptually). In the general case these conditions are stated in terms of the $[\ ]_+$ operator used in Wiener filter theory, and hence this formulation emphasizes the interconnection of modeling with Wiener filtering.

As is the case with Wiener filtering, the modeling problem is most naturally formulated as a problem in an inner product space but because of the nonlinearities inherent in this problem the optimal model is found by projecting onto a nonlinear submanifold of the space.

The nonlinear necessary conditions can be solved by a number of standard methods of numerical analysis. Engineering judgement is needed to select initial trial models which insure rapid convergence of these numerical procedures as well as to pick m and Φ_x so that the resulting optimal model is a reasonable approximation to the given system.

A major area for future investigation on this topic is the modeling of time varying and nonlinear systems. In such situations dynamic order alone is not a sufficient criterion of simplicity. Another topic for consideration is whether $m > \ell$ implies that the optimal model of order m is better than the optimal model of order ℓ. If a model of order $\ell < m$ is considered to be a degenerate model of order m then this is obviously true. This paper takes essentially the latter point of view in asking for the optimum model of order m or less, but solves the problem by finding the best (non degenerate) model for each fixed order less than or equal to m.

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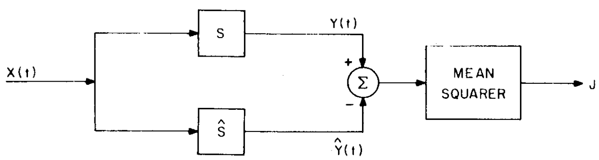

Fig. 1 Modeling Problem

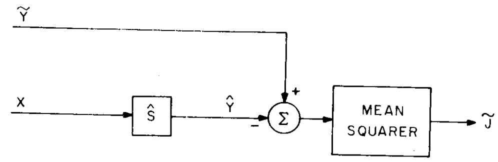

Fig. 2 Phillips-Filter Problem

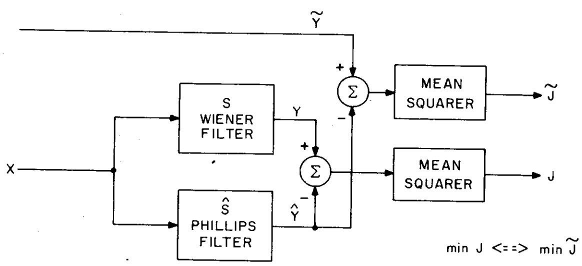

Fig. 3 Equivalence of Two Problems

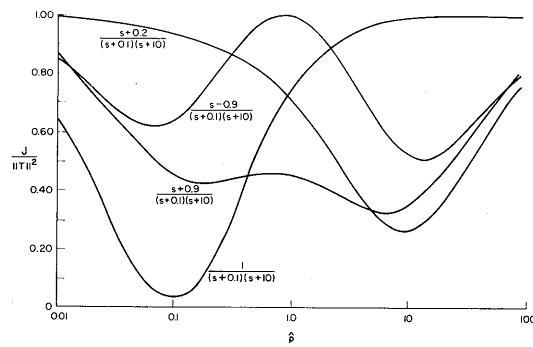

Fig. 4 First Order Examples

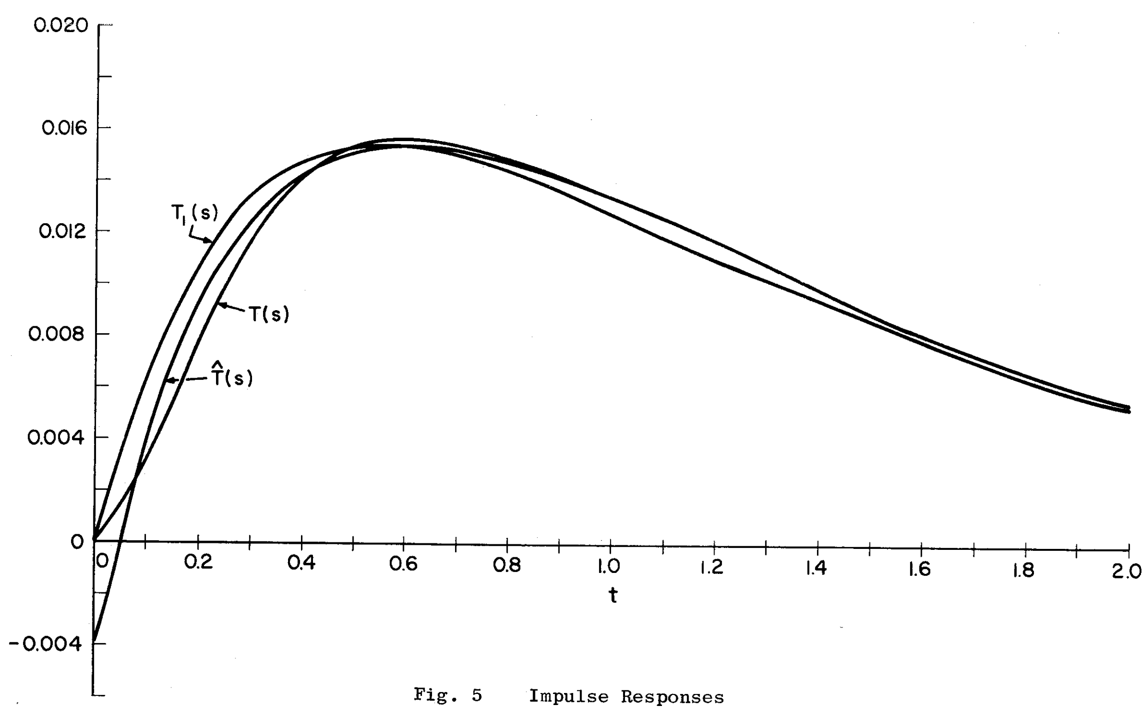

Fig. 5 Impulse Responses