

A Nonlinear Economic Control Problem with a Linear Feedback Solution

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Abstract—This paper discusses a special optimal control problem having a solution expressible in linear feedback form. The problem structure is characteristic of certain economic investment problems where the control influences the rate of deterioration of a capital good. An example of an economic problem of this type is the determination of the optimal maintenance policy for a housing unit when the effectiveness of maintenance is subject to decreasing returns to scale. Mathematically, the problem is characterized by a dynamic system that is nonlinear with respect to both the state and the control and by a linear objective function. Both the control and the state are restricted to be nonnegative and to lie below given upper bounds.

The problem possesses a unique global optimal solution expressible in linear feedback form. The associated control law is derived from the solution of an auxiliary ordinary differential equation that can be solved backward in time. In addition to the linear control law, the problem possesses other important linearity properties. The closed-loop system is linear; that is, although the system itself is nonlinear, when the optimal control law is substituted for the control, the resulting system is linear with respect to the state. And, the optimal return function is affine; that is, the optimal value that can be attained from a given point is an affine function of the value of the state at that time.

I. INTRODUCTION

IN A very real sense, the versatility and practicality of optimal control theory is not derived from any single result, but rather from the vast assortment of results and methods that potentially can be called upon for any new problem. In this respect, a very important component of the discipline is the collection of special optimal control problems that possess known solutions, expressed in some simple analytical form. This collection of solved problems often profoundly influences the way new problem situations are approached. For one thing, their simple solutions provide valuable insight into optimal control in general, indicating the type of behavior that is consistent with optimality. For another thing, these problems with their simple (and/or elegant) solutions provide tempting formats into which a problem formulator may attempt to squeeze a new problem. Therefore, these special problems enrich the art of modeling by providing an assortment of viable alternatives that can help guide the crucial modeling task of formulating the best compromise between comprehensiveness and tractability.

A new special optimal control problem is introduced in

this paper that has a linear feedback solution. The problem is suitable as a model of a class of economic problems. For example, as discussed in the next section of the paper, the problem originally arose in connection with finding economically efficient housing maintenance policies. Within the context of this paper, however, the particular original application is perhaps less significant than the abstract problem and its potential for generalization and further application. The unusually strong results that are derived for this problem suggest that it may be a useful addition to our collection of special problems.

The problem itself is characterized by a nonlinear dynamic system, a linear objective function, and upper and lower bounds on both the state and control variables. The solution can be expressed in feedback form where the control is a linear (or affine, depending on the exact definition of the state) function of the state. This feedback solution is obtained, in a manner analogous to that for the well-known linear-quadratic problem, in terms of the solution to an auxiliary ordinary differential equation with single-point boundary conditions.

In brief, the plan of the paper is as follows. In Section II the special problem is motivated in terms of an optimal maintenance problem, and then fully defined. In Section IV the actual task of solving the problem is carried out, and the structural property of the problem which leads to linear feedback control is made apparent. The method for computing the feedback law through solution of an auxiliary differential equation is also developed in that section. In the next section, Section IV, it is shown that the optimal return function of the problem is affine; that is, the optimal value that can be attained from any point is an affine function of the value of the state at that time. An alternative derivation of the solution, based on the Bellman-Hamilton-Jacobi theory, is also briefly presented in that section. In Section V the important special case corresponding to time-invariance is completely solved, with explicit algebraic formulas for the feedback law. Finally, in Section VI, extensions of the theory to other problems and general conclusions are discussed.

II. PROBLEM DESCRIPTION

The formulation of the problem posed in this paper was originally motivated by consideration of optimal maintenance policies for rental housing. Accordingly, the problem description of this section is presented in that context. The reader who is more interested in the problem

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structure itself than in its motivation can refer directly to (6) at the end of the section, where the complete formulation is summarized.

In the context of the motivating application, the problem takes the perspective of a landlord who wishes to maximize his long-term (discounted) net profit through careful property management. It is assumed that his only instrument of management is his maintenance policy, which affects both his short-term expenses (in the form of maintenance costs) and his long-term profit (in the form of increased rent from a better maintained unit). In reality, of course, there would be several other (even purely economic) aspects to the landlord's management problem, aside from maintenance. There would be the possibility of major alterations, tax considerations, uncertainty in prices, etc. These aspects are explicitly ignored (or their effects are assumed to be fixed) in order to obtain the simplest model capturing the essential features of the maintenance investment problem.

Even the maintenance investment problem alone is potentially very complex, since there are hundreds of individual items in a house that might require maintenance. To make the problem tractible, a simplifying concept, which is of common use in housing economics literature, of characterizing a housing unit by a quality level x is introduced. This level is imagined to capture and summarize a wide spectrum of specific features of the unit, which together comprise quality and determine the rent that the unit can command. Typically, the features that make up quality may include the size and design of the house, the condition of its various components, the age of appliances, its geographic location, etc. When a house of given design and location is new, or in top condition, it is characterized by a quality level of $x = X$, which is the highest quality it can achieve (without structural additions, which are not allowed in this framework). Without maintenance, the quality level would gradually decline through use and age. Maintenance, on the other hand, and this is where the control aspects enter, can increase the quality, or at least slow the rate of quality decline. The effect of maintenance expenditures, however, is subject to decreasing returns to scale, as discussed below.

The landlord's motivation for maintenance is, of course, the greater rent that a high-quality unit can command. But the additional income due to increased rent must be balanced against the additional expense for maintenance. The optimizing landlord will select his maintenance policy in such a way to maximize his discounted net profit, consisting of rent minus maintenance expense.

It is now time to turn to a specific formulation of this general problem. Assume that the quality level x is governed by a first-order differential equation of the form¹

$$\dot{x}(t) = -\beta(t)x(t) + g(x(t), u(t), t). \quad (1)$$

¹Throughout the paper the overdot means differentiation with respect to t , e.g., $\dot{x}(t) \equiv (dx(t)/dt)$.

The function g represents the effects of maintenance, and it is zero when $u(t)$ is zero. A specific form for g , which satisfies certain economic properties, will soon be proposed. The first term on the right side of this equation represents the normal decline that the unit would suffer in the absence of maintenance. If $\beta(t)$ were constant, $\beta(t) = \beta$, then the normal decline would be exponential (with a half life of $(\ln 2)/\beta$).

In addition to the differential equation, there are constraints on the allowable values of the variables x and u . These have the specific form

$$0 \leq x(t) \leq X \quad (2)$$

$$0 \leq u(t) \leq U(x(t)). \quad (3)$$

The bound (3) expresses the fact that the maximal allowable rate of maintenance may depend on the current level of quality. For instance, if $x(t) = X$, so that the unit is in top condition, it is reasonable to assume that no maintenance is possible. A specific form for U will be proposed as part of the complete problem formulation.

The conditions which seem appropriate to impose on the functions g and U for all x and u within their bounds are

- 1) $g(x, 0) = 0$
- 2) $g_u(x, u) > 0$
- 3) $g_x(x, u) < 0$
- 4) $g_{uu}(x, u) < 0$
- 5) $U_x(x) < 0$.

These conditions have the following economic interpretations: 1) With zero maintenance the quality is governed only by the linear term in (1). 2) Positive maintenance is always beneficial in that it decreases the rate of quality decline, and increasing the level of maintenance increases the effect of maintenance. 3) A given maintenance rate is less effective at high quality levels than at low ones. 4) The effectiveness of maintenance is subject to decreasing returns to scale. 5) The range of feasible maintenance rates decreases as the quality level moves closer to its upper limit.

Of these assumptions, perhaps 3) requires the most elaboration. Consider a brand new unit having quality $x = X$. After any short period without maintenance, the level will, according to our model, fall slightly below X . This corresponds, physically, to the fact the whole building (paint, appliances, roof, etc.) are all *slightly* aged. However, to restore the condition of the unit back to X would entail replacing all major items by new ones—a considerable expense for only a small improvement in quality. In this sense it is far more difficult to improve a house in good condition than one in poor condition. In terms of the maintenance function g , this is expressed by 3).

Now that the desirable properties of the functions g and U have been specified, it is appropriate to exhibit a specific combination of functions possessing these prop-

erties. One of the simplest pairs is

$$g(x, u, t) = u - \frac{1}{2} \frac{c(t)u^2}{X-x} \quad (4)$$

$$U(x, t) = \frac{X-x}{c(t)} \quad (5)$$

where $c(t) > 0$. Together these functions satisfy conditions 1)–5) in the region specified by (2), (3). (See Fig. 1, and Note 3 at the end of this section.) This particular combination is used throughout the remainder of the paper. Together (1)–(5) completely specify the dynamics and the constraints of the problem.

In addition to the dynamic equation and the constraints, the objective must be specified. Suppose that the owner of the unit is interested in optimizing his maintenance policy over the time interval $[0, T]$. During this period he collects rent, depending on the quality of the unit, and he pays maintenance costs, which depend on the level of maintenance. At the end of the period, he can sell the unit for a price that depends on the quality of the unit at that time. It is assumed that the rent obtainable from the unit at any time can be expressed in terms of a price per unit of quality; that is, the rate of rental income at time t is $p(t)x(t)$, where p is a specified function of time. The rate of maintenance expense is similarly assumed to be $s(t)u(t)$, where s is a specified function. Finally, the unit is sold at time T for a sum equal to $\bar{p}x(T) + \bar{q}$, where \bar{p} and \bar{q} are fixed nonnegative constants.

Together all the assumptions and notation developed in this section specify the following special optimal control problem. Given: 1) an initial state value $x(0)$, satisfying

$$0 \leq x(0) \leq X \quad (6a)$$

2) the dynamic system

$$\dot{x}(t) = -\beta(t)x(t) + u(t) - \frac{1}{2} \frac{c(t)u(t)^2}{X-x(t)} \quad (6b)$$

3) the constraints

$$\begin{aligned} 0 &\leq x(t) \leq X \\ 0 &\leq u(t) \leq U(x(t), t) \equiv [X-x(t)]/c(t) \end{aligned} \quad (6c)$$

find the function u on $[0, T]$ that maximizes the objective

$$J = \int_0^T [p(t)x(t) - s(t)u(t)] dt + \bar{p}x(T) + \bar{q}. \quad (6d)$$

A few final comments and technical assumptions are appropriate before turning to the detailed investigation of the problem.

Note 1: It is assumed that $c(t) > 0$, $s(t) > 0$, $p(t) > 0$, and $\beta(t) > 0$ for economic reality and for technical reasons. For convenience, all of these functions are assumed to be piecewise continuous.

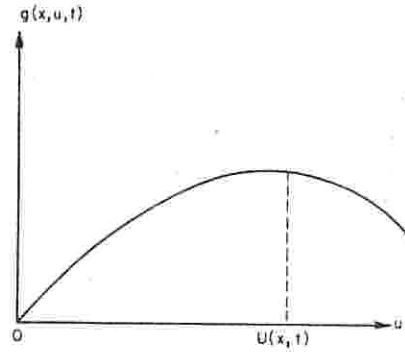


Fig. 1.

Note 2: For technical reasons it is assumed that $\beta(t)$ and $(1/c(t))$ are bounded on $[0, T]$.

Note 3: At first sight there may appear to be difficulty in the definition (4) of g when $x = X$. However, since $0 \leq u \leq (X-x)/c(t)$, it is consistent, by continuity, to simply define $g = 0$ when $x = X$.

Note 4: In an economic setting, income at future times would normally be discounted according to a discount factor r . This is implicitly accounted for in the formulation by assuming that the discounting is contained in the definition of $p(t)$, $s(t)$, \bar{p} , and \bar{q} . Indeed an important case is $p(t) = pe^{-rt}$, $s(t) = se^{-rt}$, $\bar{p} = Be^{-rT}$, $\bar{q} = Qe^{-rT}$.

Note 5: The term $\bar{p}x(T)$ represents the state-dependent part of the sales value of the unit at time T , and is the only part that can affect the solution. The \bar{q} term, although a triviality as far as defining the problem, is included in order to be consistent with the presentation in Section V.

Note 6: It is, in general, possible to introduce an additional positive parameter as a coefficient of g . By redefinition of u and s , however, the value of this parameter can always be taken as unity.

III. SOLUTION

The feedback solution to the optimal control problem can be derived by application of the Pontryagin maximum principle. Before writing down the necessary conditions, however, it is helpful to consider in some detail the role of the constraints (6c). The analysis of the control problem can be simplified by showing that most of these constraints are in fact redundant.

First, one can see that the constraint $u(t) \leq U(x(t), t)$, although important for definitional purposes, can never be a binding constraint for an optimal solution. The reason for this is that $g(x(t), u(t), t)$ is maximized with respect to $u(t)$ at $u(t) = U(x(t), t)$. Values of $u(t)$ greater than this maximum will only cost more and yield less in terms of improvement of x (through modification of \dot{x}). Second, it can be seen that the constraint $x(t) \geq 0$ is redundant. Since $\dot{x}(t) \geq -\beta(t)x(t)$ and $\beta(t)$ is bounded (say $\beta(t) \leq M$) for all $t \in [0, T]$, it follows that $\dot{x}(t) \geq -Mx(t)$ and hence that $x(t) \geq x(0)e^{-Mt} \geq 0$. Finally it can be seen that the constraint $x(t) \leq X$ is not binding. Since $u(t)$

$\leq U(x(t), t)$, it follows that

$$\dot{x}(t) \leq -\beta(t)x(t) + \frac{X - x(t)}{2c(t)} \tag{7}$$

or, equivalently, putting $y(t) = X - x(t)$,

$$\begin{aligned} \dot{y}(t) &\geq -\left[\beta(t) + \frac{1}{2c(t)}\right]y(t) + \beta(t)X \\ &\geq -\left[\beta(t) + \frac{1}{2c(t)}\right]y(t). \end{aligned} \tag{8}$$

Since $\beta(t)$ and $1/c(t)$ are assumed to be bounded, there holds, for some $N > 0$,

$$\dot{y}(t) \geq -Ny(t) \tag{9}$$

and thus $y(t) \geq 0$ for all t . By definition of $y(t)$, it then follows that $x(t) \leq X$. It can be concluded, therefore, on the basis of this subsidiary analysis, that the only constraint that may actually be binding is the constraint $u(t) \geq 0$. The other constraints, although useful in defining the problem, can be ignored in its analysis.

According to the Pontryagin maximum principle [1], there is associated with the optimal functions x and u an adjoint function λ on $[0, T]$ such that, together, the functions satisfy the three conditions:

$$1) \quad \dot{x}(t) = -\beta(t)x(t) + u(t) - \frac{1}{2} \frac{c(t)u(t)^2}{X - x(t)}, \tag{10a}$$

$x(0)$ specified as in (6a).

$$2) \quad -\dot{\lambda}(t) = -\beta(t)\lambda(t) - \frac{\lambda(t)c(t)u(t)^2}{2[X - x(t)]^2} + p(t),$$

$$\lambda(T) = \bar{p}. \tag{10b}$$

3) For each $t \in [0, T]$, $u(t)$ maximizes the (reduced) Hamiltonian

$$H(x(t), \lambda(t), u, t) = \lambda(t) \left[u - \frac{1}{2} \frac{c(t)u^2}{X - x(t)} \right] - s(t)u \tag{10c}$$

with respect to all $u \geq 0$.

Generally these three relations would be solved simultaneously to find x , λ , and u . The particular structure of the special problem, however, decouples 2) and 3) from 1).

It is helpful to first conduct a preliminary investigation of the adjoint equation (10b). It can be shown that $\lambda(t) \geq 0$ for all $t \in [0, T]$. Suppose to the contrary that $\lambda(t_1) < 0$ for some $t_1 \in [0, T]$. Then from (10b) and the fact that $\beta(t) > 0$, $c(t) > 0$, $p(t) > 0$, it follows that $\dot{\lambda}(t_1) < 0$. It then

follows that $\lambda(t) < 0$ for all $t > t_1$. In particular it follows that $\lambda(T) < 0$, which contradicts the boundary condition $\lambda(T) = \bar{p} \geq 0$. Therefore $\lambda(t) \geq 0$ for all $t \in [0, T]$.

Next it is appropriate to examine (10c). For $\lambda(t) \geq 0$, H is a differentiable concave function of u and hence the maximum can be found by setting the derivative equal to zero. If the maximum yields a negative value for u , then $u = 0$ is the nonnegative maximizer. Thus, one is led to the solution

$$u(t) = \begin{cases} 0, & \text{if } 0 < \lambda(t) \leq s(t) \\ \left[1 - \frac{s(t)}{\lambda(t)} \right] U(x(t), t), & \text{if } \lambda(t) > s(t). \end{cases} \tag{11}$$

Or, introducing the $+$ operation on real numbers defined by

$$y^+ = \begin{cases} y, & \text{if } y \geq 0 \\ 0, & \text{if } y < 0 \end{cases} \tag{12}$$

the solution is

$$u(t) = \left[1 - \frac{s(t)}{\lambda(t)} \right]^+ U(x(t), t). \tag{13}$$

The solution to (10c) can now be substituted into (10b), yielding the differential equation

$$-\dot{\lambda}(t) = -\beta(t)\lambda(t) - \left\{ \left[1 - \frac{s(t)}{\lambda(t)} \right]^+ \right\}^2 \frac{\lambda(t)}{2c(t)} + p(t)$$

$$\lambda(T) = \bar{p}. \tag{14}$$

This is a first-order differential equation for λ . It does not involve either u or x , and it can be solved by a single backward integration starting at $\lambda(T) = \bar{p}$. This is the decoupling property mentioned earlier. Equation (14) is independent of (10a).

It is appropriate to make a few comments with respect to the properties of the differential equation (14). Although the equation may at first appear to be highly nonlinear, it in fact behaves much like a linear equation. In fact it can be shown that a solution λ to (14) exists and is unique on $[0, T]$ (see [4]). Indeed, (14) can be solved backward from $t = T$ to yield a unique function λ with $\lambda(t) \geq 0$ for all t . This solution then yields the feedback control law as described below.

Main Result

Let $\lambda(t) \geq 0$, with $t \in [0, T]$, be the unique solution to the differential equation (14). Define

$$L(t) = \frac{1}{c(t)} \left[1 - \frac{s(t)}{\lambda(t)} \right]^+ \tag{15}$$

Then the optimal control for the special control problem (6) is

$$u(t) = L(t)(X - x(t)) \quad (16)$$

where $x(t)$ is the optimal value of the state.

A few observations can be made with respect to this result. First, as promised, the solution is expressed in feedback form. The value of the control at time t is expressed as a function of the state at that time, and since $L(t)$ is independent of the initial condition $x(0)$, this form is valid for all initial conditions. Second, the feedback control law is linear in $X - x(t)$. Since the state could have been defined as $X - x(t)$ rather than $x(t)$, one can say, somewhat loosely, that the control law is linear. Third, if the optimal control law is substituted into the original system, the system behaves according to

$$\dot{x}(t) = -\beta(t)x(t) + L(t)(1 - c(t)L(t)/2)[X - x(t)]$$

which is linear in x . Thus, the closed-loop system is linear. Other important linearity characteristics are examined in the next section.

One final issue that is considered here is the question as to whether $L(t)$ can equal zero. According to its definition (15) it is clear that $L(t)=0$ is not just a singular event, which may occur at isolated points t , but rather it is conceivable that $L(t)=0$ over significant intervals of t [over a period where $\lambda(t) \leq s(t)$]. An additional assumption can be imposed, however, which precludes this possibility, except near the terminal time. Specifically there is imposed the following assumption.

Nondegeneracy Assumption

The function s is differentiable, and

$$\dot{s}(t) - \beta(t)s(t) + p(t) \geq 0, \quad \text{for all } t \in [0, T]. \quad (17)$$

Under this assumption, suppose for some t that $\lambda(t) < s(t)$. It then follows that

$$-\dot{\lambda}(t) = -\beta(t)\lambda(t) + p(t) \quad (18)$$

and

$$\dot{s}(t) \geq \beta(t)s(t) - p(t). \quad (19)$$

Adding these two expressions one obtains

$$\dot{s}(t) - \dot{\lambda}(t) \geq \beta(t)[s(t) - \lambda(t)] \geq 0. \quad (20)$$

Hence, it can be concluded that $\lambda(t) < s(t)$ implies $\dot{\lambda}(t) < \dot{s}(t)$.

Suppose now that $\lambda(t_1) < s(t_1)$ for some $t_1 \in [0, T]$. Then it follows from the above result that $\lambda(t) < s(t)$ for all $t \geq t_1$. Therefore, with the nondegeneracy assumption, $\lambda(t) < s(t)$ can hold only over a final interval of the form $[t_1, T]$. It follows that $L(t)$ is also zero only on that interval. Finally, such an interval will not exist if $\bar{p} > s(T)$, since $\lambda(T) = \bar{p}$. Thus, in general with the nondegeneracy

assumption, the control is always positive, except possibly in an interval which includes $t = T$, and then only if \bar{p} is less than $s(T)$. In terms of the original economic problem, the nondegeneracy assumption essentially guarantees that a little maintenance is always cost-effective in the long run.

IV. SALES VALUE AND DYNAMIC PROGRAMMING (THE OPTIMAL RETURN FUNCTION)

The original problem formulation included a terminal time term in the objective function, representing the value at time T of owning a unit with state value $x(T)$. This term can be thought of as the salvage value, or the sales value, of the unit at time T . The overall objective is, accordingly, the sum of the profit derived from operation of the unit over the time interval $[0, T]$ plus the cash received for sale of the unit at time T . In the original formulation, the sales value at time T was specified to be affine in the state value, but this assumption has not yet been fully justified. In this section it is shown that the affine form for the sales value is consistent, in an economic sense, with the overall problem formulation.

To begin the development, it is necessary of course to explain what is meant by a consistent choice of the sales value function, for conceivably any arbitrary choice would be acceptable as long as it were "reasonable." To get at the notion of consistency, the concept of sales value is expanded to include values of t other than $t = T$. For this purpose, suppose the original owner operates the unit to a time $\tau < T$ and then considers the possibility of selling the unit at that time. How much would he be willing to accept? To answer this question, he would calculate the value of the unit to him at time τ as the maximum profit he could earn by operating the unit from τ to T and then selling it at time T . This calculation would produce a value $V(x(\tau), \tau)$ for the unit at time τ , if it had a state value $x(\tau)$. Specifically V would be calculated as

$$V(x(\tau), \tau) = \max_{u > 0} \int_{\tau}^T [p(t)x(t) - s(t)u(t)] dt + \bar{p}x(T) + \bar{q} \quad (21)$$

where for $t \geq \tau$, $x(t)$ satisfies

$$\dot{x}(t) = -\beta(t)x(t) + u(t) - \frac{1}{2} \frac{c(t)u(t)^2}{X - x(t)}. \quad (22)$$

Likewise, a prospective buyer at $t = \tau$, would (ideally) make the same calculation in order to determine how much he would be willing to pay for the unit having state value $x(\tau)$. Therefore, $V(x(\tau), \tau)$ is the implied sales value at $t = \tau$.

Through the above argument, a specified sales value function at $t = T$ can be used to deduce a sales value function V for all $\tau \leq T$. This function is the logical extension of the terminal sales value. If one makes an

argument in favor of a specific form for the sales value at some specific time (say T), then that argument should apply to other values of t as well. Thus, in order for an assumption of an affine terminal sales value to be credible, it should follow that $V(x(\tau), \tau)$ is affine in $x(\tau)$ for all $\tau \in [0, T]$. This is indeed the case for the special problem of this paper. Since the control problem has the solution

$$u(t) = L(t)(X - x(t)) \quad (23)$$

one can write explicitly

$$V(x(\tau), \tau) = \int_{\tau}^T [p(t)x(t) - s(t)L(t)(X - x(t))] dt + \bar{p}x(T) + \bar{q} \quad (24)$$

where

$$\dot{x}(t) = -\beta(t)x(t) + L(t)(1 - c(t)L(t)/2)[X - x(t)]. \quad (25)$$

Then, since all expressions are affine in $x(t)$, it can be concluded that V has the form

$$V(x(\tau), \tau) = \bar{p}(\tau)x(\tau) + q(\tau) \quad (26)$$

with $\bar{p}(\tau)$ and $q(\tau)$ determined implicitly by (24) and (25).

Apart from a justification that a reasonable form for the sales value at $t = T$ is affine in $x(T)$, consideration of the function V leads to a dynamic programming attack on the original problem. Following this through verifies the global optimality of the solution derived in the previous section and leads to the determination of $\bar{p}(\tau)$ and $q(\tau)$.

The fundamental equation for continuous dynamic programming, the Bellman-Hamilton-Jacobi equation, is the partial differential equation

$$\max_{u \geq 0} \{ V_x(x(t), t)\dot{x}(t) + V_t(x(t), t) + p(t)x(t) - s(t)u(t) \} = 0 \quad (27)$$

$$V(x(T), T) = \bar{p}x(T) + \bar{q}.$$

Within the context of dynamic programming, V is referred to as the *optimal return function*, since it gives the maximum value of the objective starting at $t = \tau$. If (27) has a solution V and a corresponding control law $u(x(t), t)$ which is the unique maximizer in (27) for every $x(t)$, $0 \leq x(t) \leq X$, then V is in fact equal to the V defined by (24), and $u(x(t), t)$ is the optimal control expressed in feedback form [5]. For the problem posed here, (27) has a solution of the form

$$V(x(t), t) = \lambda(t)x(t) + q(t) \quad (28)$$

with $\lambda(t) \geq 0$. Indeed, substituting this form in (27) and substituting (22) for $\dot{x}(t)$, the maximization in (27) reduces to exactly the maximization of (10c). After the maximiza-

tion, (27) reduces to

$$\left\{ -\beta(t)\lambda(t) - \left\{ [1 - s(t)/\lambda(t)]^+ \right\}^2 \frac{\lambda(t)}{2c(t)} + p(t) + \dot{\lambda}(t) \right\} x(t) + \dot{q}(t) + \left\{ [1 - s(t)/\lambda(t)]^+ \right\}^2 \frac{\lambda(t)}{2c(t)} X = 0 \quad (29)$$

$$\lambda(T) = \bar{p}$$

$$q(T) = \bar{q}. \quad (30)$$

If λ satisfies the adjoint equation (14), and if q satisfies

$$\dot{q}(t) = - \left\{ [1 - s(t)/\lambda(t)]^+ \right\}^2 \frac{\lambda(t)}{2c(t)} X \quad (31)$$

or, equivalently,

$$\dot{q}(t) = - \frac{c(t)L(t)^2\lambda(t)}{2} X \quad (32)$$

with $q(T) = \bar{q}$, then (27) is satisfied. Therefore, an explicit expression for the optimal return (or sales value) is

$$V(x(t), t) = \lambda(t)x(t) + q(t) \quad (33)$$

where $\lambda(t)$ is found by backward integration of (14), and then $q(t)$ is found by backward integration of (32). The feedback solution (16) is the unique maximizer in the Bellman-Hamilton-Jacobi equation and is therefore optimal.

V. THE TIME-INVARIANT CASE

An important case of the general problem considered in this paper is that where the dynamic system and objective are defined by

$$\dot{x}(t) = -\beta x(t) + u(t) - \frac{1}{2} \frac{cu(t)^2}{X - x(t)} \quad (34a)$$

$$J = \int_0^{\infty} [px(t) - su(t)] e^{-rt} dt. \quad (34b)$$

This represents a maintenance planning problem over an infinite time-horizon. The dynamic properties of the unit are time-invariant, the prices for housing service and for maintenance are constant, and current values are computed according to a constant discount factor $r > 0$. For simplicity, the parameters are assumed to satisfy the non-degeneracy assumption, which in this case reduces to

$$p \geq s(r + \beta). \quad (35)$$

Problems with an infinite time-horizon such as (34) can sometimes lead to technical difficulties associated with the boundary conditions of the adjoint equation. In this par-

ticular instance it is possible to side-step these difficulties through the artifice of a special limit procedure. Specifically, (34b) is replaced by

$$J = \int_0^T [px(t) - su(t)]e^{-rt} dt + Be^{-rT}x(T) + Qe^{-rT}. \quad (36)$$

B and Q are fixed nonnegative numbers that will be specified later. One can then consider (34b) to be the limit as $T \rightarrow \infty$ of (36). Indeed, since $0 \leq x(T) \leq X$, the second two terms on the right side of (36) become arbitrarily small as $T \rightarrow \infty$.

For the finite-time objective (36) the solution found in Section III is applicable. In view of the nondegeneracy assumption (35), it will be true that $\lambda(t) \geq se^{-rt}$ for all t , $0 < t < T$, so (14) for $\lambda(t)$ becomes

$$\begin{aligned} -\dot{\lambda}(t) &= -\beta\lambda(t) - [1 - se^{-rt}/\lambda(t)]^2 \frac{\lambda(t)}{2c} + pe^{-rt} \\ \lambda(T) &= B. \end{aligned} \quad (37)$$

One can show that, for some choice of $B > 0$, (37) has a solution of the form: $\lambda(t) = Be^{-rt}$. When this form of solution is substituted in (37) every term reduces to a constant times e^{-rt} . Indeed, executing this substitution and multiplying by e^{rt} yields

$$Br = -B\beta - [1 - s/B]^2 \frac{B}{2c} + p$$

or, equivalently,

$$\left(r + \beta + \frac{1}{2c}\right)B^2 - \left(\frac{s}{c} + p\right)B + \frac{s^2}{2c} = 0. \quad (38)$$

This quadratic equation has the two solutions

$$B = \frac{(p + s/c) \pm \sqrt{(p + s/c)^2 - 2\frac{s^2}{c}\left(r + \beta + \frac{1}{2c}\right)}}{2\left(r + \beta + \frac{1}{2c}\right)} \quad (39)$$

or, equivalently,

$$B = \frac{(p + s/c) \pm \sqrt{p^2 + \frac{2s}{c}(p - (r + \beta)s)}}{2\left(r + \beta + \frac{1}{2c}\right)}. \quad (40)$$

One must determine which of the solutions (40) is most appropriate, and show that it is nonnegative. The nondegeneracy assumption (35) guarantees that the term under the radical sign is positive, implying that the two solutions for B are real. For the solution corresponding to the minus sign in front of the radical in (40) there follows

$$\begin{aligned} B &\leq \frac{(p + s/c) - \sqrt{p^2}}{2\left(r + \beta + \frac{1}{2c}\right)} = \frac{s/c}{2\left(r + \beta + \frac{1}{2c}\right)} \\ &= \frac{s}{2rc + 2\beta c + 1} < s. \end{aligned}$$

Thus, this choice would imply $\lambda(t) < se^{-rt}$ which is not consistent with the assumption that led to (37). On the other hand, taking the solution corresponding to the plus sign yields

$$\begin{aligned} B &\geq \frac{2p + s/c}{2\left(r + \beta + \frac{1}{2c}\right)} = \frac{s(2p + s/c)}{2s(r + \beta) + s/c} \\ &\geq \frac{s[2s(r + \beta) + s/c]}{2s(r + \beta) + s/c} = s. \end{aligned}$$

Thus, this choice implies $\lambda(t) \geq se^{-rt}$ which is consistent with all earlier assumptions.

The relationship between the problem with objective (36) and that with (34b) can now be made explicit. Start by defining B in (36) by

$$B = \frac{(p + s/c) + \sqrt{p^2 + \frac{2s}{c}(p - (r + \beta)s)}}{2\left(r + \beta + \frac{1}{2c}\right)}. \quad (41)$$

Then from the analysis above it follows that

$$\lambda(t) = Be^{-rt} \quad (42)$$

is the unique solution to (36), for any $T > 0$ (and for any $Q \geq 0$). Then defining

$$L = (1 - s/B)/c \quad (43)$$

it can be concluded that

$$u(t) = L(X - x(t)) \quad (44)$$

is the corresponding optimal control law on the interval $[0, T]$. It is time-invariant.

Define Q in (36) by

$$Q = \frac{L^2 X B c}{2r}. \quad (45)$$

It then follows, from the general results (32), (33) that in this case

$$V(x(t), t) = Be^{-rt}x(t) + Qe^{-rt} \quad (46)$$

an expression, again, which is valid for the problem with objective (36) for any $T > 0$.

Since the solution (44) and optima¹ return function (46)

are independent of T , it follows that they apply to the infinite time-horizon problem (34) as well. The optimal closed-loop system is governed by

$$\dot{x}(t) = -\beta x(t) + \left(L - \frac{c}{2}L^2\right)[X - x(t)]. \quad (47)$$

By setting $\dot{x}(t) = 0$, it can be calculated that the state will asymptotically approach the value

$$\bar{x} = \frac{\left(L - \frac{c}{2}L^2\right)X}{L - \frac{c}{2}L^2 + \beta}. \quad (48)$$

The advantage of this sort of analysis, of course, is that what would otherwise involve manipulation of time functions is essentially reduced to the algebraic manipulation of coefficients. This can have important implications for the foundation of an empirical theory which seeks to infer characteristics of the problem structure from observed behavior.

VI. EXTENSIONS AND CONCLUSIONS

Many of the results presented in this paper are not limited to the single variable, continuous-time special problem described, but in fact can be generalized in several directions. One of the most obvious, and important, extensions is to the discrete-time version of the problem. Virtually everything presented in this paper has a natural analog for the problem of the form

$$x(k+1) = \phi(k)x(k) + u(k) - \frac{1}{2} \frac{c(k)u(k)^2}{X - x(k)}, \quad (49a)$$

$$0 \leq x(k) \leq X$$

$$0 \leq u(k) \leq \frac{1}{c(k)} [X - x(k)] \quad (49b)$$

$$J = \sum_{k=0}^N p(k)x(k) - r(k)u(k) + \bar{p}x(N+1) + \bar{q}. \quad (49c)$$

Paralleling the continuous-time version, one requires $\phi(k) > 0$, $c(k) > 0$, $p(k) > 0$, $r(k) > 0$, $\bar{p} > 0$. In this case, however, an additional assumption must be invoked in order to insure that $x(k) \leq X$ automatically, so that this constraint will never be binding. A sufficient assumption of this kind is $\phi(k) < 1$, $c(k) > \frac{1}{2}$.

The class of problems introduced in this paper appears to offer a potentially attractive format for modeling a diversity of economic problems. Any problem characterized by positivity constraints, a linear objective function (derived perhaps from a price), and decreasing

returns with respect to control effectiveness might be appropriately approximated by the special structure presented here. A model of this form would then lead to a linear control law that could be easily implemented. Or, conversely, in the time-invariant case, the explicit relationship between the problem parameters and the solution would allow one to infer the value of unknown parameters from observation of behavior.

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