Sequential defaults and incomplete information

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We propose a multi-firm first-passage credit model in which investors have incomplete information. In this model, investors observe neither a firm's value nor its default barrier. The model takes into account the short-term risk inherent in default events, the market-wide impact of defaults on security prices due to counterparty relations among firms, and the cyclical default dependence effects observed in credit markets. We explicitly calculate the pricing trend and the arrival intensity of the kth-to-default. These results furnish (1) tractable reduced-form formulae for arrival probabilities of sequential dependent defaults and prices of multi-name credit derivatives, and (2) an algorithm for the simulation of sequential unpredictable default times.

1 Introduction

Multi-name credit derivatives and structured credit products are increasingly popular. As a consequence, there is growing interest in quantitative models to value and to manage the risk of these securities. The models are necessarily complex since they must take into account the idiosyncratic default risk of individual firms, the sensitivity of a firm to macroeconomic and other market-wide risk factors, and the contagion generated by counterparty exposure.

Reduced-form credit models are commonly used to price credit-sensitive securities. In these models, each firm has an exogenously given *intensity* or *conditional rate of default*, and there are tractable intensity-based pricing formulae and calibration routines. Reduced-form models extend to take into account the inter-firm dependencies mentioned above. Typically, the intensity is specified as a jump–diffusion process. As described in Duffie and Singleton (1998), correlation of the diffusion terms takes into account the dependence on common

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macroeconomic factors, which induces the cyclical correlation effects we observe in credit markets. Jarrow and Yu (2001) show that correlated jumps in intensities model the contagion effects that arise from firms' interactions with each other. Contagion occurs in credit markets, for example, when a single default raises credit spreads across the board.

An alternative to reduced-form credit modeling is the structural approach, which directly models the cause of default. Structural models are very successful default forecasting tools. However, they are less effective than reduced-form models as a means to price credit-sensitive securities. This is unfortunate. Structural models have strengths that reduced-form models lack, including intuitive appeal and a natural link to equity markets, which contain a great deal of information about the financial health of firms.

A recent development is a class of structural/reduced-form hybrids that incorporate both the intuitive appeal of the structural approach and the pricing and calibration capabilities of the reduced-form approach. These hybrids are based on the economic assumption that the inputs to a structural model may not be observable. Single-firm incomplete information credit models have been studied by Duffie and Lando (2001), Giesecke (2001), Çetin *et al* (2002) and Giesecke and Goldberg (2003, 2004).

The integration of structural and reduced-form credit models is not straightforward. Reduced-form models are based on the assumption that default comes by surprise. In contrast, most structural models implicitly assume that default can be anticipated. The hypothesis of incomplete information in a structural model increases the degree of uncertainty around the default event, especially in the near term. As a result, the assumptions underlying the structural model become compatible with the reduced form. The two approaches can be integrated and their best features can be combined.

In this article, we develop a multi-firm incomplete information model of dependent defaults. Ours is a first-passage structural model: a firm defaults when its value falls below a barrier. However, unlike many first-passage models described in the literature, investors do not observe a firm's value or default barrier. We assume that there is a consensus view on the joint distribution of firms' values and default barriers. Investors observe defaults as they occur, at which time they update their assessment of firms' distances to default in accordance with the new information.

Our model assumptions imply the existence of an (endogenously defined) intensity for each default. As in all reduced-form models, compensation for the short-term default risk is described by the *pricing trend*, or cumulative intensity, discussed by Giesecke (2001) and others. We give explicit formulae for the pricing trend and its intensity in the case of dependent defaults. Both are functions of fundamental firm variables, including asset volatilities, asset correlations, reported debt, correlations in firms' debt levels and the quality of reported information.

Based on the pricing trend, we establish tractable, generalized reduced-form formulae for the prices of securities that depend on a sequence of dependent default arrivals, such as kth-to-default swaps. The trend can also be used in an algorithm for the simulation of sequential, dependent, unpredictable default times. This algorithm can be used in the simulation-based pricing of more complex products.

As in many structural models, the dependence of a firm on common factors is introduced through correlation in the firm value diffusion processes. Price jumps due to contagion arise from dependence of the default barriers across firms. This is because a firm's debt level depends on that of its counterparties. For most non-financial firms, good financial health is reflected in a low leverage ratio. If a firm defaults, its counterparties may suffer losses. To compensate the counterparties raise new debt, so their leverage increases. This creates a jump in the level of the default barrier. Thus, the default barriers of different firms are dependent.[1]

An interesting feature of our model is the *learning* processes alluded to above. The default of a firm reveals information about the value of its assets and default barrier. Investors revise their assessments of the financial health of counterparty firms in accordance with the new information. This generates a jump in model prices of credit-sensitive securities issued by the counterparty firms.[2] These "contagious jumps" can be fitted to credit market data, which show that the default of a major firm is often followed by a market-wide increase in spreads. Unlike the reduced-form models proposed in the literature, our pricing formulae take these discontinuities into account in a tractable way.

The paper is organized as follows. In Section 2, we review a one-firm incomplete information structural model. In Section 3, we consider first-to-default securities, while in Section 4 we extend to the general case of sequential dependent defaults. In Section 5, we provide examples and a calibration procedure. Section 6 concludes. All proofs are in the Appendix.

2 Pricing trend

To provide some intuition about our approach, we review the single-entity case. The results in Section 2.1 are special cases of the results derived in Giesecke (2001). Subsequent sections extend these results to the general multi-entity situation.

2.1 A structural model of default

Uncertainty is modeled by a probability space $(\Omega, \mathcal{G}, P)$ equipped with a filtration $(\mathcal{G}_t)_{t\geq 0}$ that describes the information flow over time. P is some equivalent martingale measure with respect to a given risk-free rate, r. For simplicity in the exposition we assume that r is a positive constant.

[1] There may be also other, systematic factors that drive this dependence. Conversely, dependent defaults may feed back into the joint dynamics of firm values. We do not consider these effects here.

[2] A similar learning process is present in the model of Collin-Dufresne, Goldstein and Helwege (2002), in which investors observe lagged firm values. All firms are assumed to share a common accounting quality in that they share the same, unobserved, lag. Investors update the lag distribution at firm defaults.

Investors trade securities issued by a single firm. The log of the issuer's firm value is denoted by $V \in \mathbb{R}$, and we assume that the issuer defaults when V falls to some *random* barrier $D < V_0$. We assume that D is independent of V. Formally, the firm's default time is a random variable $\tau \in (0, \infty]$ given by

$$\tau = \inf\{t > 0 : V_t \leq D\} \tag{1}$$

The associated default indicator process, N, is defined by $N_t = 1_{\{t \geq \tau\}}$. That is, N is zero before default and jumps to one at default.

Traditional structural models assume that investors can observe V and D. If V has continuous paths, then default can be anticipated: there exists a non-decreasing sequence of pre-default times that converges to τ almost surely. At a pre-default time, V falls dangerously close to the barrier. The sequence of pre-default times announces or "foretells" the default and we say that τ is *predictable* with respect to the filtration generated by V and D.

In practice it is difficult to observe either the assets of an issuer or the barrier at which the firm defaults. Therefore we drop the assumption of complete information. We assume that investors can observe the firm's default but cannot observe V and D. Accordingly, we model the information flow by the right-continuous and completed filtration $(\mathcal{G}_t)$ generated by the default indicator process N. With incomplete information, investors are uncertain about the distance of the firm to default, so that a default comes completely unexpectedly. In this situation there is no predictable time which agrees with τ on a set of positive measure and we say that τ is totally inaccessible, or *unpredictable*.

For unpredictable defaults, we can establish tractable generalized reduced-form formulae of default probabilities and prices of credit-sensitive securities in terms of the *pricing trend* (see Giesecke, 2001). The trend is a non-decreasing function A starting at zero such that the difference $N_t - A_{t\wedge\tau}$ defines a martingale with respect to the filtration $(\mathcal{G}_t)$.[3] Here $A_{t\wedge\tau}$ is the function A_t stopped at default; this process is called the *default compensator*. If the trend is absolutely continuous with respect to Lebesgue measure,

$$A_t = \int_0^t \lambda_s \mathrm{d}s \tag{2}$$

then the density λ is called the *intensity* of τ. For $t < \tau$, the intensity λ_t gives the conditional rate of default at time t. If τ is predictable as in the traditional structural models, then $A_{t\wedge\tau} = N_t$, so an intensity in the sense of (2) does not exist.

We consider the valuation of a security (T, X) that pays a bounded amount $X \in \mathcal{G}_T$ at T if there is no default by T and zero otherwise.

PROPOSITION 1 *Suppose that the pricing trend is continuous. Furthermore, suppose*

[3] We use the standard notation $a \wedge b = \min(a, b)$.

that the process defined by $E[X \mid \mathcal{G}_t]$ *is continuous at* τ, *almost surely.*[4] *The price of the security* (T, X) *at time* $t \le T$ *is given by*

$$E\left[e^{-r(T-t)} X 1_{\{\tau > T\}} \mid \mathcal{G}_t\right] = E\left[X e^{-r(T-t)+A_t - A_T} \mid \mathcal{G}_t\right]$$

for $t < \tau$ *and zero otherwise.*

Without loss of generality we normalize $V_0 = 0$. The following basic result characterizes the pricing trend in terms of the prior distribution function G of the default barrier $D \in (-\infty, 0)$ and the prior distribution function $H(t, \cdot)$ of the *running minimum log-firm value* M_t, defined by $M_t = \min_{s \le t} V_s$. We assume that $H(t, \cdot)$ admits a density $h(t, \cdot)$ for all $t > 0$.

PROPOSITION 2 *Let* $h(t, x)$ *be continuous in* t *for fixed* $x \le 0$. *Then the pricing trend is given by*

$$A_t = -\log \int_{-\infty}^{0} G(x)\, h(t, x)\, \mathrm{d}x \tag{3}$$

If the derivative $\frac{\partial}{\partial t} h(t, x)$ *is well-defined and uniformly bounded for* $x \le 0$, *then there exists an intensity given by* $\lambda_t = \frac{\partial}{\partial t} A_t$.

We note that the trend A_t and its intensity λ_t are deterministic functions of time t. The default compensator $A_{t \wedge \tau}$ is a function of τ and thus depends on the state of the world $\omega \in \Omega$. We suppress this dependence in the sequel.

Dellacherie and Meyer (1982, Theorem IV.78) show that the default time is unpredictable if and only if the default compensator is continuous. The continuity of the density function $h(\cdot, x)$ for fixed $x \le 0$ is therefore necessary and sufficient for the default to be unpredictable.

To calculate the trend with Proposition 2, we need the density $h(t, \cdot)$ of the running minimum log-firm value and the default barrier distribution G.

EXAMPLE 1 *Suppose that* V *follows a Brownian motion with drift* r *and volatility* $\sigma > 0$ *under the pricing probability* P. *Then* $h(t, x)$ *is given by*

$$h(t, x) = \frac{1}{\sigma\sqrt{t}}\, \phi\!\left(\frac{rt - x}{\sigma\sqrt{t}}\right) + e^{2rx/\sigma^2}\left[\frac{2r}{\sigma^2}\, \Phi\!\left(\frac{x + rt}{\sigma\sqrt{t}}\right) + \frac{1}{\sigma\sqrt{t}}\, \phi\!\left(\frac{x + rt}{\sigma\sqrt{t}}\right)\right]$$

for $x \le 0$ *and* $t > 0$. *Here* Φ *is the standard normal distribution function with density* ϕ. *The function* $h(\cdot, x)$ *is continuous on* $(0, \infty)$, *so that the default is unpredictable.*

EXAMPLE 2 *A family of barrier distributions under* P *is given by*

$$G(x) = e^{\alpha x}, \qquad x \le 0,\ \alpha \ge 0$$

[4] That is, the sample path $E[X \mid \mathcal{G}_t](\omega)$ is continuous on $\tau(\omega)$ for almost all $\omega \in \Omega$.

Our approach is based on a model definition of default. The corresponding trend (3) and its intensity are defined in terms of fundamental firm variables. These include the asset volatility σ embedded in the density h and the default barrier parameter α embedded in the barrier distribution G. These variables model investors' uncertainty about the unobserved asset value and default barrier, respectively. They can be used to calibrate the degree of confidence an investor feels about the information that is publicly available.

We contrast this with reduced-form models. In these models the dependence of the default intensity on fundamental firm variables is always exogenous, in the sense that the functional relationship between intensity and explanatory variable is *ad hoc*. It is not implied by a model definition of default, which is absent from the reduced-form models.

2.2 Simulating the default

In some valuation problems, the estimation of security prices by Monte Carlo methods is unavoidable. We describe a way to simulate the default time in our incomplete information model. A standard method is path simulation. Here one simulates a path of V and a large number of independent copies of D from the barrier distribution G. The average over the corresponding first-passage times provides a sample of the default time in our model. This method can be computationally expensive and the discretization of V may lead to biases.

If the distribution function p of τ is invertible, we can resort to another standard method of simulation. We just take $p^{-1}(U)$ for some standard uniform random variate U. This algorithm can be equivalently formulated in terms of the pricing trend, which we calculated in Proposition 2. For a given continuous and non-decreasing function A, the idea is to construct an unpredictable stopping time δ such that $A_{\cdot\wedge\delta}$ is its compensator in the right-continuous and completed filtration generated by the indicator process of δ.

Basic algorithm

1. Simulate an independent standard uniform random variable U.
2. Set $\delta = \inf\{t \geq 0 : A_t \geq -\log U\}$.

In our incomplete information model, the trend comes from a first-passage definition of default which is absent from the reduced-form models. In the latter the trend comes from some exogenously specified intensity, u, by setting $A_t = \int_0^t u_s \mathrm{d}s$. In this case the basic algorithm is often used to define a default time that has intensity u; see Lando (1998), for example.

3 First-to-default pricing trend

We extend our analysis to a universe of n dependent firms. In this section, we focus on the first to default. The more general kth-to-default case is described in Section 4.

3.1 A multi-firm structural model

Given the dynamics of its log-firm value V^i satisfying $V_0^i = 0$, we assume that firm $i \in \{1, 2, \ldots, n\}$ defaults when V^i falls to the *random* barrier $D^i \in (-\infty, 0)$ for the first time:

$$\tau^i = \inf\{t > 0 \colon V_t^i \leq D^i\} \tag{4}$$

As for the single-entity model, investors observe the defaults of firms as they occur. However, investors have no information on the value of the firms' assets or default barriers. Accordingly, the information flow in the bond market is modeled by the right-continuous and completed filtration $(\mathcal{G}_t)$ generated by the default indicator processes associated with the τ^i. Investors' uncertainty about the default barrier vector $D = (D^1, \ldots, D^n)$ is described by a continuous prior distribution function G. We assume that D is independent of firm values.

Suppose that if $i \neq j$, $P[\tau^i = \tau^j] = 0$. The ordered sequence (τ^i) of default times is denoted by (T^i). Let N^1 be the indicator process associated with $T^1 = \min_{1\leq i\leq n} \tau^i$. The *first-to-default pricing trend* A^1 is a non-decreasing function starting at zero such that the process defined by $N_t^1 - A_{t\wedge T^1}^1$ is a martingale with respect to the right-continuous and completed filtration $(\mathcal{G}_t^1)$ generated by the indicator process N^1. The filtration $(\mathcal{G}_t^1)$ enables us to distinguish the first default, but not which firm defaulted. The filtration $(\mathcal{G}_t)$ is finer: it enables us to distinguish the first default and which firm defaulted. If A^1 is absolutely continuous, then the corresponding density is called the intensity λ_1. Before the first default, λ_1 describes the conditional first-to-default rate.

The trend furnishes tractable reduced-form formulae for first-to-default arrival probabilities and prices of securities whose payoffs depend on the first default in a basket of names. We consider the valuation of a security $(1, T, X)$ that pays a bounded amount $X \in \mathcal{G}_T^1$ at T if $T^1 > T$ and zero otherwise.

PROPOSITION 3 *Suppose that the first-to-default pricing trend A^1 is continuous. Furthermore, suppose that the process defined by $E[X \mid \mathcal{G}_t^1]$ is continuous at T^1, almost surely. The price of the security $(1, T, X)$ at time $t \leq T$ is given by*

$$E\Big[\mathrm{e}^{-r(T-t)} X 1_{\{T^1 > T\}} \Big| \mathcal{G}_t\Big] = E\Big[X \mathrm{e}^{-r(T-t)+A_t^1 - A_T^1} \Big| \mathcal{G}_t^1\Big]$$

for $t < T^1$ and zero otherwise.

We characterize A^1 explicitly in terms of G and the prior distribution, $H(t, \cdot)$, of the running minimum log-firm value vector $M_t = (M_t^1, \ldots, M_t^n)$, where $M_t^i = \min_{s\leq t} V_s^i$. We assume that $H(t, \cdot)$ admits a density $h(t, \cdot)$ for all $t > 0$. The continuity of $h(\cdot, x)$ on $(0, \infty)$ for fixed $x \in \mathbb{R}_-^n$ is a necessary and sufficient condition for the first-to-default time to be unpredictable in $(\mathcal{G}_t)$.

PROPOSITION 4 *If the density $h(t,x)$ is continuous in t for $x \in \mathbb{R}^n_-$, the first-to-default pricing trend is given by*

$$A^1_t = -\log \int_{\mathbb{R}^n_-} G(x)\, h(t,x)\, \mathrm{d}x \qquad (5)$$

If the derivative $\frac{\partial}{\partial t} h(t,x)$ is well-defined and uniformly bounded for $x \in \mathbb{R}^n_-$, then there exists a first-to-default intensity given by $\lambda^1_t = \frac{\partial}{\partial t} A^1_t$.

The trend A^1_t and its intensity λ^1_t are deterministic functions of time, t. To calculate A^1_t, we need the density, $h(t,\cdot)$, of the running minimum log-firm value vector and the joint distribution, G, of the default barriers.

EXAMPLE 3 *Suppose that, under the pricing probability P, $(V^1,\ldots,V^n)$ follows an n-dimensional Brownian motion. If V^i has drift r and volatility σ_i as in Example 1, then the marginal density, $h^i(t,\cdot)$, of firm i's historical log-asset low, M^i_t, is given by*

$$h^i(t,x) = \frac{1}{\sigma_i\sqrt{t}}\,\phi\!\left(\frac{rt-x}{\sigma_i\sqrt{t}}\right) + \mathrm{e}^{2rx/\sigma_i^2}\left[\frac{2r}{\sigma_i^2}\,\Phi\!\left(\frac{x+rt}{\sigma_i\sqrt{t}}\right) + \frac{1}{\sigma_i\sqrt{t}}\,\phi\!\left(\frac{x+rt}{\sigma_i\sqrt{t}}\right)\right]$$

The joint density $h(t,\cdot)$ of the vector M_t in case $n = 2$ was derived by Iyengar (1995); see also Zhou (2001) and He, Keirstead and Rebholz (1998). For the higher-dimensional case see Wise and Bhansali (2004). For fixed $x \in \mathbb{R}^n_-$, the function $h(\cdot, x)$ is continuous on $(0,\infty)$, so that T^1 is unpredictable.

EXAMPLE 4 *A family of joint distributions G for the barrier vector D is specified by the marginals $G^i(x) = \mathrm{e}^{\alpha_i x}$ and a copula function C through the relation*

$$G(x_1,\ldots,x_n) = C(\mathrm{e}^{\alpha^1 x_1},\ldots,\mathrm{e}^{\alpha^n x_n}), \qquad x_i \le 0,\ \alpha^i \ge 0 \qquad (6)$$

In Proposition 4, we define the first-to-default trend in terms of individual firm variables, as mentioned in Section 2, and variables characterizing the dependence structure between firms. This includes the asset correlation matrix embedded in the joint density h and the default barrier copula C embedded in the joint distribution G. In Section 5 below we discuss parametric examples for C and study the effect of the corresponding barrier dependence on the first- and second-to-default arrival.

We construct the first-to-default intensity incorporating cyclical dependence and contagion effects without reference to the single-entity intensities that we considered in Section 2. In contrast, Duffie (1998) constructs the intensity of the first-to-default as the sum of the single-entity intensities $\lambda_t(i)$:

$$\lambda_t^1 = \sum_{i=1}^{n} \lambda_t(i), \qquad t \geq 0 \tag{7}$$

without explicit reference to the joint distribution of the default times. In the latter case, cyclical correlation can be modeled through correlation of the diffusion components of the single-entity intensities and contagion can be modeled by specifying the correlations of their jump components. As in Jarrow and Yu (2001) for example, the correlations among jump terms are specified in terms of the joint distribution of the default times. Thus, this joint distribution is implicitly represented in formula (7) although it does not explicitly appear.

3.2 Simulating the first-to-default

Based on the first-to-default trend, we simulate an unpredictable first-to-default time δ^1. That is, based on the continuous non-decreasing function A^1, we construct a stopping time δ^1 having compensator $A^1_{\cdot \wedge \delta^1}$. This is equivalent to simulating δ^1 by the standard inverse method from the distribution function of T^1. Conditional on δ^1, we simulate the identity of the first defaulter.

Let $I^1 = \sum_{i=1}^{n} i 1_{\{T^1 = \tau^i\}}$ be the index of the firm that defaults first. This is a random variable valued in $\{1, 2, \ldots, n\}$. Suppose that

$$P[I^1 = i, T^1 \leq t] = \int_0^t q^1(i,s)\,\mathrm{d}s, \qquad i \in \{1, 2, \ldots, n\}$$

Suppose furthermore

$$P[T^1 \leq t] = \int_0^t l^1(s)\,\mathrm{d}s$$

First-to-default algorithm

1. Simulate an independent standard uniform random variable U.
2. Set $\delta^1 = \inf\{t \geq 0 : A_t^1 \geq -\log U\}$.
3. Simulate a random variable $I^1 \in \{1, 2, \ldots, n\}$ with conditional distribution given by

$$P[I^1 = i \mid \mathcal{G}_t^1] = \frac{q^1(i, \delta^1)}{l^1(\delta^1)}, \qquad t \geq \delta^1 \tag{8}$$

A reduced-form alternative to this algorithm is described by Duffie (1998). It uses the representation (7) of the first-to-default intensity. In this case the trend A^1 and the distribution (8) come from the exogenously specified single-entity intensities. In our framework, these quantities are derived from a model definition of default.

To give an explicit expression for the distribution (8), suppose that each V^i is continuous so that $\{\tau^i = t\} = \{D^i = M_t^i\}$. If the partial derivative, $G_{z_i}(z_1, \dots, z_n)$, of G with respect to its ith argument is well-defined,

$$q^1(i,t) = \int_{\mathbb{R}_-^n} G_{z_i}(x)\,h(t,x)\,\mathrm{d}x$$

If the derivative $\dot{h}(t,x) = \frac{\partial}{\partial t} h(t,x)$ is well-defined and uniformly bounded for $x \in \mathbb{R}_-^n$, we can write

$$l^1(t) = -\int_{\mathbb{R}_-^n} G(x)\,\dot{h}(t,x)\,\mathrm{d}x$$

4 Sequential dependent defaults

4.1 Pricing trends

As a result of the macroeconomic and contagion effects described above, defaults often arrive in clusters. Prices of kth-to-default securities change, often abruptly, as default information becomes publicly available. Therefore, in order to price these securities, we need to define an information structure that is updated in accordance with default events.

As above, for $i = 1, 2, \dots, n$, let τ^i denote the default time of firm i. Suppose that if $i \neq j$, $P[\tau^i = \tau^j] = 0$. Let (T^k) be the ordered sequence of default times with indicator processes defined by $N_t^k = 1_{\{t \geq T^k\}}$. We set $T^0 = 0$. Let $I^k \in \{1, 2, \dots, n\}$ denote the identity of the kth defaulting entity.

We use a tower of filtrations to keep track of the information generated by the sequential defaults. Let $\mathcal{F}$ be the σ-algebra generated by Ω and the null sets in $\mathcal{G}$. For $t \geq 0$, set $\mathcal{F}_t^0 = \mathcal{F}$. For $1 \leq k \leq n$, let $(\mathcal{F}_t^k)$ be the filtration generated by the indicator processes N^i for $1 \leq i \leq k$, the random variables I^i for $1 \leq i \leq k$ and the null sets in $\mathcal{G}$. Then $\mathcal{F}_t^k$ is the smallest σ-algebra that is refined enough to distinguish the times and identities of the first k defaults that have occurred by time t.

Once $(k-1)$ defaults have occurred, investors in a kth-to-default basket are very concerned. Since the payoff is triggered by the next default, independent of its identity, we introduce the filtration $(\mathcal{G}_t^k)$, which is the filtration $(\mathcal{F}_t^{k-1})$ progressively enlarged by T^k. That is, $\mathcal{G}_t^k = \mathcal{F}_t^{k-1} \vee \sigma(N_t^k\colon s \leq t)$. This filtration is finer than $(\mathcal{F}_t^{k-1})$ but no finer than $(\mathcal{F}_t^k)$. The filtration $(\mathcal{G}_t^k)$ enables us to distinguish the first $(k-1)$ defaults and their identities and the kth default time. It is not refined enough to distinguish the kth defaulting entity.

The *kth-to-default pricing trend* A^k is a $(\mathcal{F}_t^{k-1})$-adapted, non-decreasing stochastic process starting at zero such that the process defined by $N_t^k - A_{t \wedge T^k}^k$ is a martingale with respect to the filtration $(\mathcal{G}_t^k)$. If A^k is absolutely continuous, the density λ^k is called the intensity of the kth default. Note that $A_t^k = \lambda_t^k = 0$ for $t \in [0, T^{k-1}]$.

The trend furnishes convenient and tractable reduced-form formulae for kth-to-default arrival probabilities and prices of securities whose payoffs depend on the kth default in a basket of names. We consider the valuation of a security (k, T, X) that pays a bounded amount $X \in \mathcal{G}_T^k$ at T if $T^k > T$ and zero otherwise.

PROPOSITION 5 *Suppose that the kth-to-default pricing trend A^k is continuous. Furthermore, suppose that the process defined by $E[X \mid \mathcal{G}_t^k]$ is continuous at T^k, almost surely. The price of the security (k, T, X) at time $t \leq T$ is given by*

$$E\Big[\mathrm{e}^{-r(T-t)} X 1_{\{T^k > T\}} \Big| \mathcal{G}_t\Big] = E\Big[X \mathrm{e}^{-r(T-t)+A_t^k - A_T^k} \Big| \mathcal{G}_t^k\Big]$$

for $t < T^k$ and zero otherwise.

We show that the kth-to-default pricing trend can be written in a convenient form with respect to the information generated by the first $(k-1)$ defaults. We denote by G the continuous prior distribution of D. We assume that the vector $(M_{t_1}^1, \ldots, M_{t_n}^n)$ admits a density $h(t_1, \ldots, t_n; \cdot)$.

PROPOSITION 6 *Fix some $k \geq 2$ and assume that the first default times have identities $I^i = i$ for $i \in \{1, 2, \ldots, k-1\}$. Assume that the partial derivative, $G_{z_1 \cdots z_{k-1}}$, of G with respect to its first $k-1$ arguments is well-defined. Also assume that each V^i is continuous. If $h(\tau^1, \ldots, \tau^{k-1}, t, \ldots, t; x)$ is continuous in t for $x \in \mathbb{R}_-^n$, the kth-to-default pricing trend is given by*

$$A_t^k = -\log \frac{\int_{\mathbb{R}_-^n} G_{z_1 \cdots z_{k-1}}(x) h(\tau^1, \ldots, \tau^{k-1}, t, \ldots, t; x) \mathrm{d}x}{\int_{\mathbb{R}_-^n} G_{z_1 \cdots z_{k-1}}(x) h(\tau^1, \ldots, \tau^{k-1}, \tau^{k-1}, \ldots, \tau^{k-1}; x) \mathrm{d}x}$$

on $\{t > T^{k-1}\}$. If $\dot{h}(\tau^1, \ldots, \tau^{k-1}, t, \ldots, t; x) = \frac{\partial}{\partial t} h(\tau^1, \ldots, \tau^{k-1}, t, \ldots, t; x)$ is well-defined and uniformly bounded for $x \in \mathbb{R}_-^n$, then there exists an intensity for the k-to-default time given by $\lambda_t^k = \frac{\partial}{\partial t} A_t^k$.

The assumption $I^i = i$ is made for notational convenience only. For arbitrary identities of the first $k-1$ defaulters the calculations are analogous.

4.2 Simulating sequential defaults

We simulate $m \leq n$ sequential unpredictable default times $\delta^1, \ldots, \delta^m$ by iterating the first-to-occur algorithm. Given the continuous non-decreasing function A^1 provided by Proposition 4, we construct a first-to-default stopping time δ^1. Conditional on δ^1, we simulate the identity of the first defaulter. Then we iterate using Proposition 6: at each step we use information through the $(k-1)$st default to generate the kth default. A variant of this algorithm is known in the reliability literature as the "total hazard construction" (see Shaked and Shanthikumar, 1987). It has been applied in the credit literature by Duffie and Singleton (1998) and Yu (2003).

We set $R^0 = \{1, 2, \dots, n\}$. For $1 \le k \le n$, let $R^k \subset \{1, 2, \dots, n\}$ denote the index set of the firms that survived the kth default. Let $I^k = \sum_{i \in R^{k-1}} i 1_{\{T^k = \tau^i\}}$ be the index of the kth defaulter. This is a random variable valued in R^{k-1}. Let $I^0 = 0$. Suppose that

$$P\left[I^k = i, T^k \le T \,\middle|\, \mathcal{F}_t^{k-1}\right] = \int_{T^{k-1}}^{T} q_t^k(i, s)\,\mathrm{d}s, \qquad T \ge T^{k-1}, \quad i \in R^{k-1}$$

Suppose furthermore

$$P\left[T^k \le T \,\middle|\, \mathcal{F}_t^{k-1}\right] = \int_{T^{k-1}}^{T} l_t^k(s)\,\mathrm{d}s, \qquad T \ge T^{k-1}$$

In the first-to-default case we calculated the densities $q^1(i, s) = q_0^1(i, s)$ and $l^1(s) = l_0^1(s)$ explicitly in terms of G and h. Analogous calculations lead to similar expressions in the general case.

Sequential event algorithm

1. Initialize $k = 1$.
2. Simulate an independent standard uniform random variable U^k.
3. Set $\delta^k = \inf\{t \ge 0 : A_t^k \ge -\log U^k\}$.
4. Simulate a random variable $I^k \in R^{k-1}$ with conditional distribution

$$P\left[I^k = i \,\middle|\, \mathcal{G}_t^k\right] = \frac{q_t^k(i, \delta^k)}{l_t^k(\delta^k)}, \qquad t \ge \delta^k$$

5. Set $R^k = R^{k-1} \setminus I^k$.
6. If $k = m$ then stop, else set $k = k + 1$ and go back to Step (2).

5 Examples: default baskets

The effects of firm value correlation on joint default probabilities and the pricing of default baskets have been studied, for example, by Zhou (2001) and Bluhm, Overbeck and Wagner (2003, Chapter 7). The required firm value correlations can be estimated from equity market data as described by Crouhy, Galai and Mark (2000). We focus on the less familiar topic of modeling and calibrating default barrier dependence and its effects on the prices of default baskets. The contagion effects associated with the dependence of default barriers are discussed in more detail by Giesecke (2004).

We use copulas to model default barrier dependence. The advantages and disadvantages of copulas with respect to linear correlation are widely discussed in the credit literature. The main disadvantage is the *ad-hoc* nature of the copula. There may be no intrinsic choice of copula for a given problem, and different choices can lead to wildly different model properties. Linear correlation addresses this issue by providing no choice at all. It implicitly specifies an elliptic dependence structure, which may not take account of the temporal clustering of large events. By contrast, a copula describes the complete non-linear dependence structures among the barriers, irrespective of their joint distribution type. Consequently, the copula separates the dependence from the marginal behavior. This allows us to disentangle the effects of barrier dependence on pricing characteristics from marginal effects. Further, the problem of calibrating G is partitioned into two independent sub-problems: calibration of the marginals G^i and calibration of the copula C.

5.1 Parametric copula families

We consider a basket with $n = 5$ firms, whose log-firm values, V^i, follow independent arithmetic Brownian motions with drift r and volatilities σ_i. The density of $(M^1_{t_1}, \ldots, M^n_{t_n})$ is then $h(t_1, \ldots, t_5, x) = h^1(t_1, x_1) \cdots h^5(t_5, x_5)$, where the marginal density h^i is given in Example 3. We start by modeling the barrier dependence structure by the Clayton copula family. As is not uncommon for default baskets, we suppose that the dependence structure in the basket is symmetric. We can therefore choose the one-parameter version of the Clayton family, which is given by

$$C^C_\theta(u_1, \ldots, u_5) = (u_1^{-\theta} + \cdots + u_5^{-\theta} - 4)^{-1/\theta}, \qquad u_i \in [0, 1], \quad \theta > 0 \tag{9}$$

The parameter θ controls the degree of barrier dependence: $\theta \to \infty$ reflects perfect positive dependence, and $\theta \to 0$ corresponds to independence. The degree of monotonic barrier dependence can be measured by Kendall's pairwise *rank correlation*[5] $\rho^K \in [-1, 1]$. We have $\rho^K = -1$ iff the barriers are perfectly negatively related, $\rho^K = 1$ iff they are perfectly positively related, and $\rho^K = 0$ in case of independence. For the Clayton family we have $\rho^K = \theta/(\theta + 2)$ (with $\theta > 0$, ρ^K is positive as well and (9) expresses positive dependence). Assuming that the default barrier of firm i has distribution function $G^i(x) = \mathrm{e}^x$ (Example 4 with $\alpha^i = 1$), from (6) the joint barrier distribution is

$$G^C_\theta(x_1, \ldots, x_5) = (\mathrm{e}^{-\theta x_1} + \cdots + \mathrm{e}^{-\theta x_5} - 4)^{-1/\theta}, \qquad x_i \le 0 \tag{10}$$

We consider a first-to-default basket. In Figure 1, we plot the term structure of survival probabilities

$$L^1_t = P[T^1 > t] = \int_{\mathbb{R}^5_-} G^C_\theta(x)\, h(t, \ldots, t; x)\, \mathrm{d}x$$

for varying degrees of rank barrier correlation, ρ^K (for the calculation see the

[5] See Nelsen (1999) for a formal definition.

FIGURE 1 Term structure of first-to-default survival probability, L_T^1, in per cent. We set $r = 6\%$ and $\sigma_i = 20\%$ and vary the rank barrier correlation $\rho^K \in \{0, 0.5, 0.99\}$.

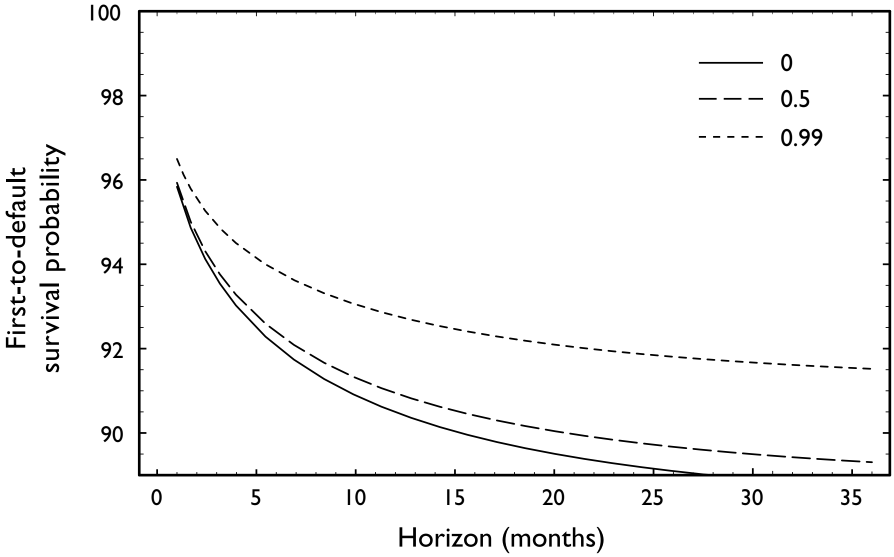
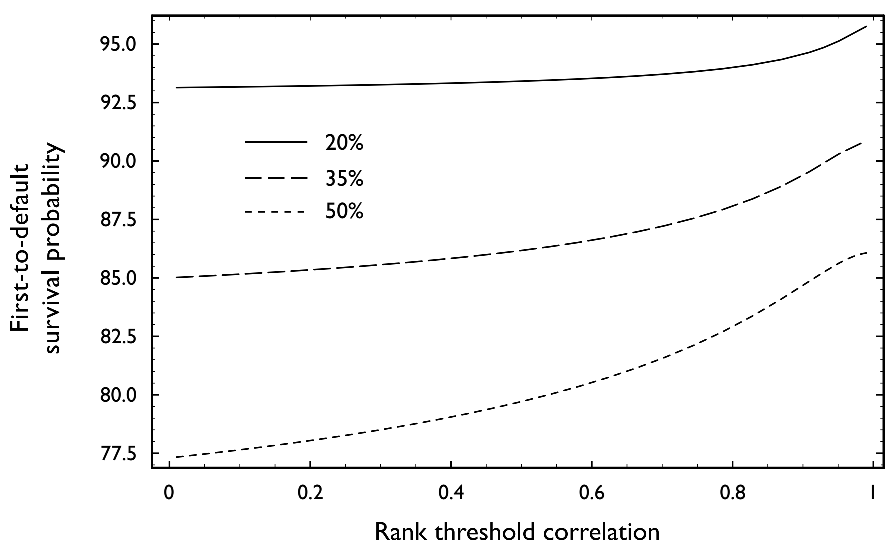


FIGURE 2 First-to-default survival probability, L_T^1, in per cent, as a function of rank barrier correlation, ρ^K. We set $r = 6\%$ and $T = 12$ months and vary the asset volatility $\sigma_i \in \{20\%, 35\%, 50\%\}$.



proof of Proposition 4 in the Appendix). We set $r = 6\%$ and $\sigma_i = 20\%$. In terms of asset volatility, the firms are of high quality, so individual default probabilities are low. Positive barrier dependence ($\rho^K > 0$) induces positive default dependence.[6] For a given horizon, L_T^1 is increasing in the degree of default dependence. This effect is more noticeable in Figure 2, which shows L_T^1 for $T = 12$ months as a function of rank barrier correlation, ρ^K, for varying asset volatilities.

Intuitively, positive monotonic barrier dependence as measured by $\rho^K > 0$ means that the default barriers of different firms are likely to cluster around a common level dictated by the common marginals G^i. In the extreme case $\rho^K = 1$, we have $D^i = D^j$ almost surely. Even with independent assets, the likelihood of several firms' assets hitting a similar barrier level before a given horizon is higher than with independent barriers, where such a clustering is not present. Positive asset correlation increases this likelihood further. Thus, holding individual default probabilities fixed, the higher the positive barrier dependence, the higher are joint default probabilities, and the higher is L^1. This can be most easily seen in case $n = 2$, where

$$L_t^1 = 1 - P[\tau^1 \leq t] - P[\tau^2 \leq t] + P[\tau^1 \leq t, \tau^2 \leq t] \tag{11}$$

We consider the value of first-to-default protection on the basket. With increasing default dependence the payoff probability $1 - L^1$ decreases and so does the value of first-to-default protection. Holding marginal default probabilities fixed, positive default dependence increases the likelihood of several firms defaulting, but the contract covers the first default only. The value of protection is maximal for perfectly negatively dependent defaults ($\rho^K = -1$). It is minimal for perfectly positively dependent defaults ($\rho^K = 1$).

For comparison, we consider the value of second-to-default protection on the basket. For our calculations we assume that the first-to-default time $T^1 = 6$ months and that the identity of the first defaulter is $I^1 = 1$. For the second-to-default survival probability we have $L_t^2 = 1$ for $t \leq \tau^1$ and

$$L_t^2 = P\left[T^2 > t \,\middle|\, \mathcal{F}_t^1\right] = \frac{\int_{\mathbb{R}_-^5} \frac{\partial}{\partial z_1} G_\theta^C(x) h(\tau^1, t, \ldots, t; x)\,\mathrm{d}x}{\int_{\mathbb{R}_-^5} \frac{\partial}{\partial z_1} G_\theta^C(x) h(\tau^1, \ldots, \tau^1; x)\,\mathrm{d}x}$$

for $t > \tau^1$ (see the proof of Proposition 6 in the Appendix). Figure 3 shows the term structure of second-to-default survival probabilities; we set $r = 6\%$ and $\sigma_i = 20\%$ as in the first-to-default case. We observe that for a given horizon, L_T^2 is decreasing in the degree of default dependence as measured by the rank correlation coefficient ρ^K. This is the opposite of what we found for first-to-default

[6] For a formal statement of the relationship between barrier copula C and the copula of $(\tau^1, \ldots, \tau^5)$ as a measure of default dependence, see Giesecke (2004).

FIGURE 3 Term structure of second-to-default survival probabilities, $L_2(T)$, in per cent. We set $T^1 = 6$ months, $r = 6\%$ and $\sigma_i = 20\%$ and vary the rank barrier correlation $\rho^K \in \{0, 0.5, 0.99\}$.

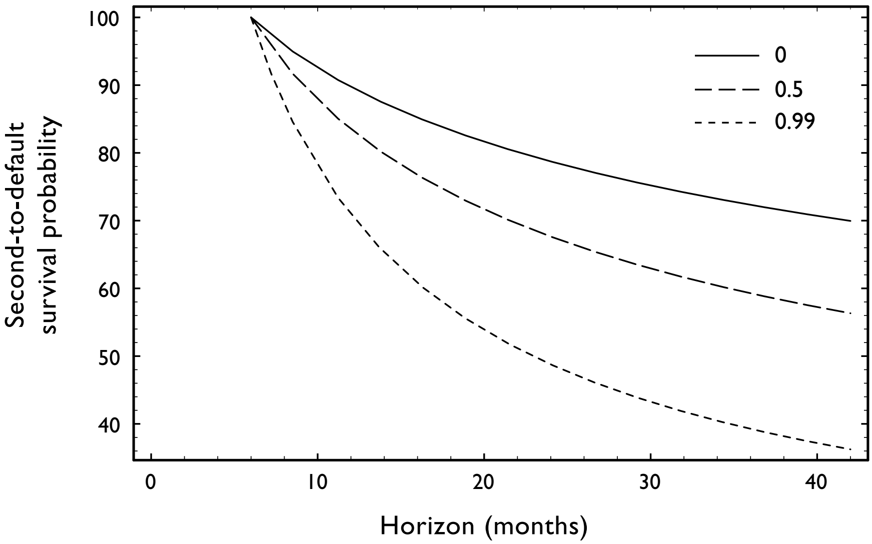


FIGURE 4 Term structure of second-to-default survival probabilities, L_T^2, in per cent. We set $T^1 = 6$ months, $r = 6\%$ and $T = 12$ months and vary the asset volatility $\sigma_i \in \{20\%, 35\%, 50\%\}$.

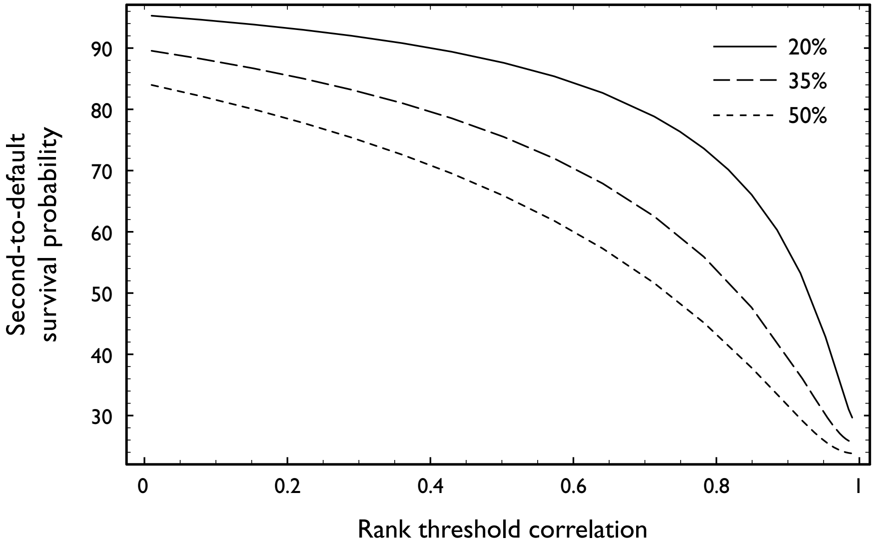
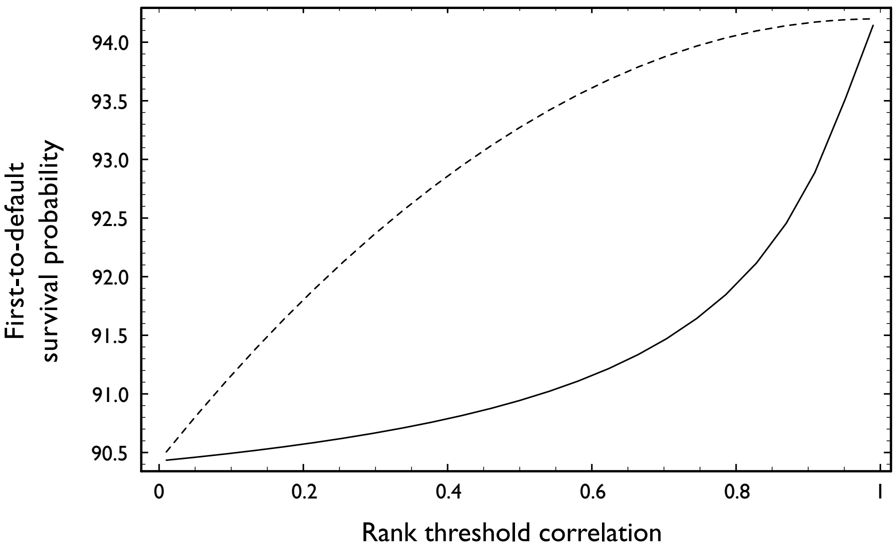


FIGURE 5 First-to-default survival probability, L_T^1, as a function of rank barrier correlation, ρ^K, for different copula families (*solid line*: Clayton; *dashed line*: Gumbel). We set $r = 6\%$, $\sigma_i = 20\%$ and $T = 12$ months.



survival probabilities L^1 – compare Figure 4 and Figure 2. With increasing default dependence the payoff probability $1 - L^2$ increases and so does the value of second-to-default protection.

Survival probabilities are sensitive to the choice of the copula family. To get some intuition for this, we consider the Gumbel family with parameter $\theta \geq 1$, given by

$$C_\theta^G(u_1, \dots, u_5) = \exp\left(-\left[(-\log u_1)^\theta + \dots + (-\log u_5)^\theta\right]^{1/\theta}\right), \qquad u_i \in [0, 1]$$

The value $\theta = 1$ corresponds to independence, while $\theta \to \infty$ reflects perfect positive dependence. For the Gumbel family the pairwise rank correlation is $\rho^K = 1 - 1/\theta$. With $G^i(x) = \mathrm{e}^x$, we have for the joint barrier distribution

$$C_\theta^G(x_1, \dots, x_5) = \exp\left(-\left[(-x_1)^\theta + \dots + (-x_5)^\theta\right]^{1/\theta}\right), \qquad x_i \leq 0 \tag{12}$$

The choice of the copula family has significant effects indeed on the resulting arrival probabilities. Figure 5 displays the 12-month first-to-default survival probability as a function of rank barrier correlation, ρ^K, for both Clayton and Gumbel barrier copulas. The asset volatility σ_i is set to 20% for all names. The differences in the survival probability for the two families are due to their tail dependence properties. The Gumbel copula exhibits upper tail dependence, which refers to the pronounced tendency of a copula to generate high values in all marginals simultaneously. All else being equal, this implies in turn an increased likelihood of joint defaults, which leads to higher survival probabilities, see (11). The Clayton copula exhibits lower tail dependence, which leads to opposite effects. Consequently, for a given horizon the first-to-default survival probability with the Gumbel barrier copula is at least as high as with the Clayton copula.

5.2 Calibrating the barrier copula

We calibrate a copula family C_θ with parameter vector $\theta \in \mathbb{R}^m$. The barrier dependence described by C_θ controls the effects of contagion in the underlying multi-firm structural model. If we observe price jumps in traded credit-sensitive securities due to contagion effects, we can calibrate θ. Instead of using market prices, we can also use anticipated spread or price jumps. This allows us to calibrate the contagion parameters to subjective beliefs. In fact, it is often more intuitive to estimate the jump in spreads given certain default scenarios in the market than to come up with an estimate of asset correlation.

Given the default of firm j at time t, let

$$\Delta_i^j(t,T), \qquad t \le T, \quad j \neq i$$

be the jump in the $\mathcal{G}_t$-conditional probability of survival of firm i by time T. Given appropriate assumptions on recovery rates and risk-free interest rates, $\Delta_i^j(t,T)$ can be backed out from quotes of traded bonds maturing at T or credit swap spreads of firm i around the default of issuers comparable to j. Another way is to calculate the jump in the survival probability, $\Delta_i^j(t,T)$, from the anticipated jump in bond prices or swap spreads under the appropriate default scenario.

We consider the instructive case with $n = 2$ issuers. Using Bayes' rule, we have for conditional survival probabilities

$$p^1(t,T) = P\left[\tau^1 > T \,\middle|\, \mathcal{G}_t\right] = \frac{p(T,t)}{p(t,t)} \qquad \text{on } \{T^1 > t\}$$

where $T^1 = \tau^1 \wedge \tau^2$ and $p(t,s) = P[\tau^1 > t,\, \tau^2 > s]$ is the joint survival probability

$$p(t,s) = \int_{-\infty}^{0} \int_{-\infty}^{0} C_\theta\big(G^1(x), G^2(y)\big)\, h(t,s;x,y)\, \mathrm{d}x\mathrm{d}y \tag{13}$$

If h is sufficiently smooth,

$$p^1(t,T) = \frac{\left.\dfrac{\partial}{\partial s} p(T,s)\right|_{s=t}}{\left.\dfrac{\partial}{\partial s} p(t,s)\right|_{s=t}} \qquad \text{on } \{T^1 = \tau^2 = t\}$$

For the jump in conditional default probabilities we therefore obtain

$$\Delta_1^2(t,T) = \frac{\left.\dfrac{\partial}{\partial s} p(T,s)\right|_{s=t}}{\left.\dfrac{\partial}{\partial s} p(t,s)\right|_{s=t}} - \frac{p(T,t)}{p(t,t)} \tag{14}$$

Suppose that we have calibrated a model for the joint asset dynamics and the barrier margin G^i. Given $\Delta_1^2(t,T)$, Equation (14) allows us to calibrate C_θ under the assumption that the barrier dependence structure is symmetric.

6 Conclusion

We have described a multi-firm structural credit model that is based on incomplete information. In this model investors observe neither a firm's value nor its default barrier. Cyclical default correlation effects are modeled through correlation in the diffusions of firm value. Contagion effects due to counterparty relations are modeled through dependence of default barrier processes. We explicitly constructed the pricing trend and the arrival intensity of the kth-to-default in terms of fundamental firm variables and provided sufficient conditions for the existence of that intensity. These results furnish tractable reduced-form formulae for the probabilities of sequential defaults and prices of multi-name credit derivatives. We have also presented an algorithm for the simulation of sequential dependent and unpredictable default times.

Appendix A Proofs

PROOF OF PROPOSITION 1 See Giesecke (2001, Theorem 4.5). □

PROOF OF PROPOSITION 2 The result is a special case of Giesecke (2001, Proposition 5.2 and Theorem 5.3). We give a direct proof that uses the specific structure of the filtration $(\mathcal{G}_t)$.

Let $L_t = P[\tau > t]$ denote the survival function of τ. Assume $L_t > 0$ for all $t > 0$ and $P[\tau = 0] = 0$. It is a classic result, due to Dellacherie (1970), that the compensator of N in the filtration $(\mathcal{G}_t)$ generated by N is given by $A_{\cdot \wedge \tau}$, with A defined by the Stieltjes integral

$$A_t = -\int_0^t \frac{\mathrm{d}L_s}{L_{s-}} \tag{A1}$$

Noting the independence of D and V and that $\{\tau > t\} = \{M_t > D\}$, we have for the survival function

$$L_t = P[D < M_t] = \int_{-\infty}^0 G(x)\,h(t,x)\,\mathrm{d}x \tag{A2}$$

If $h(t,x)$ is continuous in t for $x \leq 0$, then L is continuous, and we obtain from (A1) and (A2) that

$$A_t = -\log L_t = -\log \int_{-\infty}^{0} G(x)h(t,x)\,\mathrm{d}x \tag{A3}$$

which proves the first statement.

Under our smoothness hypothesis on h, Aven's (1985) conditions are satisfied, so there exists a function λ such that $A_t = \int_0^t \lambda_s \mathrm{d}s$. □

PROOF OF PROPOSITION 3 This is a special case of Proposition 5 below. □

PROOF OF PROPOSITION 4 We appeal to the results of Chou and Meyer (1975), who construct the compensator of a point process (T^k) in its own right-continuous and completed filtration. This is the completed filtration generated by the indicator processes N^k for $1 \leq k \leq n$. It enables us to distinguish the default arrivals, but not which firms defaulted.

Let $L_t^1 = P[T^1 > t]$. Assume $L_t^1 > 0$ for all $t > 0$ and $P[T^1 = 0] = 0$. Consider the function A^1 given by the Stieltjes integral

$$A_t^1 = -\int_0^t \frac{\mathrm{d}L_s^1}{L_{s-}^1} \tag{A4}$$

Proposition 3 of Chou and Meyer (1975) implies that $N_t^1 - A_{t\wedge T^1}^1$ defines a martingale in the filtration $(\mathcal{G}_t^1)$ generated by the first-to-default indicator N^1. In other words, $A_{t\wedge T^1}^1$ is the $(\mathcal{G}_t^1)$-compensator of N^1.

Noting that $\{\tau^i > t\} = \{M_t^i > D^i\}$ and that D is independent of V,

$$L_t^1 = P\left[\tau^1 > t, \ldots, \tau^n > t\right] = \int_{\mathbb{R}_-^n} G(x)h(t,x)\,\mathrm{d}x \tag{A5}$$

If $h(t,x)$ is continuous in t for fixed $x \leq 0$, then L_t^1 is continuous and (A4) implies that the trend is given by $A_t^1 = -\log L_t^1$.

If h is smooth, Aven's (1985) conditions are satisfied, so there exists a function λ^1 such that $A_t^1 = \int_0^t \lambda_s^1 \mathrm{d}s$. □

PROOF OF PROPOSITION 5 Without loss of generality we set $r = 0$. Let $K_t = E[X\mathrm{e}^{-A_T^k} \mid \mathcal{G}_t^k]$. Note that $K_t = \mathrm{e}^{-A_T^k} E[X \mid \mathcal{G}_t^k]$. Define $Y_t = K_t \mathrm{e}^{A_t^k}$. Since A^k is non-decreasing, it is of finite variation. Thus the quadratic covariation $[K, \mathrm{e}^{A_k}]$ is zero and integration by parts yields

$$\begin{aligned} \mathrm{d}\left(K_t \mathrm{e}^{A_t^k}\right) &= K_{t-}\mathrm{d}\left(\mathrm{e}^{A_t^k}\right) + \mathrm{e}^{A_t^k}\mathrm{d}K_t + \mathrm{d}\left[K, \mathrm{e}^{A_k}\right]_t \\ &= Y_{t-}\,\mathrm{d}A_k(t) + \mathrm{e}^{A_t^k}\mathrm{d}K_t \end{aligned}$$

Since N^k is a quadratic pure jump semimartingale, we have that

$$[1-N^k, Y]_t = (1-N_0^k)Y_0 + \sum_{0<s\leq t} \Delta(1-N_s^k)\Delta Y_s$$

see Protter (1990, Chapter II, Theorem 29). Since we assume that the process defined by $E[X \mid \mathcal{G}_t^k]$ does not jump at T^k, the process Y does not jump at T^k. Hence $\mathrm{d}[1-N^k, Y]_t = \Delta(1-N_t^k)\Delta Y_t = 0$ for $t \leq T$.

Consider the process $U = (1-N^k)Y$. Integrating by parts,

$$\begin{aligned}
\mathrm{d}U_t &= -Y_{t-}\mathrm{d}N_t^k + (1-N_{t-}^k)\mathrm{d}Y_t + \mathrm{d}[1-N^k, Y]_t \\
&= (1-N_{t-}^k)\,\mathrm{e}^{A_t^k}\mathrm{d}K_t - Y_{t-}(\mathrm{d}N_t^k - (1-N_{t-}^k)\,\mathrm{d}A_t^k) \\
&= (1-N_{t-}^k)\,\mathrm{e}^{A_t^k}\mathrm{d}K_t - Y_{t-}(\mathrm{d}N_t^k - \mathrm{d}A_{t\wedge T^k}^k) \\
&= (1-N_{t-}^k)\,\mathrm{e}^{A_t^k - A_T^k}\mathrm{d}Z_t - Y_{t-}\mathrm{d}(N_t^k - A_{t\wedge T^k}^k)
\end{aligned}$$

for $t \leq T$. Here $Z_t = E[X \mid \mathcal{G}_t^k]$ and $N_t^k - A_{t\wedge T^k}^k$ define bounded $(\mathcal{G}_t^k)$-martingales. Since the integrands $(1-N_{t-}^k)\exp(A_t^k - A_T^k)$ and Y_{t-} are bounded and $(\mathcal{G}_t^k)$-predictable, U is a $(\mathcal{G}_t^k)$-martingale. Thus

$$U_t = (1-N_t^k)Y_t = E[U_T \mid \mathcal{G}_t^k] = E[X(1-N_T^k) \mid \mathcal{G}_t^k] \tag{A6}$$

Our assertion follows by noting that

$$E[X(1-N_T^k) \mid \mathcal{G}_t^k] = E[X 1_{\{T^k > T\}} \mid \mathcal{G}_t]$$

on the set $\{t < T^k\}$. □

The proof of Proposition 4 relies on a result of Chou and Meyer (1975), which constructs the compensator of the point process (T^k) in its own filtration. We consider the compensator of (T^k) in the larger filtration that is generated by the sequence of pairs (T^k, I^k). We extend Chou and Meyer's (1975) representation result and then prove Proposition 6.

In order to provide this extension, we need to keep track of the identities of the defaulters. Denote by J the set of ordered tuples of distinct integers in $\{1, 2, \ldots, n\}$. An element of J is a candidate for an ordering of defaults of the firms in our basket. We include an extra symbol to identify states of the world in which no default occurs. Let J^k denote the tuples of length at most k and the no-default symbol and let $\mathcal{J}^k$ be the σ-algebra where each of the finitely many elements of J^k is measurable.

LEMMA A1 *The filtrations* $(\mathcal{F}_t^k)$ *and* $(\mathcal{G}_t^k)$ *are complete and right-continuous.*

PROOF The filtrations are complete by construction. Our proof of right-continuity is a trivial extension of the argument in Protter (1990, Chapter I, Theorem 25), which shows that the natural filtration of a counting process is right-continuous. We include the argument for readers' convenience.

Let $\mathcal{B}$ be the Borel sets of $E = [0, \infty]$. For each $s > 0$, let $E_s = E$ and let Γ denote the marked path space

$$\Gamma = \left(J^k \times \prod_{s \in [0,\infty)} E_s, \mathcal{J}^k \otimes \bigotimes_{s \in [0,\infty)} \mathcal{B}_s \right)$$

Let N denote the counting process of the first k defaults $T^1, T^2, \ldots, T^k$. For each $t \geq 0$, define a map

$$\pi_t : \Omega \to \Gamma$$

that assigns to ω the path $s \mapsto N_{s \wedge t}$ and the sequence $i(\omega)$ that lists the identities of the first k defaults that occur before time t. The σ-algebra $\mathcal{F}_t^k$ is generated by π_t.

Let Λ be an event $\cap_{n \geq 1} \mathcal{F}_{t+1/n}$. We need to show that $\Lambda \in \mathcal{F}_t$. Our assumption implies that, for each n, there is a set $A_n \subset I^k \times \prod_{s \in [0,\infty)} E_s$ such that $\Lambda = \{\pi_{t+1/n} \in A_n\}$. Let W_n denote the set $\{\pi_t = \pi_{t+1/n}\}$. For almost every ω, there is an n such that $s \mapsto N_s$ is constant on $[t, t + 1/n]$; therefore, $\Omega = \cup_{n \geq 1} W_n$, where W_n is an increasing sequence of events. Ignoring sets of measure zero,

$$\begin{aligned}
\Lambda &= \lim_{n\to\infty} (W_n \cap \Omega) \\
&= \lim_{n\to\infty} W_n \cap \{\pi_{t+1/n} \in A_n\} \\
&= \lim_{n\to\infty} W_n \cap \{\pi_t \in A_n\} \\
&= \lim_{n\to\infty} \{\pi_t \in A_n\}
\end{aligned}$$

It follows that $\Lambda \in \mathcal{F}_t^k$, demonstrating the right-continuity of $(\mathcal{F}_t)$.

If we replace J^k and $\mathcal{J}^k$ with J^{k-1} and $\mathcal{J}^{k-1}$ in the preceding argument,we get a proof of the right-continuity of $(\mathcal{G}_t^k)$. □

The $(\mathcal{G}_t^k)$-adapted indicator process N^k is right-continuous and bounded. Therefore it admits a Doob–Meyer decomposition. We can write

$$N^k - A^{k,T} = M^k \tag{A7}$$

where M^k is a $(\mathcal{G}_t^k)$-martingale and $A^{k,T}$ is the non-decreasing, $(\mathcal{G}_t^k)$-predictable compensator of N^k. We show that this compensator can be written in a convenient form with respect to the information generated by the first $(k - 1)$ defaults.

Let L_t^k denote the conditional probability that at least k firms survive time t given information generated by the first $(k-1)$ defaults:

$$L_t^k = E\left[1 - N_t^k \mid \mathcal{F}_t^{k-1}\right] = P\left[T^k > t \mid \mathcal{F}_t^{k-1}\right] \tag{A8}$$

Suppose that L^k is monotone and $(\mathcal{F}_t^{k-1})$-predictable. Suppose further that $L_t^k > 0$ for all $t > 0$ and $P[T^k = 0] = 0$. The $(\mathcal{F}_t^{k-1})$-predictable, non-decreasing process given by

$$A_t^k = -\int_0^t \frac{\mathrm{d}L_s^k}{L_{s-}^k} \tag{A9}$$

is called the *kth-to-default pricing trend*. Note that $A_t^k = 0$ for $t \in [0, T^{k-1}]$. If L^k is continuous, the definition (A9) of the pricing trend simplifies to

$$A_t^k = -\log L_t^k \tag{A10}$$

THEOREM A1 *The $(\mathcal{G}_t^k)$-compensator $A_t^{k,T}$ to N^k is equal to $A_{t \wedge T^k}^k$.*

For the proof of Theorem A1 we need the following key lemma.

LEMMA A2 *For $h > 0$, $t \geq 0$ and $1 \leq k \leq n$ we have*

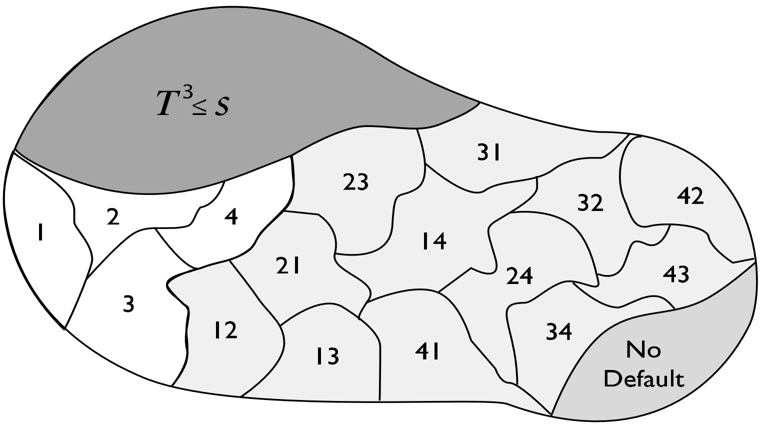
$$P\left[T^k \leq t + h \mid \mathcal{G}_t^k\right] = 1_{\{t \geq T^k\}} + \frac{P\left[t < T^k \leq t + h \mid \mathcal{F}_t^{k-1}\right]}{P\left[t < T^k \mid \mathcal{F}_t^{k-1}\right]} 1_{\{t < T^k\}} \tag{A11}$$

PROOF OF LEMMA A2 Since both the left and right hand sides of (A11) are $\mathcal{G}_t^k$-measurable, it suffices to show that they have the same integral over any set $A \in \mathcal{G}_t^k$.

Ignoring sets of measure zero, we can write $A \cap \{t < T^k\}$ as a finite disjoint union of sets $A_j \in \mathcal{F}_t^{k-1}$, where $j \in I^{k-1}$ is an ordered tuple that labels defaults up to the $(k-1)$st that occur by time t (see Figure 6). We have

$$\begin{aligned}
\int_{A \cap \{T^k > t\}} 1_{\{T^k \leq t+h\}} \mathrm{d}P &= \sum_{j \in I^{k-1}} \int_{A_j} 1_{\{T^k \leq t+h\}} \mathrm{d}P \\
&= \sum_{j \in I^{k-1}} P\left[T^k \leq t + h \mid A_j\right] \\
&= \sum_{j \in I^{k-1}} \frac{P\left[\{T^k \leq t + h\} \cap A_j\right]}{P[A_j]}
\end{aligned}$$

FIGURE 6 The σ-algebra $\mathcal{F}_t^k$ is generated by the sets $\{T^j \wedge s\}$ for $j = 1, 2, \ldots, k$ and $s \leq t$ along with the identities, I^j, of defaulters. For each s, there are finitely many new generating sets. The generating sets for a fixed time s are illustrated here.



$$= \int_{A \cap \{T^k > t\}} \frac{P\left[t < T^k \leq t + h \mid \mathcal{F}_t^{k-1}\right]}{P\left[t < T^k \mid \mathcal{F}_t^{k-1}\right]} 1_{\{t < T^k\}} \mathrm{d}P$$

On the set $A \cap \{T^k \leq t\}$,

$$P\left[T^k \leq t + h \mid \mathcal{G}_t^k\right] = E\left[1_{\{T^k \leq t+h\}} \cdot 1_{\{T^k \leq t\}} \mid \mathcal{G}_t^k\right] = 1_{\{T^k \leq t\}}$$

This completes the proof. □

PROOF OF THEOREM A1 We verify the martingale property of the process defined by $N_t^k - A_{t \wedge T^k}^k$ with respect to the filtration $(\mathcal{G}_t^k)$. For some $t > 0$ we let $\pi = (t_0, \ldots, t_m)$ be a partition of $[0, t]$ with mesh $\|\pi\| = \max_{1 \leq i \leq m}(t_{i+1} - t_i)$. By Lemma A2 we have

$$A_t^k(\pi) := \sum_{i=1}^{n} E\left[N_{t_i+1}^k - N_{t_i}^k \mid \mathcal{G}_{t_i}^k\right] = \sum_{i=1}^{n} \frac{P\left[t_i < T^k \leq t_{i+1} \mid \mathcal{F}_{t_i}^{k-1}\right]}{P\left[t_i < T^k \mid \mathcal{F}_{t_i}^{k-1}\right]} 1_{\{t_i < T^k\}}$$

Using the definition (A8), we get

$$A_t^k(\pi) = -\sum_{i=1}^{n} \frac{E\left[L_{t_{i+1}}^k - L_{t_i}^k \mid \mathcal{F}_{t_i}^{k-1}\right]}{L_{t_i}^k} 1_{\{t_i < T^k\}}$$

These Riemann–Stieltjes sums converge as follows:

$$\lim_{\|\pi\|\to 0} A_t^k(\pi) = -\int_0^{t\wedge T^k} \frac{\mathrm{d}L_s^k}{L_{s-}^k}$$

□

PROOF OF PROPOSITION 6 If $t \le T^{k-1}$, then $L_t^k = 1$. We consider the case $t > T^{k-1}$. Our hypothesis is that $T^i = \tau^i$ for $1 \le i \le k-1$. With the definition (4) of a default event, from (A8) we get

$$L_t^k = P\Big[\min_{k\le i\le n} \tau^i > t \,\Big|\, \{T^i = \tau^i\}_{i=1}^{k-1}, \{\tau^i > \tau^{k-1}\}_{i=k}^{n}\Big]$$

$$= P\Big[D^k < M_t^k, \dots, D^n < M_t^n \,\Big|\, \{D^i < M_{\tau_i}^i\}_{i=1}^{k-1}, \{D^i < M_{\tau^{k-1}}^i\}_{i=k}^{n}\Big]$$

Using the fact that M^i has decreasing paths, with Bayes' rule we get

$$L_t^k = \frac{\int_{\mathbb{R}_-^n} G_{z_1\cdots z_{k-1}}(x)\, h(\tau^1,\dots,\tau^{k-1}, t,\dots,t; x)\,\mathrm{d}x}{\int_{\mathbb{R}_-^n} G_{z_1\cdots z_{k-1}}(x)\, h(\tau^1,\dots,\tau^{k-1}, \tau^{k-1},\dots,\tau^{k-1}; x)\,\mathrm{d}x}$$

for $t > \tau^{k-1}$. Now Theorem A1 implies the first statement.

Under our smoothness hypothesis on h, Aven's (1985) conditions are satisfied, so there exists a process λ^k such that $A_t^k = \int_0^t \lambda_s^k \mathrm{d}s$. This process is zero on $[0, T^{k-1}]$. □

REFERENCES

Aven, T. (1985). A theorem for determining the compensator of a counting process. *Scandinavian Journal of Statistics* **12**, 69–72.

Bluhm, C., Overbeck, L., and Wagner, C. (2003). *An introduction to credit risk modeling.* Chapman & Hall/CRC, London.

Çetin, U., Jarrow, R., Protter, P., and Yildirim, Y. (2002). Modeling credit risk with partial information. Working paper, Cornell University.

Chou, C. and Meyer, P. A. (1975). Sur la représentation des martingales comme intégrales stochastiques dans la processus ponctuels. In *Séminaire de probabilités IX, Lecture Notes in Mathematics*, pp. 60–70. Springer-Verlag Berlin.

Collin-Dufresne, P., Goldstein, R., and Helwege, J. (2002). Are jumps in corporate bond yields priced? Modeling contagion via the updating of beliefs. Working paper, Carnegie Mellon University.

Crouhy, M., Galai, D., and Mark, R. (2000). A comparative analysis of current credit risk models. *Journal of Banking and Finance* **24**, 59–117.

Dellacherie, C. (1970). Une exemple de la théorie générale des processus. In *Séminaire de probabilités IV, Lecture Notes in Mathematics 124*. Springer-Verlag, Berlin.

Dellacherie, C., and Meyer, P. A. (1982). *Probabilities and potential.* North-Holland, Amsterdam.

Duffie, D. (1998). First-to-default valuation. Working paper, GSB, Stanford University.

Duffie, D., and Lando, D. (2001). Term structures of credit spreads with incomplete accounting information. *Econometrica* **69**(3), 633–64.

Duffie, D., and Singleton, K. J. (1998). Simulating correlated defaults. Working paper, GSB, Stanford University.

Giesecke, K. (2001). Default and information. Working paper, Cornell University.

Giesecke, K. (2004). Correlated default with incomplete information. *Journal of Banking and Finance* **28**, 1521–45.

Giesecke, K., and Goldberg, L. (2003). The market price of credit risk. Working paper, Cornell University.

Giesecke, K., and Goldberg, L. (2004). Forecasting default in the face of uncertainty. *Journal of Derivatives* **12**(1), 11–25.

He, H., Keirstead, W. P., and Rebholz, J. (1998). Double lookbacks. *Mathematical Finance* **8**, 201–28.

Iyengar, S. (1985). Hitting lines with two-dimensional Brownian motion. *SIAM Journal of Applied Mathematics* **45**(6), 983–9.

Jarrow, R. A., and Yu, F. (2001). Counterparty risk and the pricing of defaultable securities. *Journal of Finance* **56**(5), 555–76.

Lando, D. (1998). On Cox processes and credit risky securities. *Review of Derivatives Research* **2**, 99–120.

Nelsen, R. (1999). *An introduction to copulas.* Springer-Verlag, New York.

Protter, P. (1990). *Stochastic integration and differential equations.* Springer-Verlag, New York.

Shaked, M., and Shanthikumar, G. (1987). The multivariate hazard construction. *Stochastic Processes and Their Applications* **24**, 241–58.

Wise, M., and Bhansali, V. (2004). Correlated random walks and the joint survival probability. Working paper, Caltech and PIMCO.

Yu, F. (2003). Dependent default in intensity based models. Working paper, University of California at Irvine.

Zhou, C. (2001). An analysis of default correlation and multiple defaults. *Review of Financial Studies* **14**(2), 555–76.