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AN EFFICIENCY HIERARCHY FOR CONSTRAINED ECONOMIES*

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This paper extends the efficiency theory for contrained economies. We augment the usual first and second-best dichotomy with a new type of efficiency, called hybrid efficiency. First-best efficiency has a useful interpretation with respect to a hypothetical planner. Second-best efficiency accounts for the equilibria that can be attained by the actual allocation mechanism. Hybrid efficiency combines these features. The three efficiency criteria form a hierarchy, as hybrid efficiency is stronger than second-best efficiency but weaker than first-best efficiency. Hybrid efficiency yields a precise interpretation of the equal welfare weight condition found in first-order analysis of constrained economies.

I. INTRODUCTION

In competitive (Arrow-Debrue) economies, equilibrium allocations are first-best efficient. In other words, the competitive allocation mechanism fully utilizes the resources and production technology available in the economy. Many economies, however, do not satisfy the assumptions of the competitive model. These economies may have institutional constraints, non-competitive agents, missing markets, or some other complication. Furthermore, the government may influence the allocation mechanism by selecting a vector of controls – for example, prices and commodity taxes – that firms and consumers use in their decisionmaking. We use the general term constrained economies to refer to economies that contain one or more of these deviations from the competitive model. An example is an economy in which the government collects sales taxes and uses the revenue to provide public goods. Another example is an economy in which profit from government production is constrained to be zero.

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Equilibrium allocations in constrained economies will generally not be first-best efficient. The allocation mechanism does not fully utilize the resources and production technology available in the economy. Although there exist other allocations that are preferred by everyone in the economy to a given equilibrium allocation, such allocations cannot be reached by the allocation mechanism. Since first-best efficiency is unattainable, Boiteux (1956) and Guesnerie (1979) define a weaker efficiency criterion for constrained economies called second-best efficiency. Second-best efficiency considers only equilibrium allocations. Hence second-best efficiency is based on allocations that can actually be reached by the allocation mechanism.

A comparison of the following two examples suggests that effiency in constrained economies requires a richer characterization than the traditional first and second-best dichotomy. To simplify the discussion, we assume in these examples that the equilibrium in the economy is a smooth function of the government controls. First, consider an otherwise competitive economy where the government collects commodity taxes and returns a fixed fraction of the tax revenue to each consumer in a lump-sum fashion. If the government selects a zero tax vector, then there is no tax revenue, and the economy satisfies all the requirements of a competitive economy. Hence the equilibrium corresponding to a zero tax vector is first-best efficient. It is also second-best efficient, because no other equilibrium allocation can make one consumer better off and no other consumer worse off. On the other hand, since any non-zero tax vector creates deadweight loss, the equilibria corresponding to a non-zero tax vector cannot be first-best efficient. Some of these equilibria are second-best efficient but others are not. In particular, very large tax vectors create so much deadweight loss that all consumers would be better off if taxes were reduced. The government may find one of the second-best efficient equilibrium allocations to be more equitable than the zero tax equilibrium allocation. There is a trade-off between equity and first-best efficiency.

Now consider an economy where the government provides public goods and pays for them by collecting commodity taxes. While some equilibrium allocations will be second-best efficient, none will be first-best efficient. (In this economy the zero tax vector does not lead to a first-best equilibrium allocation because none of the public good is provided.) There is, however, a trade-off between equity and "efficiency". This trade-off is usually viewed through the lens of first-order analysis. Suppose that the government selects a tax vector such that the resulting equilibrium maximizes a social welfare function. The first-order necessary conditions for this problem, often called optimal tax rules, are first-order necessary conditions for second-best efficiency. Following Starrett (1988), one can define an efficiency concept by setting all welfare weights[1] in the optimal tax rules equal to a common value. This type of efficiency

[1] Consumer i's welfare weight is equal to the product of the marginal social welfare of her utility and the Lagrange multiplier of her budget constraint.

requires that taxes satisfy a set of "inverse-elasticity" rules; goods with low elasticity of demand should be heavily taxed. The government may want to relax the inverse-elasticity rules to attain a more equitable second-best efficient equilibrium allocation.

These examples suggest that a trade-off between equity and *some* type of efficiency is a general phenomenon for constrained economies. Furthermore, the relevant efficiency concept is neither first-best nor second-best, but at times may intersect with both (as in the first example) or second-best only (as in the second). The second example illustrates a likely form for the first-order necessary conditions for this type of efficiency in differentiable economies. This paper rigorously defines and characterizes a new efficiency criterion for constrained economies, called *hybrid efficiency* that does indeed have these properties.

To fully understand the differences between the three efficiency criteria, we must go back to the primitive concept of a feasible set of allocations of goods to individuals, for it is the comparison of the elements in such sets that forms the basis for efficiency concepts. First-best efficiency is defined with respect to the most inclusive feasible set for the economy, which we call the first-best feasible set. The first-best feasible set is constructed from the consumption possibility sets, the resource endowment, and the production technology in the economy. For convenience, the resource endowment and production technology are often combined into a single entity called the transformation set. An element of the first-best feasible set is formed by selecting an aggregate bundle of goods from the transformation set and then dividing this bundle into individual bundles for each consumer in a manner that is consistent with the consumption possibility sets. The collection of all such allocations yields the first-best feasible set. We say an allocation is first-best efficient if it is Pareto efficient with respect to the first-best feasible set.

This definition enables us to give a precise interpretation to the observation that when an equilibrium allocation is first-best efficient, then the allocation mechanism is fully utilizing the resources and production technology of the economy. Consider a *hypothetical* allocation mechanism directed by an omniscient planner who has full control over the transformation of resources and distribution of goods to consumers. The hypothetical planner can direct the economy to any allocation in the first-best feasible set. When an equilibrium allocation is first-best efficient, the actual allocation mechanism is doing as well as the hypothetical one. Even the omniscient planner cannot devise another way to transform the resources in the economy and allocate the results to consumers in such a way that one consumer is better off and no other consumer is worse off.

First-best efficiency is well-defined for both competitive and constrained economies. The allocation mechanism in the latter, of course, is not likely to attain first-best efficient allocations. One must employ a weaker criteria to evaluate the equilibria in constrained economies. Boiteux (1956) and Guesnerie (1979) define second-best efficiency with respect to another feasible set, which we naturally refer to as the second-best feasible set. This set contains all equilibrium allocations. The second-best feasible set is a subset of the first-best feasible set. (The hypothetical planner can always duplicate the actual transformation and distribution performed by the constrained economy allocation mechanism.) We say an allocation is second-best efficient if it is Pareto efficient with respect to the second-best feasible set. Clearly second-best efficiency has operational significance for all constrained economies. At least one of the equilibrium allocations must be Pareto efficient with respect to the set of all equilibrium allocations. Like first-best efficiency, second-best efficiency is well-defined for both competitive and constrained economies. (In competitive economies the distinction between the two efficiency criteria is trivial because equilibrium allocations are both first and second-best efficient.) Second-best efficiency differs from first-best efficiency, however, in that the second-best feasible set is not constructed from a transformation set. Hence second-best efficiency does not have an alternative interpretation with respect to a hypothetical planner that controls transformation and distribution. Second-best efficiency only considers those allocations that can be attained by the *actual* allocation mechanism.

As the name implies, hybrid efficiency combines the salient characteristics of both first and second-best efficiency. First-best efficiency is based on a feasible set with an underlying transformation set and thus has an alternative interpretation with respect to a hypothetical planner. Second-best efficiency accounts for the equilibria that can be attained by the actual allocation mechanism. Correspondingly, we define a hybrid feasible set based on a *constrained* transformation set. The constrained transformation set is the collection of aggregate bundles of goods that correspond to some equilibrium of the actual allocation mechanism. We say an equilibrium allocation is hybrid efficient if it is Pareto efficient with respect to the hybrid feasible set. We can interpret the hybrid feasible set as the set of allocations that can be attained by a *constrained* hypothetical planner. The constrained planner has full control over distribution but must select an aggregate bundle of goods from the constrained transformation set.

The three feasible sets form a hierarchy; any allocation in the second-best feasible set is an element in the hybrid feasible set, and any allocation in the hybrid feasible set is an element of the first-best feasible set. Hence the three efficiency criteria also form a hierarchy. Second-best efficiency is the lowest member of the hierarchy. Some second-best efficient allocations may be hybrid efficient. And some hybrid efficient allocations may be first-best efficient. Like first-best and second-best, hybrid efficiency is well-defined for both competitive and constrained economies. In competitive economies, of course, the distinction between the three efficiency criteria is trivial. For example, in the first economy described above, the zero tax equilibrium allocation is second-best efficient, hybrid efficient, and first-best efficient. Of course, in most constrained economies, equilibrium allocations will not be first-best efficient, and we will primarily be interested in identifying which equilibrium allocations are hybrid efficient.

In the following sections, we formally define hybrid efficiency and compare it to first-best and second-best efficiency. We give a detailed numerical example of a constrained economy in which we identify the hybrid efficient equilibrium allocation. We also characterize hybrid efficiency by describing its relationship to Lindahl equilibria in an economy with a decentralized allocation system. Finally, we relate hybrid efficiency to an optimization principle based on the benefit function. This principle yields first-order necessary conditions for hybrid efficiency. These conditions are indeed equal to the first-order nesessary conditions for second-best efficiency when the welfare weights are equal. This result verifies, in a precise way, the intuition that the equal welfare weights condition indicates "efficiency".

Thus constrained economies do indeed contain a general trade-off between hybrid efficiency and equity. Once we identify hybrid efficient equilibria, we can then specify this trade-off. Once again consider the two examples discussed above. In the first example, the zero tax vector leads to a hybrid efficient equilibrium allocation. Other equilibrium allocations are not hybrid efficient, and deviations from the zero tax vector must be justified on equity grounds. Likewise, in the second example, a tax vector that satisfies the inverse-elasticity rules leads to a hybrid efficient equilibrium allocation. Other tax vectors do not lead to hybrid efficient allocations, although they may be desirable for equity reasons.

II. THE EFFICIENCY HIERARCHY

Consider a constrained economy with m private goods and m_p public goods. There are n consumers, k firms, and one government agent. Each consumer i has a consumption possibility set $\mathcal{X}_i = \mathcal{R}_+^{m_p+m}$ and a preference ordering on $\mathcal{X}_i$ represented by a continuous utility function u_i. An *allocation* is a vector $X = (x_0, x_1, x_2, ..., x_n)$ where x_0 is a vector of public goods, x_i is a vector of private goods for consumer i, and $(x_0, x_i) \in \mathcal{X}_i$ for each i. Private firms do not produce public goods. For notational simplicity, we assume that public goods are not needed as inputs for production of any goods. Each firm j has a production possibility set $\mathcal{Y}_j \subset \mathcal{R}^{m_p+m}$ where $\mathcal{Y}_j = (0, \tilde{\mathcal{Y}}_j)$ and $\tilde{\mathcal{Y}}_j \subset \mathcal{R}^m$. A *production plan* is a vector $Y = (y_1, y_2, ..., y_k)$ where $y_j \in \mathcal{Y}$ for every j. The aggregate private production set is $\mathcal{Y} = \sum_{j=1}^{k} \mathcal{Y}_j$. There is also a public production possibility set $\mathcal{Z} \subset \mathcal{R}^{m_p+m}$. The total resource endowment of the economy is $(0, w)$ where w is a strictly positive vector of private goods. We will frequently utilize a reference bundle of private goods, denoted by $g = (0, \tilde{g}) \in \mathcal{R}^{m_p+m}$ where $\tilde{g} \in \mathcal{R}_+^m$ and $\tilde{g} \neq 0$.

We now specify the allocation mechanism. The government selects a vector of controls (c, z), where z is a public production plan chosen from the public production set, and $c \in \mathcal{C} \subset \mathcal{R}^l$ contains both private goods prices and policy variables such as commodity taxes and resource endowments for consumers. The fact that the price vector is an element under the control of government deserves further explanation. We might normally expect that prices are influenced by the actions of firms and consumers, the resource endowment, and the policy choices of the government. The equilibrium values of such prices are often considered to be determined by an artificial "market maker"; but to make the model as general as possible, we allow for the possibility the government may directly control some of the prices in the economy. Thus it is convenient to assign the government a real role as policy setter and an artificial role as market maker for prices that are not explicit policy variables. Treating government in this way keeps the model compact and allows efficiency criteria to be easily defined.

Consumers maximize utility in a manner consistent with the controls of the government. Let $\eta_i(c, z) \subset \mathcal{X}_i$ contain all solutions to consumer i's utility maximization problem when government controls are (c, z). Firms also respond to government controls according to behavioral choice rules. Let $\psi_j(c, z) \subset \mathcal{Y}_i$ represent the set of netput vectors consistent with firm j's choice rules and the government control vector (c, z). Profit maximization with respect to prices is a special case of this formulation.

In many constrained economies, the government's use of controls is limited by one or more institutional constraints. Let $\mathcal{P} \subset \mathcal{R}^\alpha$ be a fixed set that describes the constraints. An allocation, production plan, and government control vector satisfy the institutional constraints if $(c, z, X, Y) \in \mathcal{P}$. An example, mentioned earlier, is an economy where public production must break even. Other constraints may limit the government's ability to manipulate the consumers' income – for example requiring that consumers' resource endowments and firm ownership shares remain fixed at exogenous levels. Our model is similar to Guesnerie's [1979] model. Using his terminology, we have a single controlled firm (the public firm) and no controlled consumers.

An equilibrium is a rest point of the constrained economy allocation mechanism.

Definition. A vector $q = (c, z, X, Y)$ (where (c, z) is a government control vector, X is an allocation, and Y is a production plan) is an *equilibrium* of the constrained economy if:

$$\begin{aligned} (x_0, x_i) &\in \eta_i(c, z) \quad \text{for every } i = 1, 2, ..., n, \\ y_i &\in \psi_j(c, z) \quad \text{for every } j = 1, 2, ..., k, \\ (x_0, \sum_{i=1}^{n} x_i) &= \sum_{i=1}^{n} y_j + z + (0, w), \end{aligned}$$
$$(c, z, X, Y) \in \mathcal{P}, \quad \text{and} \quad z \in \mathcal{Z}.$$

We adopt a notational convention to refer to various components of an equilibrium vector. For an equilibrium q, the corresponding allocation is signified by $X(q)$, the private production plan by $Y(q)$, the public production plan by $z(q)$, and the other controls by $c(q)$. In addition, an equilibrium allocation $X(q)$ has an associated utility profile $U(q^*) = [u_1(q^*), u_2(q^*), ..., u_n(q^*)]$ where $u_i(q^*) = u_i[x_0(q^*), x_i(q^*)]$. Let

Q denote the set that contains all equilibria of the economy. We assume for simplicity that for each consumer i the allocation $[x_0(q), x_i(q)]$ is in the interior of $\mathcal{X}_i$ for all $q \in Q$.

We now turn to the efficiency properties of the equilibria. Pareto efficiency is a fundamental concept in this discussion.

Definition. Consider a feasible set of allocation $\mathcal{F}$. An allocation $X^* \in \mathcal{F}$ is *Pareto efficient with respect to* $\mathcal{F}$ if there is no other allocation $X \in \mathcal{F}$ such that $u_i(x_0, x_i) \geq u_i(x_0^*, x_i^*)$ for all $i = 1, 2, ..., n$, with at least one of these inequalities being strict.

It is natural to characterize efficiency using sets defined in the utility space. Consider the set of all utility profiles $U(X) = (u_1(x_0, x_1), u_2(x_0, x_2), ..., u_n(x_0, x_n))$ corresponding to allocations in a feasible set $\mathcal{F}$. This set is called the *utility possibility set,* denoted by $U(\mathcal{F})$. Suppose an allocation X^* is Pareto efficient with respect to $\mathcal{F}$. Then the utility profile $U(X^*)$ is on the outer boundary, or frontier, of the utility possibility set $\mathcal{U}(\mathcal{F})$.

The first-best feasible set for the constrained economy is formed from the consumption possibility sets, the resource endowment, and the production technology in the economy. First we combine the latter elements into the first-best transformation set

$$\mathcal{T}_1 = \mathcal{Y} + \mathcal{Z} + (0, w).$$

Then we incorporate the consumption possibility sets. We have

$$\mathcal{F}_1 = \{X : (x_0, \sum_{i=1}^{n} x_i) \in \mathcal{T}_1, (x_0, x_i) \in \mathcal{X}_i \text{ for every } i\}.$$

An allocation $X \in \mathcal{F}_1$ is *first-best efficient* if it is Pareto efficient with respect to $\mathcal{F}_1$. It is unlikely, however, that an equilibrium allocation in a constrained economy will be first-best efficient. The constraints in the economy prevent the allocation mechanism from fully utilizing the ability of the economy to transform resource endowments into goods for the consumers. In other words, the utility profile $U[X(q^*)]$ is usually not on the frontier of $\mathcal{U}(\mathcal{F}_1)$. Thus a hypothetical planner, working with the first-best transformation set and the consumption possibility sets $\mathcal{X}_i$, can find an allocation that makes every consumer better off.

Second-best efficiency is based on a more restrictive feasible set than the first-best feasible set. Let the second-best feasible set be defined by

$$\mathcal{F}_2 = \{X : X = X(q) \text{ for some } q \in Q\}.$$

Clearly the second-best feasible set is a subset of the first-best feasible set. An allocation $X \in \mathcal{F}_2$ is *second-best efficient* if it is Pareto efficient with respect to $\mathcal{F}_2$.

Hybrid efficiency combines the features of first-best and second-best efficiency. From first-best efficiency we utilize the idea of defining a feasible set based on a transformation set. From second-best efficiency we utilize the idea of restricting the feasible set to account for the effect of the institutional constraints and the other deviations from the competitive model in the economy. Often the effect of these constraints cannot be divorced from the other equilibrium conditions. In response to this difficulty, second-best efficiency considers only equilibrium allocations. For hybrid efficiency, we construct a transformation set from equilibrium aggregate bundles of goods.

As a preliminary step, consider the transformation set corresponding to an equilibrium q. Let

$$\mathcal{T}_h(q) = \sum_{j=1}^{k} y_j(q) + z(q) + (0, w).$$

A hypothetical planner working with $\mathcal{T}_h(q)$ must take all production as fixed at the equilibrium level. We form the hybrid transformation set by aggregating over the entire set Q. Let

$$\mathcal{T}_h = \bigcup_{q \in Q} \mathcal{T}_h(q).$$

Notice that $\mathcal{T}_h \subset \mathcal{T}_1$. The set $\mathcal{T}_h$ represents the transformation possibilities available to the economy, given the equilibrium implications of constraints. Hybrid efficiency, defined formally below, shows when the actual allocation mechanism is effectively utilizing the constrained transformation potential of the economy.

We now define two feasible sets that account for the new transformation sets and the original consumption possibility sets. We have:

$$\mathcal{F}_h(q) = \{X : (x_0, \sum_{i=1}^{n} x_i) \in \mathcal{T}_h(q), (x_0, x_i) \in \mathcal{X}_i \text{ for every } i\}$$

$$\mathcal{F}_h = \{X : (x_0, \sum_{i=1}^{n} x_i) \in \mathcal{T}_h, (x_0, x_i) \in \mathcal{X}_i \text{ for every } i\},$$

It is trivial to show that $\mathcal{F}_h = \bigcup_{q \in Q} \mathcal{F}_h(q)$ and $\mathcal{F}_h \subset \mathcal{F}_1$. We refer to $\mathcal{F}_h$ as the hybrid feasible set.

We can now define hybrid efficiency.

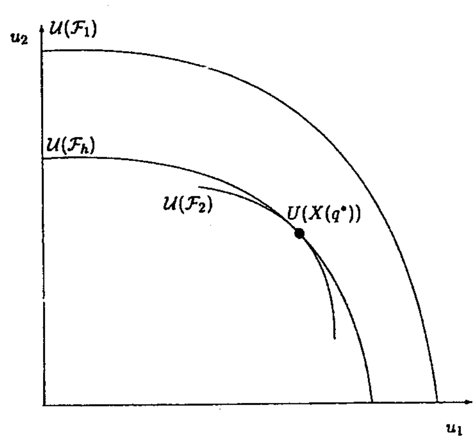
Definition. An allocation $X \in \mathcal{F}_h$ *is hybrid efficient* if X is Pareto efficient with respect to $\mathcal{F}_h$.

If an equilibrium allocation $X(q^*)$ is hybrid efficient, then $U[X(q^*)]$ is on the frontier of $\mathcal{U}(\mathcal{F}_h)$.

Since the equilibrium allocation $X(q)$ must be an element of $\mathcal{F}_h(q)$, it follows that the feasible sets form a hierarchy: $\mathcal{F}_2 \subset \mathcal{F}_h \subset \mathcal{F}_1$. The efficiency hierarchy is a direct consequence of the feasible set hierarchy. Suppose $X(q^*)$ is first-best efficient. By definition, $X(q^*)$ is an element of the other feasible sets. It is also efficient with respect to these sets because they are subsets of $\mathcal{F}_1$. Likewise, a hybrid efficient equilibrium allocation is also second-best efficient. On the other hand, a second-best efficient equilibrium allocation may or may not be hybrid efficient; and a hybrid efficient allocation may or may not be first-best efficient.

Consider once again an economy where the government uses commodity tax revenue to provide public goods. In this case, hybrid efficient allocations will not be first-best efficient. The utility profile for a hybrid efficient allocation $X(q^*)$ is inside the frontier of $\mathcal{U}(\mathcal{F}_1)$ but on the frontier of $\mathcal{U}(\mathcal{F}_h)$ and $\mathcal{U}(\mathcal{F}_2)$. This situation is illustrated in Fig. 1. A hypothetical planner working with the first-best transformation set can make an efficiency improvement relative to $X(q^*)$; but the same hypothetical planner, when restricted to the hybrid transformation set $\mathcal{T}_h$, can no longer make such an improvement. Notice that the efficiency hierarchy only works for equilibrium allocations. For example, many hybrid efficient allocations are not elements of $\mathcal{F}_2$. Such allocations can be reached by the constrained planner but not by the actual allocation mechanism. We are, of course, primarily interested in the efficiency properties of equilibrium allocations.

FIGURE 1
Hybrid Efficiency



A detailed numerical example of another constrained economy will further illustrate the construction of the hybrid transformation set and the efficiency hierarchy. Suppose there are three goods, two consumers, and one public firm. The consumers' utility functions are given by

$$u_i = \sum_{l=1}^{3} \alpha_{i,l} ln(x_{i,l}).$$

The public firm produces goods two and three from good one with the production technology $z_2^{\frac{1}{2}} + z_3^{\frac{1}{2}} = (-z_1)^2$. The resource endowment of the economy is $w =$

$[2, 0, 0]$. The government control vector is (p, e, z) where p is a price vector, $e = [e_1, e_2]$ is a vector of resource endowments for the consumers, and $z = (z_1, z_2, z_3)$ is a public production plan. A policy constraint forces the profit from public production to be zero. Another policy constraint requires that consumers' resource endowments be fixed at w_1 and w_2, respectively.

The set $\eta_i(p, e, z)$ consists of all solutions to:

$$\begin{aligned} \max \quad & u_i(x_i) \\ s.t. \quad & p \cdot x_i = p \cdot e_i \\ & x_i \in \mathcal{X}_i. \end{aligned}$$

The institutional constraints are satisfied when $p \cdot z = 0$, $e_1 = w_1$, and $e_2 = w_2$. With $\alpha = \begin{bmatrix} \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\ \frac{2}{3} & \frac{1}{6} & \frac{1}{6} \end{bmatrix}$, $w_1 = w_2 = [1, 0, 0]$, and the normalization $p_1 = 1$, the equilibrium conditions reduce to

$$\begin{aligned} 1 &= z_1 + 2 \\ \frac{2}{3p_2} &= z_2 \\ \frac{1}{3p_3} &= z_3 \\ p \cdot z &= 0 \\ z_2^{\frac{1}{2}} + z_3^{\frac{1}{2}} &= (-z_1^2). \end{aligned}$$

Thus any z in the set

$$\Pi = \{z : z_1 = -1, z_2^{\frac{1}{2}} + z_3^{\frac{1}{2}} = 1, z_2 \geq 0, \text{ and } z_3 \geq 0\}$$

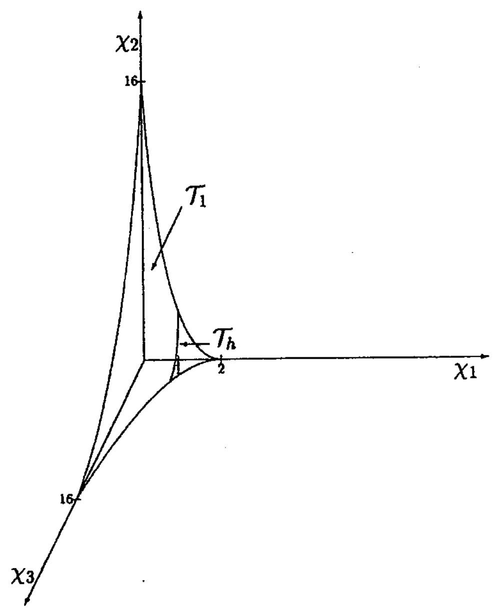
yields an equilibrium allocation $X(z)$. The corresponding equilibrium price vector is $(1, \frac{2}{3z_2}, \frac{1}{3z_3})$. Equivalently, equilibrium allocations can be described by any z_2 in the interval $[0, 1]$ because the choice of z_2 uniquely determines z_3 through the equation $z_2^{\frac{1}{2}} + z_3^{\frac{1}{2}} = 1$. Only some of these equilibrium allocations are second-best efficient. In particular, one can show that any z_2 in the interval $[0.25, 0.562]$ leads to an equilibrium allocation that is also second-best efficient. Traditional constrained efficiency theory stops at this point. Any further choice among the second-best efficient allocations requires explicit distributional considerations.

Hybrid efficiency offers an alternative view. In this example, the set $\mathcal{T}_h$ is given by

$$\mathcal{T}_h = [\chi = (\chi_1, \chi_2, \chi_3) : \chi_1 = 1, \chi_2^{\frac{1}{2}} + \chi_3^{\frac{1}{2}} = 1].$$

This is a smooth curve on the surface of the first-best transformation set. Each point on the curve corresponds to a public production plan that breaks even at equilibrium prices. The first-best transformation set and the hybrid transformation set are shown in Fig. 2. Once the hybrid transformation set is identified we can construct the hybrid feasible set $\mathcal{F}_h$ and then determine the hybrid efficient equilibrium allocations. Only a subset of second-best efficient allocations are hybrid efficient. These hybrid efficient allocations are the proper choice set for a government concerned only with obtaining efficiency subject to the constraints on the economy. Later we show that one equilibrium allocation does indeed satisfy the necessary conditions for hybrid efficiency; but first we present other properties of hybrid efficiency.

FIGURE 2
Transformation Sets



III. EFFICIENCY AND LINDAHL EQUILIBRIUM

Guesnerie (1979) shows that with any second-best efficient equilibrium there is an associated vector of social values of commodities. In this section we show that a similar result is true for hybrid efficiency. The social value vector results from the relationship between hybrid efficiency and Lindahl equilibria in an economy with a decentralized allocation mechanism.

Let q^* be an equilibrium in the constrained economy, and let $\tilde{E}(q^*)$ denote a distribution economy that has a transformation set $\tilde{\mathcal{T}} = \mathcal{T}_h(q^*)$. Each consumer i in $\tilde{E}(q^*)$ has the same consumption possibility set and utility function as consumer i in the original constrained economy. The allocation mechanism is decentralized througha special set of prices $P = (p_1, p_2, ..., p_n, \tilde{p})$ where each $p_i \in \mathcal{R}^{m_p}, \tilde{p} \in \mathcal{R}^m$, and $P \neq 0$. Consumer i maximizes utility with respect to the personalized vector of public good prices p_i and the common set of private good prices $\tilde{p}$. A rest point of the allocation mechanism is called a Lindahl equilibrium.

Definition. A *Lindahl equilibrium* is a pair (X^*, P^*) where X^* is an allocation and P^* is a price vector such that:

(1) $(x_0^*, \sum_{i=1}^n x_i^*) \in \tilde{T}$.

(2) For each $i = 1, 2, ..., n$, if $(x_0, x_i) \in X_i$ and $u_i(x_0, x_i) > u_i(x_0^*, x_i^*)$ then $(p_i^*, \tilde{p}^*) \cdot (x_0, x_i) > (p_i^*, \tilde{p}^*) \cdot (x_0^*, x_i^*)$.

The next proposition formally links hybrid efficiency and Lindahl equilibria. The proof utilizes an extension of the Second Theorem of Welfare Economics to economies with public goods.

Proposition 1 *Suppose that all utility functions are quasi-concave and strongly monotonic. Suppose that $X(q^*)$ is hybrid efficient. Then there exists a $P^* = (p_1^*, p_2^*, ..., p_n^*, \tilde{p}^*) > 0$ such that $[X(q^*), P^*]$ is a Lindahl equilibrium in the distribution economy $\tilde{E}(q^*)$.*

Proof: Condition (1) is trivial because $X(q^*)$ is an equilibrium allocation in the original constrained economy. The economy $\tilde{E}(q^*)$ satisfies the conditions of Theorem 2.1 in Milleron (1972). Therefore, there exists a P^* (strictly positive by strong monotonicity) such that $(x_0(q^*), x_i(q^*))$ minimizes $(p_i^*, \tilde{p}^*) \cdot (x_0, x_i)$ on $\{(x_0, x_i) \in X_i : u_i(x_0, x_i) \geq u_i(q^*)\}$ for each i. Condition (2) follows directly since $(x_0(q^*), x_i(q^*))$ is in the interior of X_i for every i. ∎

The price vector P^* in Proposition 1 can be interpreted as a social value vector. It represents the value of commodities to consumers at a hybrid efficient equilibrium allocation. The vector P^* also forms a Lagrange multiplier vector for the optimization problem discussed below. Since $P^* > 0$, we can normalize such that $\tilde{p}^* \cdot \tilde{g} = 1$. Consider $\tilde{p}^*$, the part of the social price vector that corresponds to private goods. Intuitively, if the utility functions are "smooth", then $\tilde{p}^*$ should be equal to the equilibrium prices in the original economy $p(q^*)$. (Both represent the value of commodities to consumers at the same allocation.) We first state a smoothness assumption and then prove a proposition that gives simple conditions under which this intuition is correct.

C.1 Consider any utility level u_i^* in the range of u_i over X_i. Let $Z_i(u_i^*) = [(x_0, x_i) : (x_0, x_i) \in X_i, u_i(x_0, x_i) \geq u_i^*]$. For each $(x_0, x_i) \in \partial Z_i[u_i^*]$ (the boundary of $Z_i(u_i^*)$) there exists a unique supporting hyperplane.

The sets $Z_i[u_i^*]$ are called upper contour sets. Assumption C.1 ensures that the upper contour sets do not have "kinks".

Proposition 2 *Suppose that the utility functions are quasi-concave, strongly monotonic, and satisfy C.1. Suppose that each consumer i in the original constrained economy*

We use Lemma 1, along with Proposition 4, to write the first order necessary conditions for hybrid efficiency. Suppose $X(q^*)$ is hybrid efficient. Then it is zero-maximal for (B). From Proposition 4, we know that the vector $\tilde{p}^*$ is a Lagrange multiplier vector for (b). By Lemma 1, there exists a μ_h such that q^* satisfies $\tilde{p}^*\nabla_q l(q^*) - \mu_h \nabla_q \Phi(q^*) = 0$. From the definition of $l(q)$, we have

$$\tilde{p}^*\nabla_q\Big(\sum_{j=1}^{k} y_j(q^*) + z(q^*) + w\Big) - \mu_h \nabla_q \Phi(q^*) = 0.$$

If the economy satisfies the conditions of Proposition 2, then $p(q^*) = \tilde{p}^*$. By the market clearing equilibrium condition, we have

$$p(q^*)\nabla_q\Big(\sum_{i=1}^{n} x_i(q^*)\Big) - \mu_h \nabla_q \Phi(q^*) = 0. \tag{2}$$

The system (2) represents the pricing rules for hybrid efficiency.

It is interesting to compare the first-order necessary conditions for second-best and hybrid efficiency. If the welfare weights satisfy $\alpha_i\theta_i = \rho$ for every i, then the systems (1) and (2) are the same (with $\mu_h = \frac{\mu}{\rho}$). To a first order approximation, a second-best efficient allocation that maximizes social welfare and results in an optimal income distribution is also a hybrid efficient allocation. Thus we would not expect hybrid efficiency to yield new pricing rules. We do, however, get a precise statement of the relationship between the equal welfare weight condition and efficiency. A planner, restricted to the use of the constrained transformation possibilities of the economy, cannot make a Pareto improvement with respect to a second-best efficient allocation that yields equal welfare weights.

Consider once again the example defined in Section II. Since the economy satisfies the conditions of Proposition 2, the pricing rules for hybrid efficiency are given by (2), where $\Phi(q) = \begin{bmatrix} (-z_i)^2 - (z_2^{\frac{1}{2}} + z_3^{\frac{1}{2}}) \\ z_1 + 1 \end{bmatrix}$. Substituting in the equilibrium relations yields:

$$\begin{bmatrix} 1 & \dfrac{2}{3z_2} & \dfrac{1}{3z_3} \end{bmatrix} \nabla_z \begin{bmatrix} z_1 + 2 \\ z_2 \\ z_3 \end{bmatrix} - \mu\nabla_z \begin{bmatrix} (-z_1)^2 - (z_2^{\frac{1}{2}} + z_3^{\frac{1}{2}}) \\ z_1 + 1 \end{bmatrix} = 0.$$

This system of equations, combined with the requirement that $\Phi(q) = 0$, yields a system of five equations and five unknown variables (z_1, z_2, z_3, μ_1, and μ_2). Solving for the unknown variables yields $(z_1^*, z_2^*, z_3^*) = (-1, \frac{4}{9}, \frac{1}{9})$ and $(\mu_1^*, \mu_2^*) = (-2, -2)$. Recall that any z_2 in the interval [0.25, 0.562] leads to a unique choice of z_3 and a corresponding second-best efficient equilibrium allocation $X(z)$. So $X(z^*)$ is a second-best efficient equilibrium allocation that also satisfies the first order necessary conditions for the stronger efficiency criterion hybrid efficiency. In Yates (1997) it is verified that $X(z^*)$ is indeed hybrid efficient. A planner, working with the hybrid transformation set $\mathcal{T}_h$, cannot make a Pareto improvement relative to $X(z^*)$. Thus hybrid efficiency identifies, without the use of a social welfare function and the attendant interpersonal comparisons of utility, a unique public production plan in this example.

VI. CONCLUSION

The traditional analysis of efficiency in constrained economies offers a choice between first and second-best efficiency. The strength of second-best efficiency is that it offers operational significance; it can actually be attained by the constrained economy allocation mechanism. The new efficiency criterion presented in the paper, hybrid efficiency, enriches the efficiency theory of constrained economies. Although we have not fully proven the operational significance of hybrid efficiency, we have several results that suggest that constrained economies can attain hybrid efficiency. The numerical example verifies that hybrid efficiency can indeed be attained by at least one constrained economy allocation mechanism. The first-order analysis indicates that certain second-best allocations satisfy the necessary conditions hybrid efficiency. Further study of the hybrid feasible set $\mathcal{F}_h$ yields insight into general conditions under which the necessary conditions for hybrid efficiency are sufficient. See Yates (1997).

Hybrid efficiency identifies the trade-off between efficiency and equity in constrained economies. A government may want to deviate from a hybrid efficient allocation to promote equity. Hybrid efficient allocations are the proper choice, however, for a government that does not want to make the explicit interpersonal comparisons of utility required for equity considerations.

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