A COMBINED PARALLEL-TANGENTS/PENALTY-FUNCTION APPROACH TO SOLVING TRAJECTORY OPTIMIZATION PROBLEMS*

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ABSTRACT

A combined parallel-tangents/penalty-function approach to solving trajectory optimization problems with terminal state constraints is discussed. The procedure has been designed to give accelerated convergence when using penalty-functions to handle terminal constraints. An attractive feature of the procedure is that its implementation is only slightly more complicated than the standard steepest-descent procedure. An example is presented illustrating the improved convergence properties afforded by this procedure.

I INTRODUCTION

This paper describes a combined parallel-tangents/penalty-function approach to solving trajectory optimization problems with terminal state constraints. The method of parallel-tangents is a rapidly converging minimization procedure that in all aspects is superior to the standard steepest-descent method. Parallel-tangents is a member of a family of general methods that, when applied to a quadratic objective functional, generates a sequence of conjugate directions during the descent procedure.§ In this paper a modified parallel-tangents procedure is used to accelerate convergence when penalty-functions are used to handle terminal constraints in trajectory optimization problems. An attractive feature of the combined parallel-tangents/penalty-function approach is that its implementation is only slightly more complicated than the standard steepest-descent procedure.

In the next section, the parallel-tangents procedure is discussed from a geometric viewpoint and its convergence properties are described. Section III considers the effect of large penalty-function terms on the contours of the objective function and the convergence rate. A modified parallel-tangents procedure to accelerate convergence when penalty functions are used is then described. For ease of explanation, the discussion in the above sections is basically in terms of a finite-dimensional minimization problem. Section IV describes the combined parallel-tangents/penalty-function approach to solving trajectory optimization problems with terminal constraints. Next, the results of applying the procedure to calculate the optimum boost-glide trajectory for a missile are discussed.

II PARALLEL-TANGENTS

In this section the method of parallel-tangents is described** and its convergence properties are discussed in the context of conjugate direction methods and compared to the method of steepest-descent.

A. The Procedure

Consider the problem of minimizing the functional $f(\underline{x})$ over all $\underline{x} \in E^n$ (where E^n is n-dimensional real Euclidean space). The parallel-tangents procedure can be described by the following recursive algorithm (see Fig. 1):*

$\underline{k = 0}$:

$$\begin{aligned} &\underline{x}^1 = \underline{x}^0 - \mu^0 \nabla f(\underline{x}^0) \\ &\text{where } \mu^0 = \arg\min_{\mu} \{f[\underline{x}^0 - \mu \nabla f(\underline{x}^0)]\} \end{aligned} \tag{1}$$

$\underline{k \geq 1}$:

$$\begin{aligned} &\underline{v}^k = \underline{x}^k - \mu^k \nabla f(\underline{x}^k) \\ &\text{where } \mu^k = \arg\min_{\mu} \{f[\underline{x}^k - \mu \nabla f(\underline{x}^k)]\} \end{aligned} \tag{2}$$

$$\begin{aligned} &\underline{x}^{k+1} = \underline{x}^{k-1} + \omega^k[\underline{v}^k - \underline{x}^{k-1}] \\ &\text{where } \omega^k = \arg\min_{\omega} \{f[\underline{x}^{k-1} + \omega(\underline{v}^k - \underline{x}^{k-1})]\} \end{aligned} \tag{3}$$

Referring to Fig. 1, the above algorithm states that to find $\underline{x}^{k+1}$, given $\underline{x}^k$ and $\underline{x}^{k-1}$, one first starts at $\underline{x}^k$, moving in the direction of the negative gradient until the minimum point $\underline{v}^k$ is reached. The point $\underline{x}^{k+1}$ is then found by starting at $\underline{x}^{k-1}$ and moving in the direction $\underline{v}^k - \underline{x}^{k-1}$ to the minimum point. It is to be emphasized that the above algorithm decreases the cost functional at each step. More specifically, if the gradient $\nabla f(\underline{x}^k)$ is nonzero, then the definition of $\underline{v}^k$ guarantees that $f(\underline{v}^k) < f(\underline{x}^k)$. But the definition of $\underline{x}^{k+1}$ guarantees that $f(\underline{x}^{k+1}) \leq f(\underline{v}^k)$, and consequently $f(\underline{x}^{k+1}) < f(\underline{x}^k)$ for $\nabla f(\underline{x}^k)$ nonzero.

A heuristic motivation for the method can be obtained by examining the procedure for a quadratic functional in E^2 (see Fig. 2). It is known that for a quadratic functional, any line emanating from its minimum point intersects the contours at equal angles. Consequently, in seeking the minimum from a given initial starting point, one might search for a second point such that the tangents to the contours at both points are parallel (the name parallel-tangents arises from this heuristic motivation). If this is accomplished, then the line passing through both points will pass through the minimum of the quadratic functional. Referring to

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§ Two other methods belonging to this family are conjugate gradients [1,2] and Davidon's method [3,4].

** For a more detailed discussion of parallel-tangents, see [5 to 8].

* The notation $\nabla f(\underline{x})$ represents the gradient of $f(\underline{x})$ at $\underline{x}$. The quantities μ and ω are scalars, which determine the step-size. The notation arg min { } represents the value of the argument that minimizes { }. If this minimizer is not unique, it is to be understood that the smallest positive value is chosen.

the algorithm and Fig. 2, since $\underline{x}^1 - \underline{x}^0$ is along the negative gradient, then $\underline{x}^1 - \underline{x}^0$ is perpendicular to the contour tangent at $\underline{x}^0$. Since $\underline{x}^1$ is at the minimum along $\underline{x}^1 - \underline{x}^0$ and $\underline{v}^1 - \underline{x}^1$ is in the negative gradient direction, then $\underline{v}^1 - \underline{x}^1$ is perpendicular to $\underline{x}^1 - \underline{x}^0$. Consequently, $\underline{v}^1 - \underline{x}^1$ is parallel to the contour tangent at $\underline{x}^0$. Since $\underline{v}^1$ is the minimum along $\underline{v}^1 - \underline{x}^1$, the contour tangent at $\underline{v}^1$ contains the line $\underline{v}^1 - \underline{x}^1$, which implies that the contour tangent at $\underline{x}^0$ is parallel to the contour tangent at $\underline{v}^1$. Therefore, $\underline{x}^2$, which is the minimum along the direction $\underline{v}^1 - \underline{x}^0$, is the minimum of the quadratic functional.

B. Convergence

Two vectors $\underline{p}^1$ and $\underline{p}^2$ are said to be orthogonal if $\langle \underline{p}^1, \underline{p}^2 \rangle = 0$.* Two vectors $\underline{p}^1$ and $\underline{p}^2$ are said to be conjugate with respect to the quadratic form $\langle \underline{x}, Q\underline{x} \rangle$ if $\langle \underline{p}^1, Q\underline{p}^2 \rangle = 0$. It is seen that for $Q = I$, where I is the identity matrix, the notions of orthogonality and conjugacy coincide. If we define the direction vectors $\underline{p}^k = \underline{x}^k - \underline{x}^{k-1}$, for $k = 1, \ldots, n$, where the $\underline{x}^k$'s are generated by the parallel-tangent procedure in (1) to (3), and if $f(\underline{x}) = \langle \underline{x}, Q\underline{x} \rangle - 2\langle \underline{x}, \underline{b} \rangle$ is a positive-definite symmetric quadratic functional (i.e., Q is a positive-definite symmetric matrix), then it can be shown that the $\underline{p}^k$'s generated by the method of parallel-tangents form a Q-conjugate system.†

It is well known that the minimum of a positive-definite symmetric quadratic functional can be obtained in n steps if one successively minimizes the functional along n conjugate directions, where n is the dimension of the vector $\underline{x}$. (A simple demonstration of this fact is given in the Appendix.) Consequently, for a finite-dimensional problem, the method of parallel-tangents will find the minimum of a positive-definite symmetric quadratic functional in n steps. It can be shown for the infinite-dimensional quadratic problem (i.e., $n = \infty$) that the convergence rate for parallel-tangents is given by

$$\|\underline{x}^k - \underline{x}^*\|^2 \leq C \cdot \left(\frac{1 - \sqrt{\beta/\alpha}}{1 + \sqrt{\beta/\alpha}}\right)^{2k} \cdot \|\underline{x}^0 - \underline{x}^*\|^2 \qquad (4)$$

where for all $\underline{x}$, α and β satisfy $\beta \cdot \langle \underline{x}, \underline{x} \rangle \leq \langle \underline{x}, Q\underline{x} \rangle \leq \alpha \cdot \langle \underline{x}, \underline{x} \rangle$, $\|\underline{x}\|^2 \triangleq \langle \underline{x}, \underline{x} \rangle$, and C is a constant depending on $\underline{x}^0$, and $\underline{x}^*$ is the minimum point. For steepest-descent the best estimate on convergence is

$$\|\underline{x}^k - \underline{x}^*\|^2 \leq \hat{C} \cdot \left(\frac{1 - \beta/\alpha}{1 + \beta/\alpha}\right)^{2k} \cdot \|\underline{x}^0 - \underline{x}^*\|^2 \qquad (5)$$

where $\hat{C}$ is a constant depending on $\underline{x}^0$. Consequently, in both the finite-dimenstional and infinite-dimensional cases, the convergence properties of the parallel-tangents procedure is superior to the steepest-descent procedure. In particular, for an n-dimensional quadratic problem the error tends to zero within n-steps using parallel-tangents while only geometrically with steepest-descent.

* The notation $\langle \underline{y}, \underline{z} \rangle$ represents the inner product.
† A proof of this fact is given in [6].

(See [5] for a more detailed discussion of convergence rates.)

III PENALTY-FUNCTIONS

In this section the effect of large penalty-function terms on the contours of the objective function and the convergence rate is first discussed. This is followed by a modification of the parallel-tangents procedure to accelerate convergence when penalty-functions are used to handle constraints.

A. Eccentricity of the Contours

To understand the effect of penalty-functions on the contours of the objective function and the convergence rate consider the problem of minimizing the positive-definite symmetric quadratic functional $f(\underline{x}) = \langle \underline{x}, Q\underline{x} \rangle - 2\langle \underline{x}, \underline{b} \rangle$, subject to the one-dimensional linear constraint $H(\underline{x}) = \langle \underline{d}, \underline{x} \rangle = \theta$, where $\underline{x}, \underline{d} \in E^n$. Using the penalty-function approach, the objective function $f(\underline{x})$ is augmented by a penalty term involving the constraint $H(\underline{x}) = \theta$, so that the original problem is solved by minimizing the modified objective function $f(\underline{x}) + \mu \cdot \|H(\underline{x})\|^2$, where μ is a suitably large scalar. For the simple problem under consideration, the modified objective function to be minimized is $g(\underline{x}) = \langle \underline{x}, Q'\underline{x} \rangle - 2\langle \underline{x}, \underline{b} \rangle$, where $Q' = Q + \mu \cdot \underline{d}\underline{d}^T$. Since $\mu \cdot \underline{d}\underline{d}^T$ is of rank one*, this implies that $\mu \cdot \underline{d}\underline{d}^T$ has only one nonzero eigenvalue. Consequently, for large μ, the largest eigenvalue of Q' is very large while the other eigenvalues of Q' remain near the eigenvalues of Q. Geometrically this implies that the contours of $g(\underline{x})$ become more eccentric or elongated as the weighting factor on the penalty term becomes larger.

It is known that the convergence of steepest descent slows down considerably as the contours become more eccentric. Referring to Fig. 3, the steepest-descent procedure essentially "bounces off the walls of a long, narrow valley" without significantly improving the cost at each step. This behavior of steepest descent can be explained analytically by examining its convergence properties with respect to the quadratic functional $\langle \underline{x}, Q\underline{x} \rangle - \langle \underline{x}, \underline{b} \rangle$. In particular, for the finite-dimensional case the constants α and β in (5) correspond to the largest and smallest eigenvalues of Q. It is seen by (5) that the convergence rate for steepest-descent depends on the relative magnitudes of the largest and smallest eigenvalues. For $\alpha = \beta$ the contours are circular and convergence occurs in one step. However, if $\alpha \gg \beta$ the contours are highly eccentric and convergence can be extremely slow.

By contrast, referring to Fig. 4, it is seen that the method of parallel tangents uses intermediate gradient steps to essentially align the direction of the next step in the direction of the eccentric contour. Furthermore, as was shown in the previous section, the method of parallel tangents obtains the minimum of a quadratic form in n steps, where n is the dimension of the problem.

*The matrix $\underline{d}\underline{d}^T$ is of rank one, since its columns are scalar multiples of $\underline{d}$.

B. A Modification to Accelerate Convergence

As pointed out above, the parallel-tangents procedure is considerably more effective than the standard steepest-descent procedure in handling the highly eccentric contours that arise when employing penalty-functions. One can, however, improve on the parallel-tangents procedure by noting that the cost contours are highly eccentric only in a subspace smaller than the full space E^n. It is proved in [10] that the dimension of this smaller subspace corresponds to the dimension of the constraint equations. To take advantage of this fact in minimizing $f(\underline{x}) + \mu\|\underline{H}(\underline{x})\|^2$, where $f(\underline{x})$ is the cost, $\underline{H}(\underline{x}) = \underline{\theta}$ is the constraint, $\underline{x} \in E^n$, and $\underline{H}(\underline{x}) \in E^m$, one should re-initialize the parallel-tangents procedure given in (1) to (3) every m+1 steps. If this is done, the convergence rate in minimizing $f(\underline{x}) + \mu\|\underline{H}(\underline{x})\|^2$ does not become arbitarily slow for large μ, but is independent of μ and depends only on properties of the Lagrangian $f(\underline{x}) + \langle\underline{\lambda},\underline{H}(\underline{x})\rangle$ for the problem (see [10]). Indeed, the rate is approximately equal to the rate that would be obtained by eliminating the subspace causing the highly eccentric contours.

It should be pointed out that most of the underlying theory for parallel-tangents has been developed for the quadratic minimization problem. Because in practice the objective functions of ultimate interest are nonquadratic and to take into account previously imperfect steps, it is desirable to re-initialize the parallel-tangents procedure at regular iteration intervals. The above discussion (paraphrased from [10]) gives a rational basis for re-initializing the parallel-tangents procedure to accelerate the convergence rate when dealing with penalty-functions.

IV TRAJECTORY OPTIMIZATION PROBLEMS

The discussion in the previous sections was basically centered about optimization problems in finite-dimensional vector spaces. In this section the combined parallel-tangents/penalty-function procedure will be described for trajectory-optimization problems. The trajectory-optimization problem can be formulated as follows:

> Find the control history $\underline{u}(t)$, $t_o \le t \le t_f$, and the final time t_f that minimizes the objective functional $\psi[\underline{x}(t_f), t_f]$, while satisfying the state equations $\dot{\underline{x}}(t) = \underline{f}[\underline{x}(t), \underline{u}(t), t]$ with $\underline{x}(t_o)$ given, the terminal state constraints $\underline{H}[\underline{x}(t_f)] = \underline{\theta}$, and the control constraints $\underline{u}(t) \in U[\underline{x}(t), t]$.

In applying the penalty-function approach to solving the above problem, one considers a modified objective functional

$$J[\underline{x}(t_f),t_f] \triangleq \psi[\underline{x}(t_f),t_f] + \sum_{i=1}^{m} K_i \cdot \{H_i[\underline{x}(t_f)]\}^2 \quad (6)$$

where $H_i[\underline{x}(t_f)]$ is the ith component of the m-dimensional constraint $\underline{H}[\underline{x}(t_f)]$ and the K_i's are suitably large positive constants. The combined parallel-tangents/penalty-function approach, described earlier, can be applied once the gradient of the modified objective functional $J[\underline{x}(t_f), t_f]$ with respect to the control $\underline{u}(t)$ is identified.

The calculation of this gradient is well known and consists of the following steps (see [11]):

(1) With a nominal control history $\underline{u}(t)$ integrate forward in time the equations of motion

$$\dot{\underline{x}}(t) = \underline{f}[\underline{x}(t), \underline{u}(t), t] \quad (7)$$

from the initial state $\underline{x}(t_o)$. This yields a nominal trajectory $\underline{x}(t)$ and a corresponding value for the modified objective functional $J[\underline{x}(t_f), t_f]$.

(2) Integrate backwards in time the ajoint equations

$$\dot{\underline{\lambda}}(t) = - F(t)^T \cdot \underline{\lambda}(t) \quad (8)$$

from the final condition

$$\underline{\lambda}(t_f) \triangleq \left.\frac{\partial J[\ ]}{\partial \underline{x}}\right|_{\underline{x}(t_f),t_f} \quad (9)$$

where the matrix $F(t)$ is given by

$$F(t) \triangleq \left.\frac{\partial \underline{f}[\ ]}{\partial \underline{x}}\right|_{\underline{x}(t),\underline{u}(t),t} \quad (10)$$

(3) The gradient of the modified objective functional with respect to the control is calculated as

$$\nabla_{\underline{u}} J(t) = G(t)^T \cdot \underline{\lambda}(t) \quad (11)$$

where the matrix $G(t)$ is given by

$$G(t) \triangleq \left.\frac{\partial \underline{f}[\ ]}{\partial \underline{u}}\right|_{\underline{x}(t),\underline{u}(t),t} \quad (12)$$

The control constraints and the free final time are handled in the following manner. Before integrating the equations of motion forward in time, each candidate control history is limited in magnitude over those intervals of time that the control constraints would be violated. The final time t_f for the forward integration of the state equations is chosen for each candidate control history to minimize the modified objective functional $J[\underline{x}(t_f), t_f]$. As in [12], this is accomplished by determining when the total time derivative of $J[\underline{x}(t_f), t_f]$ changes sign from minus to plus.

The parallel-tangents procedure is described by the recursive algorithm (the superscript represents the iteration)

$$\underline{v}^k(t) = \underline{u}^k(t) - \mu^k \nabla_{\underline{u}} J(t) \quad (13)$$

$$\underline{u}^{k+1}(t) = \underline{u}^{k-1}(t) + \omega^k[\underline{v}^k(t) - \underline{u}^{k-1}(t)]. \quad (14)$$

Figure 5 shows a simplified flow chart of the combined parallel-tangents/penalty-function procedure. The optimum step size for both the standard steepest-descent step and the parallel-tangents step is calculated by a one-dimensional line search over μ^k and ω^k, respectively (see Fig. 5). This line search essentially consists in approximating the modified objective functional along the line by a parabola. The optimum step size is obtained by evaluating the minimum of the parabola. It is also seen from Fig. 5 that the parallel-tangents procedure is re-initialized every N steps. As described in Sec. III-B, the number N is equal to m+1, where m is the dimension of the constraint $\underline{H}[\underline{x}(t_f), t_f]$.

The flow chart in Fig. 5 shows the logic for the combined parallel-tangents/penalty-function procedure to be essentially the same as the standard steepest-descent procedure, except that the parallel-tangents step given by Eq. (14) is added after the standard steepest-descent step given by Eq. (13). Consequently, for a small additional programming effort and moderate additional computing time, a superior method to the steepest-descent procedure can be obtained.

V NUMERICAL RESULTS

In this section, the combined parallel-tangents/penalty-function approach described in Sec. IV will be applied to the computation of the optimal trajectory and guidance history for a two-dimensional missile problem. This example considers the optimization of the boost-glide trajectory for a two-stage, intermediate-range missile.

The forces acting upon the missile are illustrated in Fig. 6, where the y and z axes define the horizontal and vertical directions, respectively, at the launch site. Assuming a spherical, nonrotating earth, the equations of motion for the missile are given by:

$$\ddot{y} = \frac{1}{m}\left[(-D + T\cos u)\frac{\dot{y}}{V} - (L + T\sin u)\frac{\dot{z}}{V}\right] - g\frac{y r_o^2}{r_1^3} = f_3 \tag{15}$$

$$\ddot{z} = \frac{1}{m}\left[(-D + T\cos u)\frac{\dot{z}}{V} + (L + T\sin u)\frac{\dot{y}}{V}\right] - g\frac{(z + r_o)\, r_o^2}{r_1^3} = f_4 \tag{16}$$

where

u = angle of attack (control variable)

$V = \sqrt{\dot{y}^2 + \dot{z}^2}$ = velocity

g = gravitational acceleration

$r_1 = \sqrt{y^2 + (z + r_o)^2}$

r_o = earth radius

$D = \frac{1}{2}\rho(h)V^2 S_R C_D(M,u)$ = drag force

$L = \frac{1}{2}\rho(h)V^2 S_R C_L(M,u)$ = lift force

$T = T_{VAC}(t) - S_E\, p(h)$ = thrust

in which

$h = r_1 - r_o$ = altitude

$\rho(h)$ = atmospheric mass density, given in tabular form

S_R = missile reference area = $\begin{cases} 15.43 \text{ ft}^2, & \text{for first stage} \\ 3.09 \text{ ft}^2, & \text{for second stage} \end{cases}$

M = Mach number, obtained from atmospheric tables

$C_D(M,u)$ = drag coefficient, given in tabular form

$C_L(M,u)$ = lift coefficient, given in tabular form

$T_{VAC}(t)$ = vacuum thrust = $\begin{cases} 185{,}185 \text{ lbs}, & \text{for first stage} \\ 46{,}296 \text{ lbs}, & \text{for second stage} \end{cases}$

$p(h)$ = ambient pressure, given in tabular form

S_E = nozzle exit area = 9.26 ft^2,

and

$$\dot{m} = -\frac{T_{VAC}(t)}{g\, I_{SP}} \tag{17}$$

with

I_{SP} = specific impulse = 260 s.

Given the vacuum thrust $T_{VAC}(t)$, the missile's mass history can be precomputed by integrating Eq. (17). Shortly after first stage ignition (t=0 s), the missile's mass is 1152 slugs; at first stage burnout (t=35 s), the mass is 374 slugs; at second stage ignition (t=35 s), the mass is 287 slugs; and at second stage burnout (t=77 s), the mass is 55 slugs. It should be noted, from the above numbers for thrust and mass, that missile staging is assumed to occur instantaneously. In addition, it has been assumed that the missile responds instantaneously to commands in u, the angle of attack; that is, the equations of motion (15) and (16) consider the missile as a mass point, with its attitude dynamics neglected. Finally, the angle of attack u is subject to the following constraint:

$$|u(t)| \leq 30^\circ \quad . \tag{18}$$

If the four-dimensional state vector of the missile is defined as

$$\underline{x} = [y, z, \dot{y}, \dot{z}]^T \quad , \tag{19}$$

then the differential equations (15) and (16) can be rewritten concisely in state-variable form as

$$\dot{\underline{x}}(t) = \underline{f}[\underline{x}(t), u(t), t] \quad ; \tag{20}$$

if f_i is the ith element of the four-dimensional vector function $\underline{f}$, then $f_1 = \dot{y}$, $f_2 = \dot{z}$, and f_3 and f_4 are given by Eqs. (15) and (16), respectively.

The problem as formulated is to find the control history u(t), $0 \leq t \leq t_f$ (where the final t_f is fixed at 77 s, which is second stage burnout), that optimizes the boost-glide trajectory of the missile. This optimum is achieved by maximizing the terminal velocity, subject to the terminal constraints that the velocity is horizontal (i.e. the flight path angle is zero) and the desired altitude is reached. For the relatively short ranges considered in this problem, it is reasonable to assume that the horizontal and vertical components of velocity are $\dot{y}$ and $\dot{z}$, respectively. Hence, the optimization problem can be stated as: maximize $\dot{y}(t_f)$, subject to $\dot{z}(t_f) = 0$ and $h(t_f) = h_o$, where h_o is the desired altitude. In terms of the notation used in Sec. IV, the objective functional is (recall that ψ is to be minimized)

$$\psi[\underline{x}(t_f)] = -[\dot{y}(t_f)]^2 \quad , \tag{21}$$

and the two-dimensional terminal state constraint is

$$\underline{H}[\underline{x}(t_f)] = \begin{bmatrix} h(t_f)-h_o \\ \dot{z}(t_f) \end{bmatrix} = \underline{\theta} \quad . \tag{22}$$

From Eq. (6), the modified objective functional is given by

$$J[\underline{x}(t_f)] = -[\dot{y}(t_f)]^2 + K_1[h(t_f)-h_o]^2 + K_2[\dot{z}(t_f)]^2 \quad , \tag{23}$$

where the weighting coefficients on the penalty terms were chosen to be

$$K_1 = 10^2 \qquad K_2 = 10^5 \quad , \tag{24}$$

the desired altitude is

$$h_o = 200{,}000 \text{ ft}, \tag{25}$$

and the final time is

$$t_f = 77 \text{ s}. \tag{26}$$

To compute the gradient of the modified objective functional with respect to control, as given by Eq. (11), requires the following matrices of partial derivatives: $\partial J/\partial \underline{x}$, $\partial \underline{f}/\partial \underline{x}$, and $\partial \underline{f}/\partial u$, which are defined in Eqs. (9), (10), and (12). The elements comprising these matrices are given in [13] and, hence, will not be repeated here.

For the results presented below, the following initial conditions, corresponding to a time shortly after first-stage ignition, were used for the missile:

$$\underline{x}(0) \begin{cases} y(0) = 0 \text{ ft} \\ z(0) = 0 \text{ ft} \\ \dot{y}(0) = 9 \text{ ft/s} \\ \dot{z}(0) = 100 \text{ ft/s}. \end{cases} \tag{27}$$

The parallel-tangent procedure and the standard steepest-descent technique have been programmed in FORTRAN and run on a CDC-6400 computer for purposes of comparison. The results for the optimization of the boost-glide trajectory for this missile example are presented in Figs. 7, 8, and 9. In Fig. 7, the modified objective functional (in normalized form) is plotted versus computation time; the dots represent successive iterations on the solid curve, and alternate iterations on the dashed curve. Both iterative methods were initialized with $u(t) = 0$, $0 \le t \le t_f$, which corresponds to zero-lift control. As can be seen from Fig. 7, the convergence of the parallel-tangent procedure is much faster than that of the standard steepest-descent technique. The parallel-tangent procedure has converged to the optimum by iteration $k = 18$ (which corresponds to a computation time of 167 s); for this iteration, the terminal value of the horizontal velocity is

$$\dot{y}(t_f) = 19{,}046 \text{ ft/s} \quad , \tag{28}$$

and the constrained terminal states are

$$h(t_f) = 200{,}066 \text{ ft} \qquad \dot{z}(t_f) = 5 \text{ ft/s}. \tag{29}$$

Whereas, the standard steepest-descent technique is not even close to the optimum by iteration $k = 28$ (which corresponds to a computation time of 169 s); for this iteration,

$$\dot{y}(t_f) = 19{,}038 \text{ ft/s} \quad , \tag{30}$$

and

$$h(t_f) = 145{,}522 \text{ ft} \qquad \dot{z}(t_f) = 1{,}060 \text{ ft/s}. \tag{31}$$

Figures 8 and 9 show control histories and trajectories (altitude vs. range) for itermediate iterations of the parallel-tangent procedure. Recall that

$$\text{altitude } h = \sqrt{y^2 + (z+r_o)^2} - r_o$$

and

$$r = r_o \tan^{-1}(y/z+r_o),$$

where r_o is the earth radius. It should be noted that for iteration $k = 0$, $u(t) = 0$.

VI CONCLUSIONS

The trajectory optimization technique described in Sec. IV has been developed into a computer program that is applicable to a wide variety of problems. Computational experience on several missile trajectory optimization problems with terminal state constraints has shown the combined parallel-tangents/penalty-function procedure to be rapidly convergent and superior to the standard steepest-descent method. One such example was discussed in Section V. An attractive feature of the approach is that from a programming viewpoint, the logic required to implement the combined parallel-tangents/penalty-function approach is essentially the same as the standard steepest-descent procedure. The difference lies in the addition of a parallel-tangents step after completing a standard steepest-descent step. Consequently, the combined parallel-tangents/penalty-function approach is reasonably simple to implement.

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APPENDIX

Minimizing a Quadratic Functional Along Conjugate Directions

Assume for the moment that one would like to minimize a special quadratic functional of the form $f'(\underline{y}) = \langle \underline{y}, I\underline{y}\rangle - 2\langle \underline{y}, \underline{a}\rangle$, where $\underline{y}, \underline{a} \in E^n$ and I is the identity matrix. Since $f'(\underline{y})$ is a separable sum in each coordinate variable (i.e., no cross terms), the minimum can be found by minimizing $f'(\underline{y})$ successively in each of the n coordinate directions. Now consider a transformation of $f'(y)$ to any mutually orthogonal coordinate system. More specifically, if $\underline{z} = B\underline{y}$, where B preserves orthogonality, then $B^T = B^{-1}$ and $f'(\underline{y}) = f''(\underline{z}) = \langle \underline{z}, I\underline{z}\rangle - \langle \underline{z}, B\underline{a}\rangle$* Since $f''(\underline{z})$ is also a separable sum in each coordinate variable, this implies the quadratic functional $f'(y)$ can be minimized by successively minimizing in n mutually orthogonal directions.

Now assume a general quadratic functional of the form $f(\underline{x}) = \langle \underline{x}, Q\underline{x}\rangle - 2\langle \underline{x}, \underline{b}\rangle$ is to be minimized, where Q is positive definite and symmetric. Since Q is positive definite and symmetric, there exists a matrix A such that $A^TA = Q$.† Consequently, using the transformation $\underline{y} = A\underline{x}$, the general quadratic functional $f(\underline{x})$ can be mapped into the special quadratic functional $f(\underline{x}) = f'(\underline{y}) = \langle \underline{y}, I\underline{y}\rangle - 2\langle \underline{y}, (A^{-1})^T\underline{b}\rangle$. However, the transformation $\underline{y} = A\underline{x}$ transforms Q-conjugate vectors into orthogonal vectors (i.e., $\langle x^j, Qx^i\rangle = 0$, $i \neq j$ implies that $\langle \underline{y}^i, \underline{y}^j\rangle = 0$, $i \neq j$). Thus, using the previous development, to minimize $f(\underline{x}) = \langle \underline{x}, Q\underline{x}\rangle - 2\langle \underline{x}, \underline{b}\rangle$ one could transform variables and minimize $f'(\underline{y}) = \langle \underline{y}, I\underline{y}\rangle - 2\langle \underline{y}, (A^{-1})^T \underline{b}\rangle$ by sucessively minimizing along n orthogonal directions, or minimize $f(\underline{x}) = \langle \underline{x}, Q\underline{x}\rangle - 2\langle \underline{x}, \underline{b}\rangle$ directly by sucessively minimizing along n Q-conjugate directions.

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* See [9] on orthogonal transformations.
† See [9] on positive transformations.

FIGURES

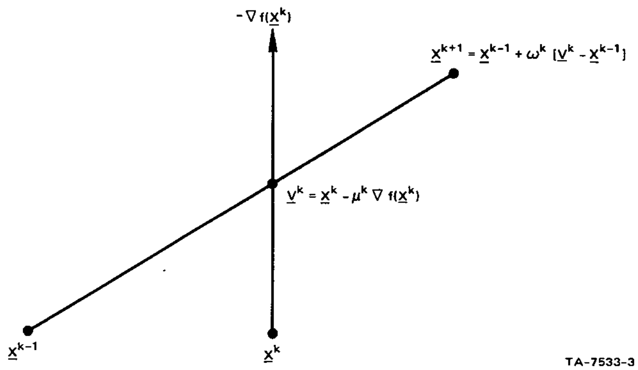

FIGURE 1 THE PARALLEL-TANGENTS PROCEDURE

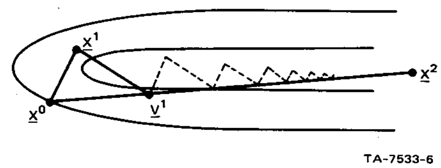

FIGURE 4 PARALLEL-TANGENTS PROCEDURE WITH HIGH-ECCENTRICITY CONTOURS

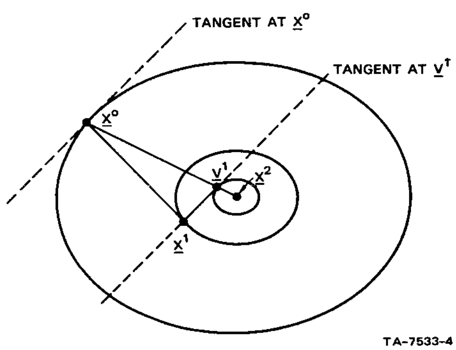

FIGURE 2 PARALLEL-TANGENTS PROCEDURE FOR A QUADRATIC FORM IN E^2

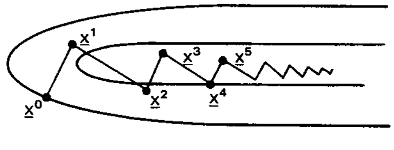

FIGURE 3 STEEPEST-DESCENT TECHNIQUE WITH HIGH-ECCENTRICITY CONTOURS

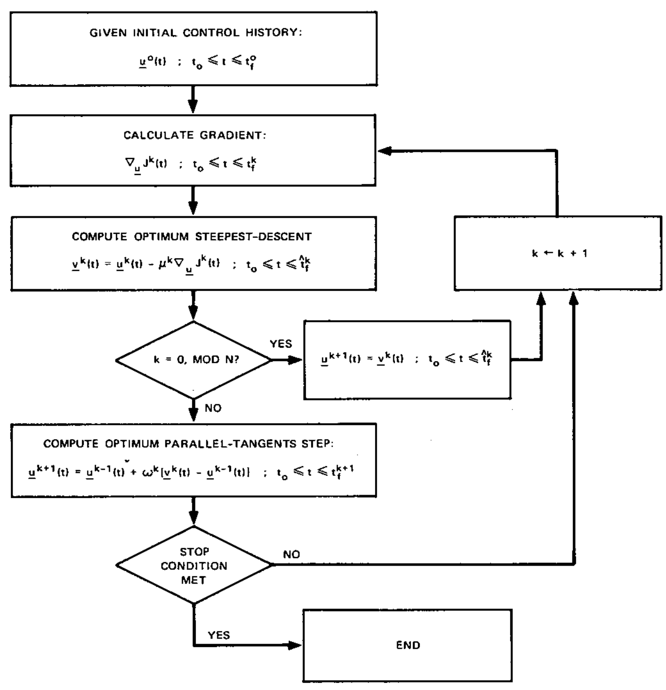

FIGURE 5 SIMPLIFIED FLOW CHART OF THE COMBINED PARALLEL-TANGENTS/PENALTY-FUNCTION PROCEDURE

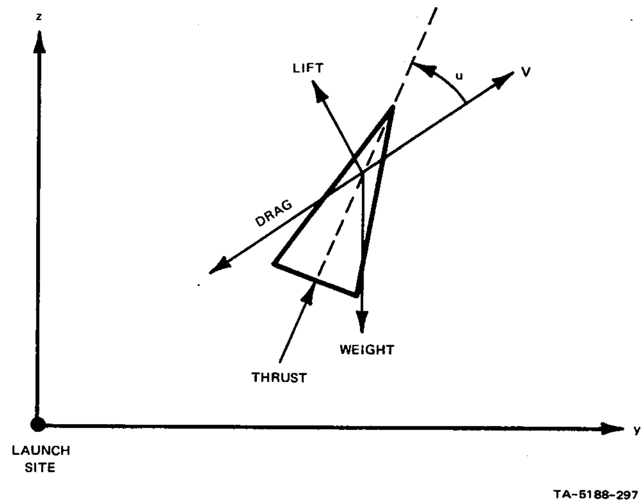

FIGURE 6 MISSILE GEOMETRY

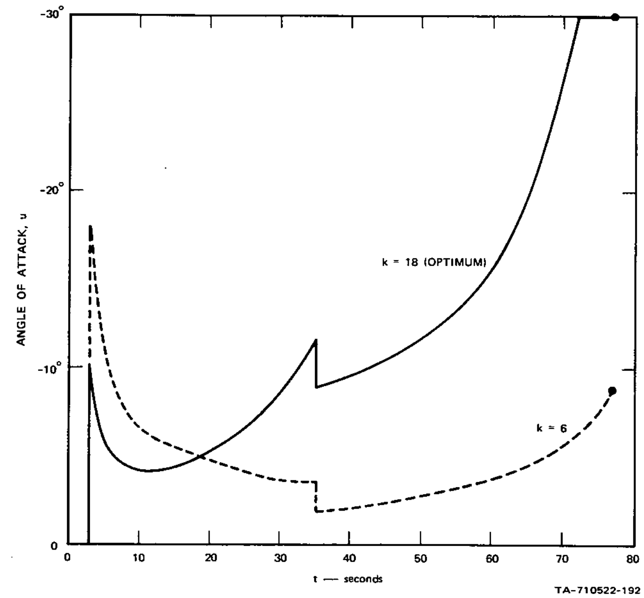

FIGURE 8 CONTROL HISTORIES FOR EXAMPLE

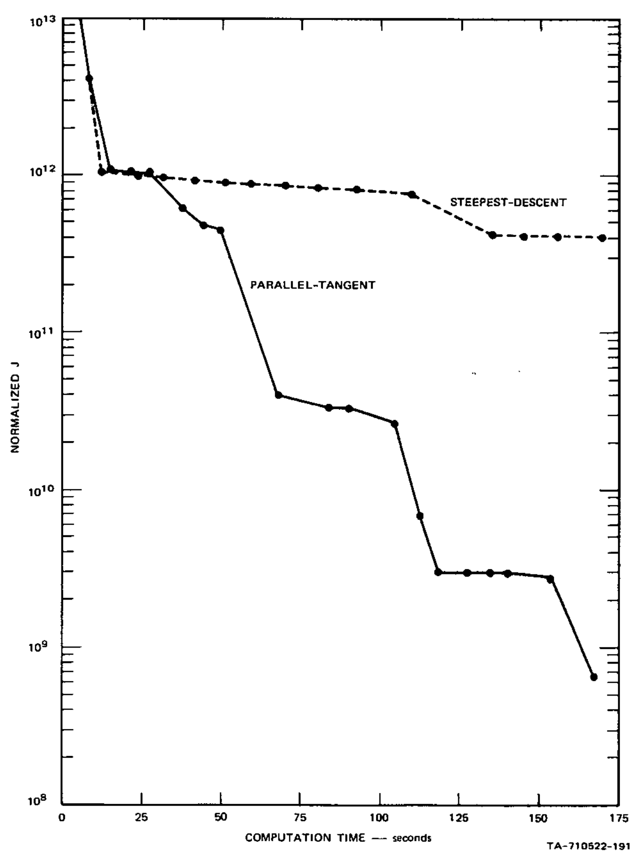

FIGURE 7 CONVERGENCE PROPERTIES OF PARALLEL-TANGENTS PROCEDURE VERSUS STEEPEST-DESCENT TECHNIQUE

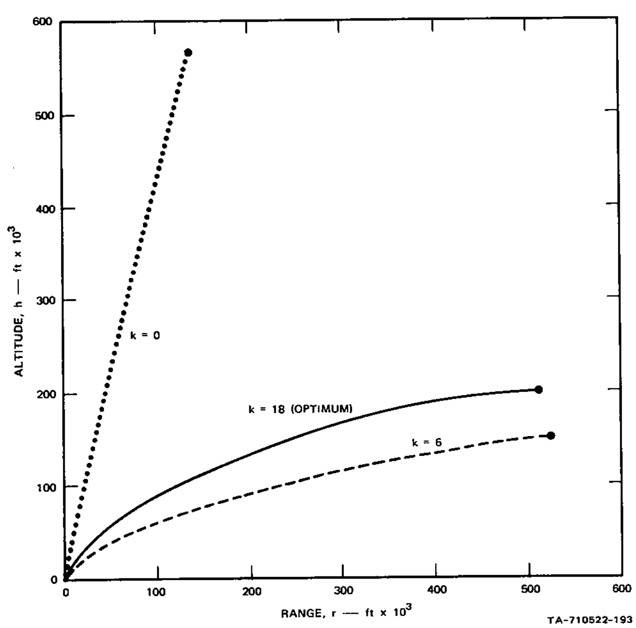

FIGURE 9 TRAJECTORIES (ALTITUDE VERSUS RANGE) FOR EXAMPLE