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Correlated default with incomplete information

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Abstract

The recent accounting scandals at Enron, WorldCom, and Tyco were related to the misrepresentation of liabilities. We provide a structural model of correlated multi-firm default, in which public bond investors are uncertain about the liability-dependent barrier at which individual firms default. Investors form prior beliefs on the barriers, which they update with the default status information of firms arriving over time. Whenever a firm suddenly defaults, investors learn about the default threshold of closely associated business partner firms. This updating leads to "contagious" jumps in credit spreads of business partners. We characterize joint default probabilities and the default dependence structure as assessed by investors, where we emphasize the the modeling of dependence with copulas. A case study based on Brownian asset dynamics illustrates our results.

JEL classification: G12; G13
Keywords: Correlated defaults; Contagion; Incomplete information; Copulas

1. Introduction

A number of studies have investigated historical bond price and default data. They found, quite plausibly, that credit spreads as well as aggregate default rates are strongly related to general macro-economic factors such as the level of default-free interest rates (see, for example, Duffee, 1998; Keenan, 2000). Due to their joint dependence on smoothly varying common variables, credit spreads across firms are smoothly correlated through time. This might be called *cyclical default correlation*.

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Another observation is that sudden large changes or jumps in spreads across several issuers do not seem to appear independently from each other. In its extreme form, a sudden spread change of one issuer, possibly due to a rating downgrade, a news announcement, or a default, leads to a simultaneous market-wide response in spreads. That is, an event specific to one entity in the first place may have an immediate economic or perceived impact on other entities, which finds its expression in a cascade of subsequent spread changes. In analogy to the literature on financial crises, for example Allen and Gale (2000), we will refer to this particular pattern as *contagion*. A recent example is the default of Argentina and the associated immediate spread widening on debt of other South American countries, most notably Brazil. On the corporate debt markets, similar phenomena can be observed: the default of Enron caused substantial adverse changes in the spreads on several utility companies throughout the US. One explanation of such contagion effects is the existence of close ties between firms. In general, these links are of legal (e.g. parent–subsidiary), financial (e.g. trade credit), or business nature (e.g. buyer–supplier). It is easy to imagine that through these channels economic distress of one firm can have an immediate adverse effect on the financial health of that firm's business partners. This distress propagation is accompanied by a prompt jump in spreads on the involved firms' debt, which reflects the market's re-assessment of their default probabilities.

In this paper we provide a model for correlated defaults, where we emphasize the contagion phenomena associated with direct ties between firms. A thorough understanding of the mechanisms behind contagion is of crucial importance in several respects. From a banking regulatory perspective, contagion phenomena may endanger the stability of the financial system. Suddenly appearing large-scale adverse spread changes cause typically excessive losses in banks' credit portfolios. The distress of lending institutions has typically adverse effects on the entire financial system. Here it is the non-smooth nature of contagion processes in comparison with cyclical default correlations, which generates suddenly large losses that need to be buffered promptly. The design of effective capital provisions requires explicit models for contagion phenomena.

The existence of contagion plays also an important role in the pricing and hedging of multi-name credit derivatives, which offer insurance against spread changes and defaults of several underlying instruments, bonds for example. Suppose we want to hedge a position in a first-to-default swap, which provides a payoff upon the first default in some basket of bonds, in return for a periodic payment. We then sell protection via plain-vanilla default swaps on each bond in the basket. In case a default event occurs, we receive the first-to-default payment, pay the default premium in the plain-vanilla swap corresponding to the first defaulter, and unwind the remaining vanilla swaps. Given direct ties between the issuers, for the latter transaction we have to account for the fact that a spread jump in the remaining bonds in the basket may follow the first defaulter. This would adversely affect the economics of the hedge. For the control of this hedging risk we need again an explicit model for the contagion effects in defaults and spreads.

Corporate defaults and credit spreads on corporate debt can be basically modeled via two distinct approaches, the intensity-based approach and the structural approach. In the former, default is an unpredictable event whose stochastic structure is prescribed by an intensity process. The intensity models the conditional default arrival rate per unit time. To induce correlation between defaults, one would typically introduce correlation between the intensity processes (Duffie and Singleton, 1998; Duffie and Garleanu, 2001, Jarrow et al., 2000, for example). One way to do this is by letting intensities of several firms be driven by common variables, for instance riskless interest rates, which would correspond to the observed credit spread correlation across firms due to dependence on general economic factors (cyclical correlation). A disadvantage of this setup is that defaults are conditionally independent given the intensities; contagion effects cannot arise as long as intensities are smoothly varying. To impose a stronger degree of default correlation and model contagion effects, one can let intensities experience correlated jumps, or directly construct a firm's intensity to jump upon defaults of other firms. This is essentially the path followed by several recent contributions, for example Jarrow and Yu (2001), Davis and Lo (2001), Avellaneda and Wu (2001), Schönbucher and Schubert (2001), Collin-Dufresne et al. (2002), and Giesecke and Weber (2002).

While in the ad hoc intensity-based approach the default event and the associated intensity is typically given exogenously, the structural approach takes the cause and effect nature of default as a starting point. Taking the dynamics of a firm's asset value as given, a default takes place if the assets fall to some lower threshold. Here, with a typically continuous asset process, defaults are, however, predictable, meaning that they are not surprise events any more. Indeed, at any time investors know the nearness of the assets to the default threshold, so that they are warned in advance when a default is imminent. This implies that investors would not demand a default premium for short-term debt, which is not plausible and also at odds with empirical observations. In the structural approach, cyclical default correlation between issuers is easily introduced through asset correlation; this is the approach taken by the standard industry models KMV (Kealhofer, 1998) and CreditMetrics (Gupton et al., 1997). With predictable defaults, however, jumps in bond prices and credit spreads cannot appear at all: prices converge continuously to their default-contingent values. This means that, although the existing structural approaches provide important insights into the relation between firms' fundamentals and correlated default events as well as practically most valuable tools, they fail to be consistent in particular with the observed contagion phenomena.

In this paper we provide a structural model of correlated default which is consistent with several significant credit spread characteristics: the implied short-term credit spreads are typically non-zero, credit spreads display cyclical correlations across firms, and, most importantly, contagion effects are predicted. Our key assumption is that issuers' fundamental characteristics are not completely transparent to public bond investors on the secondary market. Unlike the incomplete observation model of Duffie and Lando (2001), we suppose that the threshold level at which a firm's management liquidates the firm is not publicly known. The threshold level is some function of the firm's liabilities, whose total amount is inside firm information. This sensitive information is usually not fully disclosed to the public market; only rarely there is an incentive to do so. A case in point are the accounting scandals at Enron, WorldCom, or Tyco. Similar informational assumptions appear in Lambrecht and Perraudin (1996) and in the CreditGrades Model, which was recently published by the RiskMetrics Group (2002).

With incomplete information investors are not able to precisely assess the nearness of the assets to the default threshold, so that the default becomes a totally unpredictable event, as in the intensity-based approach. This implies that credit spreads are generally non-zero, even for short maturities. Non-vanishing short-term spreads can be seen as "transparency spreads", and recent empirical evidence in this direction is presented by Yu (2002). The specific shape of the term structure of credit spreads depends on the relation of the current asset value to the historical low of assets; see Giesecke (2001) for an analysis. Since defaults are unanticipated surprise events, jumps in spreads and bond prices at or around default will appear, in line with empirical observations. In addition, time series of default probabilities implied by our assumptions seem to fit historical data better than those implied by the usual perfect-information model (as employed by KMV's standard industry model, for example). For more details in that respect we refer to Giesecke and Goldberg (2003).

In lack of perfect information, bond investors form a prior distribution on firms' default thresholds. This prior reflects the publicly known fact that firms can have close ties between each other, in the sense that the default thresholds of directly linked firms are *not* independent of each other. For example, it is easy to imagine that for two firms in a parent–subsidiary relationship the respective debt levels, which are intimately connected to the default threshold, are not independent. For two firms in a trade credit, the credit-granting supplier has to appropriately fund the credit to the buyer, which affects its liability side. This means the liabilities of both firms in the transaction cannot be considered independent, giving rise to dependence in the respective default thresholds as functions of liability levels.

In view of the connections between firms, bond investors will rationally use any available information on the public market to assess the default threshold of a particular firm. From observing assets and defaults of *all* firms in the market, investors *learn* over time about the characteristics of *individual* firms. Put another way, investors update their threshold belief with the information continuously arriving over time. Specifically, the non-default status of a firm reveals that the firm's unobservable (non-time varying) default threshold must be below the historic low of the firm's asset value. In the event of a default, the precise value of the threshold level becomes public knowledge.

Together with the updated a posteriori default threshold belief, an assumption on the (joint) asset dynamics provides a model for the evolution of (joint) default probabilities and bond prices from the viewpoint of the incompletely informed public bond market. Over time, default probabilities and hence credit spreads display a characteristic jump pattern, which corresponds to the idea of contagion of distress between firms. If some firm in the market unpredictably defaults, investors immediately update their threshold belief on all those remaining firms which have close ties to the defaulted firm. This discontinuity in the information arrival leads to a jump in default probabilities and spreads of all affected remaining firms, corresponding to a reassessment of those firms' financial health. Indeed, if we imagine two firms in a trade credit, the default of the buyer can immediately adversely affect the supplier, whose default probability is therefore likely to jump upwards.

The paper is organized as follows. In Section 2, we formulate the model. We begin in Section 3 with an analysis of the complete information case. In Section 4, we consider the incomplete information case. Describing dependence between random variables by copula functions, we study joint conditional default probabilities, the default dependence structure, and the implied contagion effects. We illustrate our general results in an example in Section 5. Section 6 concludes. Appendix A contains the proofs.

2. Model setup

We consider an economy with a financial market. Uncertainty in the economy is modeled by a probability measure P. Investors are assumed to be risk-neutral; on the financial market they can trade in bonds issued by several firms. The finite index set of all firms is denoted $I = \{1, 2, \dots, n\}$. We take as given some $\mathbb{R}^n$-valued stochastic process $V = (V^1, \dots, V^n)$, on which investors agree, and we denote by $(\mathscr{F}_t)_{t \geqslant 0}$ the filtration generated by V. The process $V^i = (V^i_t)_{t \geqslant 0}$ is continuous and normalized to satisfy $V^i_0 = 0$. V^i_t is a sufficient statistic for the expected discounted future cash flows of firm i as seen from time t. We will therefore call V^i asset process. The running minimum asset process $(M^i_t)_{t \geqslant 0}$ is defined by

$$M^i_t = \min\{V^i_s \,|\, 0 \leqslant s \leqslant t\},$$

so that M^i_t denotes the *historical low* of the asset value in the period $[0, t]$.

A firm has issued non-callable consol bonds, paying coupons at some constant rate as long as the firm operates. When a firm stops servicing the coupon, we say it defaults. It then enters financial distress and some form of corporate reorganization takes place. Consistent with this time-homogeneous capital structure, bond investors suppose that there is a vector of random *default thresholds* $D = (D_1, \dots, D_n)$, such that firm i defaults as soon as its asset value falls to the level $D_i < 0$. The thresholds D are assumed to be independent of assets V. The random *default time* is thus given by

$$\tau_i = \min\{t > 0 \,|\, V^i_t \leqslant D_i\} \tag{1}$$

and we set $\tau = (\tau_1, \dots, \tau_n)$. By convention, $V^i_t = V^i_{\tau_i}$ for $t \geqslant \tau_i$. With the running minimum asset process M^i we obtain immediately

$$\{\tau_i \leqslant t\} = \{M^i_t \leqslant D_i\}, \tag{2}$$

meaning that the event of default before time t is equivalent to the event that the assets of the firm have been below the default barrier at least once in $[0, t]$.

3. Complete information

We begin by considering the case where the public bond market is assumed to have perfect information about the default characteristics of firms; this is the assumption made in all existing structural models of correlated default. We thus describe the public information flow by the filtration $(\mathscr{G}_t)_{t \geqslant 0}$ defined by

$$\mathscr{G}_t = \mathscr{F}_t \vee \sigma(D), \tag{3}$$

meaning that at time t, the market knows firms' asset values up to t and their default thresholds. This implies, given the continuity of the asset process, that defaults are *predictable* by public bond investors. This means that the default is announced by a sequence of observable phenomena, such as the asset value falling dangerously close to the default boundary. Loosely, over sufficiently short-time horizons there is no uncertainty about the default any more. It is proved in Giesecke (2001) that this implies credit spreads going to zero with maturity going to zero, meaning that investors will not demand a default risk premium for short-maturity debt. Moreover, the predictability implies that bond prices converge continuously to their recovery values; there are no surprise jumps at or around default announcement. These predictions are neither intuitively nor empirically plausible.

Let us turn to the modeling of default correlation in this setup of complete information. *Cyclical* correlation arising from the dependence of firms on common smoothly varying (macro-) economic factors, is naturally introduced through mutually dependent asset values. For example, the asset value of firms heavily dependent on the oil price will vary with the oil price over time, leading to correlation in the assets of these firms over time. The correlated evolution of firms' assets is described by the process V, whose specific form is unanimously fixed by investors. The chosen dynamics determine immediately the distribution function $H_i(\cdot, t)$ of the asset's historical low M_t^i for $t > 0$, which is the relevant quantity in our model, see (2). The dependence between assets also immediately induces the dependence structure between the historical low processes M^i. This dependence structure is formally described by the *copula* [1] $C^M_{t_1,\dots,t_n}$ of the vector $(M^1_{t_1}, \dots, M^n_{t_n})$. Together with the marginals $H_i(\cdot, t)$, the copula C^M specifies the joint distribution function of $(M^1_{t_1}, \dots M^n_{t_n})$ by

$$P[M^1_{t_1} \leqslant x_1, \dots, M^n_{t_n} \leqslant x_n] = C^M_{t_1,\dots t_n}(H_1(x_1, t_1), \dots H_n(x_n, t_n)) \tag{4}$$

for all fixed $t_i > 0$ and all $x_i \leqslant 0$. This representation is unique if the marginals $H_i(x, t)$ are continuous in x for fixed t.

Example 3.1. Suppose V is an n-dimensional Brownian motion with some covariance matrix Σ. Then $H_i(x, \cdot)$ is the well-known inverse Gaussian distribution function for fixed $x \leqslant 0$, and C^M is the corresponding inverse Gaussian copula [2] with covariance matrix Σ. Here the joint distribution of firms' historical asset lows is uniquely specified by (4).

There is an interesting relation to an important default model class. In fact, for fixed horizon T, we can view the historical asset low M_T^i as a latent variable with associated cut-off level D_i, so that $(M_T^i, D_i)_{i \in I}$ can be considered as a *latent variable de-*

[1] We refer to Nelsen (1999) for a comprehensive introduction to copula functions.

[2] Given the multivariate inverse Gaussian distribution function, a corollary to Sklar's theorem (see Nelsen, 1999) allows to derive the inverse Gaussian copula function. The former can be found in Iyengar (1985) or Rebholz (1994), for example.

fault model in the sense of Frey and McNeil (2001) and Frey and McNeil (2002). The choice of the asset's running minimum M_T^i as the latent variable provides a new example for this popular class of models, which includes, for instance, the asset-based models of KMV and CreditMetrics. The latter consider the asset value V_T^i at the horizon (debt maturity) T as the latent variable, implying that the path the asset value has taken before T does not matter. In fact, the asset value may be well below the cut-off level D_i before T without causing a default, if it only exceeds D_i at T. From bond investors' perspective, this is very unsatisfactory since it can lead to very low recovery values unless they can intervene and force a liquidation of the firm if assets first cross some lower threshold (Black and Cox, 1976). From equity investors' or management's perspective, this path independence is unrealistic as well, since it may be optimal for them to liquidate well before the debt matures at T (see Leland, 1994, for example). We address this criticism by allowing a firm to default at any time before the horizon T, given its asset value V^i falls to the threshold D_i. This choice leads to a latent variable model based on the assets' historical low M instead on the assets V itself.

The conceptual advantage of separating the dependence structure C^M from the marginals H_i of the latent variables M^i becomes apparent if we consider joint conditional default probabilities, given the complete information of the public bond market:

$$\begin{aligned} F_t(T_1, \dots T_n) &= P[\tau_1 \leqslant T_1, \dots, \tau_n \leqslant T_n \,|\, \mathcal{G}_t] \\ &= P[M_{T_1}^1 \leqslant D_1, \dots, M_{T_n}^n \leqslant D_n \,|\, \mathcal{G}_t] \end{aligned} \tag{5}$$

for $T_i > t$ (the second line follows immediately from (2)).

Example 3.2. Suppose there has been no default by time t. Assuming that assets V follow a Markov process with stationary increments, by (4) we get

$$F_t(T_1, \dots, T_n) = C^M_{T_1 - t, \dots, T_n - t}(H_1(D_1 - V_t^1, T_1 - t), \dots, H_n(D_n - V_t^n, T_n - t))$$

for $T_i > t$. This formula offers in fact a general multivariate copula representation of Zhou's (2001) recent result, who directly computed $F_0(T, T)$ in case $n = 2$. Conditional default probabilities of individual firms are simply given by

$$F_t^i(T) = P[\tau_i \leqslant T \,|\, \mathcal{G}_t] = H_i(T - t, D_i - V_t^i), \quad t < \tau_i, \quad t < T. \tag{6}$$

It is obvious that joint default probabilities F_t critically depend on the nature of the copula C^M, i.e. the dependence structure of the latent variables M^i. Moreover, we have the following result relating C^M to the copula C^τ of the default time vector τ, which describes the complete non-linear *default dependence structure* (see Section 4.3).

Proposition 3.3. *Assume that $H_i(t, D_i)$ is continuous* [3] *in t and let $J_i = H_i^{-1}(\cdot, D_i)$ denote its generalized inverse. The copula C^τ of the default times is for all $u_i \in [0, 1]$ given by*

[3] This is satisfied, for instance, with Brownian asset dynamics as in Example 3.1.

$$C^{\tau}(u_1,\ldots,u_n) = C^M_{J_1(u_1),\ldots,J_n(u_n)}(u_1,\ldots,u_n).$$

Given the marginal default probabilities $F^i_0(T) = H_i(T, D_i)$, the default dependence structure C^{τ} is given by the copula C^M of the historical asset low. This means that the statistical properties of C^M determine the likelihood of joint defaults, and in particular the tail of the aggregate default loss distribution, cf. Frey and McNeil (2001). We mention here copulas C^M displaying *lower tail dependence*, i.e. copulas having a pronounced tendency to generate low values in all marginals simultaneously (for a formal definition see Nelsen, 1999). In comparison with copulas displaying asymptotic independence, such copulas would induce a higher likelihood of all firms' historical asset lows being simultaneously low, corresponding to a higher likelihood of observing joint defaults and large aggregate default losses. From the viewpoint of the default copula C^{τ} a similar conclusion can be made: lower tail dependence implies a higher likelihood of early default for a large number of firms simultaneously, meaning a higher probability of large aggregate losses before a given horizon.

The sensitivity of the latent variable model $(M^i_T, D_i)_{i\in I}$ to the dependence structure of the latent variables M^i suggests different choices of the copula $C^M_{T,\ldots,T}$ for given fixed horizon T. We recall that together with given marginals $H_i(\cdot, T)$ of M^i_T, any copula $C^M_{T,\ldots,T}$ defines a proper joint distribution function for the latent variables M^i_T via (4). Moreover, if the marginals $H_i(x, T)$ are continuous in x, then the joint distribution is uniquely defined. Following Example 3.1, a natural choice for continuous marginals appears to be inverse Gaussian, which would correspond to the standard assumption of assets V^i following a Brownian motion. While the corresponding latent variable copula $C^M_{T,\ldots,T}$ would be inverse Gaussian, alternative choices seem equally plausible. Obvious examples include the Gaussian or the Student-t-copula. Frey and McNeil (2001) analyze alternative latent variable copulas for the latent variable model $(V^i_T, D_i)_{i\in I}$ for V^i following a Brownian motion.

Finally, we observe that with perfect information about assets and default thresholds contagion effects *cannot* appear in this model. As discussed in the Introduction, contagion phenomena correspond to the conditional default probability $F^i_t(T)$ of firm i experiencing a jump upon the default of some closely related firm. This jump would reflect the immediate re-assessment of firm i's financial health in light of the default of some closely linked firm. Inspection of (6) shows, however, that other firms' default status does not matter for $F^i_t(T)$, ruling out cascading reactions. Moreover, given assets follow a sufficiently irregular process (so that $H_i(t,x)$ is continuous in t), the predictability of defaults rules out any jumps in the default probability.

4. Incomplete information

We now consider the situation when the default characteristics of firms are not completely transparent to investors in the public bond market. The recent accounting scandals surrounding Enron, WorldCom, and Tyco, revealed that the level of the liabilities is often (intentionally) not fully disclosed to the public. Motivated by this, we relax the unrealistic perfect information assumption of the previous section and suppose that the default threshold of a firm, being some function of liabilities, is unknown to bond investors. Again in line with the existing models, investors can observe the asset values of the firms, as well as defaults in the market as they occur. [4] From now on, the corresponding public information filtration $(\mathscr{G}_t)$ is therefore defined by

$$\mathscr{G}_t = \mathscr{F}_t \vee \sigma(\tau_1 \wedge t) \vee \ldots \vee \sigma(\tau_n \wedge t), \tag{7}$$

where $a \wedge b$ denotes the minimum of a and b. Similar informational assumptions, though in different one-firm setups, appear in Lambrecht and Perraudin (1996) and in the CreditGrades Model, which was recently published by the RiskMetrics Group (2002). Hull and White (2000a) and Avellaneda and Zhu (2001) take the threshold to be an unknown deterministic function of time and estimate it from observed defaultable bond prices.

In contrast to the perfect information case, with incomplete information investors do not know the exact distance of a firm's assets to its default threshold. Therefore the default hits the market by surprise, it is totally *unpredictable*. It is shown in Giesecke (2001), that this implies positive short-term credit spreads whenever the firm's assets are close to their historical lows. Here short spreads can be seen as a transparency premium; see Yu (2002) for a recent empirical analysis in this direction. Moreover, a transparency premium applies to all maturities, so that spreads are generally above those implied by a comparable complete information model. In a comprehensive empirical analysis Huang and Huang (2000) found that the existing complete information models generate counterfactually low credit spreads once they are calibrated to historical data. Incomplete information thus provides a potential solution to these difficulties with structural models. In Giesecke (2001), it is shown that the specific shape of the term structure of credit spreads depends on the riskiness of the firm, measured in terms of the distance of assets to their all-time low. Due to the surprise nature of default, bond prices and credit spreads will jump to their default-contingent values upon the default event; they do not continuously converge to these values as with complete information. As for time series of default probabilities, the empirical analysis of Giesecke and Goldberg (2003) shows that incomplete default threshold observation leads to plausible default probability forecasts.

4.1. Threshold prior and learning

Lacking precise information on the default thresholds of the firms, for each firm $i \in I$ investors agree at time $t = 0$ on a continuous and strictly increasing prior distribution function G^i on the threshold D_i, which we take as given. This prior

[4] The perfect observability of assets can be relaxed additionally, but the conclusions drawn below are not changed. Observability of defaults is a realistic assumption, since default events are publicly announced, and bond investors recognize if debt service ceases. Therefore, in order to keep our analysis simple and instructive, we have chosen to focus on threshold uncertainty.

describes the uncertainty surrounding firm i's threshold D_i with respect to *idiosyncratic* factors, such as those related to the firm's (perceived) capital structure.

Besides idiosyncratic factors, default thresholds of several firms may be driven by common, or partly *systematic* factors. This is closely related to the existence of direct ties between firms, which are often of legal (e.g. parent–subsidiary), financial (e.g. trade credit), or business (e.g. buyer–supplier) nature. Such direct ties give rise to dependence between firms' default thresholds. For example, it is easy to imagine that for two firms in a parent–subsidiary relationship the respective debt levels, which are intimately connected to the default threshold, are not independent. For two firms in a trade credit, the credit-granting supplier has to appropriately fund the credit to the buyer, which affects its liability side. This means the liabilities of both firms in the transaction cannot be considered independent, giving rise to dependence in the respective default thresholds as functions of liability levels.

Besides on the marginal priors G^i, at time $t = 0$ investors agree also on the copula C^D of the random vector $(D_1, \dots, D_n)$. This copula describes the complete non-linear dependence structure between firms' default thresholds D_i, as arising from the direct links between firms. We take this copula as given. Together with the marginal priors G^i, the prior copula C^D corresponds to a unique representation of investors' joint prior threshold distribution G across firms:

$$G(x_1, \dots, x_n) = C^D(G^1(x_1), \dots, G^n(x_n)), \quad x_i \leqslant 0. \tag{8}$$

At time $t = 0$, investors start with the prior G, which describes the individual and joint uncertainty surrounding the firms' default thresholds. With the passage of time, information on the default characteristics of firms becomes publicly available. Bond investors will rationally use all available public information to update their prior on any individual non-defaulted firm's default threshold. To study this learning procedure, we introduce the process $(S_t)_{t \geqslant 0}$, taking values in the power set $\mathbb{P}(I)$ of the firm index set I, by

$$S_t = \{i \in I : \tau_i \leqslant t\}. \tag{9}$$

The set S_t is called the default scenario at t, since it collects the indices of all those bonds having defaulted by time t. By (2) and the continuity of the asset process V^i, we get

$$\{S_t = s\} = \bigcap_{i \in s} \{D_i = M^i_{\tau_i}\} \cap \bigcap_{i \in I - s} \{D_i < M^i_t\}.$$

Suppose some scenario $s \in \mathbb{P}(I)$ appears at time t. Then investors know that $D_i = M^i_{\tau_i}$ for all defaulted firms $i \in s$ and that $D_i < M^i_t$ for all operating firms $i \in I - s$. That is, the default of a firm reveals the precise value of its threshold to the market, while the non-default status of a firm reveals that its threshold must be below the historical low of assets. With any default scenario appearing at time t we can hence associate a set $B(M_t, \cdot) \in \mathscr{B}^n_-$ such that

$$\{D \in B(M_t, s)\} = \{S_t = s\}.$$

In the following we relate the a posteriori belief to the prior.

Lemma 4.1. *Let L denote the joint law of D. On the set $\{S_t = s\}$, the a posteriori threshold belief is represented by the distribution*

$$P[D \in A \,|\, \mathscr{G}_t] = \frac{L(A \cap B(M_t, s))}{L(B(M_t, s))}, \quad A \in \mathscr{B}^n_-.$$

This Lemma allows to deduce the posteriori belief, i.e. the $\mathscr{G}_t$-conditional distribution function G_t of the threshold vector D, from the prior G.

Example 4.2. Consider the no-default scenario $S_t = \emptyset$. Then the thresholds of all firms satisfy $D_i < M^i_t$ or, put another way, $D \in B(M_t, \emptyset) = (-\infty, M^1_t) \times \cdots \times (-\infty, M^n_t)$. Noting that $G(x_1, \ldots, x_n) = L((-\infty, x_1) \times \cdots \times (-\infty, x_n))$, Lemma 4.1 implies that

$$G_t(x_1, \ldots, x_n) = \frac{G(x_1 \wedge M^1_t, \ldots, x_n \wedge M^n_t)}{G(M^1_t, \ldots, M^n_t)}, \quad x_i \leqslant 0.$$

We now use the familiar idea of a copula to separate the dependence structure from the conditional threshold distribution. We thus introduce the process $(C^D_t)_{t \geqslant 0}$, where $C^D_t : \Omega \times [0,1]^n \times [0, \infty) \to [0,1]$ is the *conditional threshold copula*, representing the conditional threshold dependence structure:

$$G_t(x_1, \ldots, x_n) = C^D_t(G^1_t(x_1), \ldots, G^n_t(x_n)), \quad x_i \leqslant 0,$$

where G^i_t is the marginal conditional threshold distribution. Let us denote by $I^i_t(u) = \inf\{x \geqslant 0 : G^i_t(x) \geqslant u\}$ the generalized inverse of G^i_t. Since the G^i_t are continuous for all fixed times $t < \tau_i$, the copula C^D_t can be constructed via

$$C^D_t(u_1, \ldots, u_n) = G_t(I^1_t(u_1), \ldots, I^n_t(u_n)), \tag{10}$$

where $u_i \in [0,1]$ is $\mathscr{G}_t$-measurable. Let us observe that C^D and C^D_t are not equal in general; for an example we refer to Section 5.

The (''ordinary'') copula couples marginal distributions with the joint distribution of some random vector. The copula linking the marginal survival functions with the joint survival function is called survival copula, cf. Nelsen (1999). Ordinary copula and survival copula express in an equivalent way the dependence structure of a given random vector. In analogy to $(C^D_t)_{t \geqslant 0}$, we introduce the conditional survival threshold copula process $(\bar{C}^D_t)_{t \geqslant 0}$.

Lemma 4.3. *The conditional survival threshold copula can be constructed from the conditional threshold copula via*

$$\bar{C}^D_t(u_1, \ldots, u_n) = \sum_{i_1=1}^{2} \cdots \sum_{i_n=1}^{2} (-1)^{i_1 + \cdots + i_n} C^D_t(v_{1i_1}, \ldots, v_{ni_n}),$$

where $v_{j1} = 1 - u_j$ and $v_{j2} = 1$ and $u_i \in [0,1]$ is $\mathscr{G}_t$-measurable.

4.2. Default distribution

Viewing the historical asset low M_T^i for some fixed horizon T as a latent variable, $(M_T^i, D_i)_{i\in I}$ can be considered as a latent variable model with *random* cut-off levels D_i. Here, default dependence arises from both dependence across the latent variables and dependence between the cut-off levels. We now characterize joint conditional default probabilities (5) implied by this model. These are the key for the solution of a variety of problems, such as the aggregation of correlated default risks, or the analysis of multi-name credit derivative instruments.

Proposition 4.4. *The conditional joint default probability is for $T_i > t$ given by*

$$F_t(T_1,\dots,T_n) = E[\bar{C}_t^D(1 - G_t^1(M_{T_1}^1),\dots,1 - G_t^n(M_{T_n}^n)) \,|\, \mathcal{G}_t].$$

The marginal default probability of a firm i is for $t < \tau_i$ and $t < T$ given by

$$F_t^i(T) = P[\tau_i \leqslant T \,|\, \mathcal{G}_t] = E[1 - G_t^i(M_T^i) \,|\, \mathcal{G}_t].$$

More explicit expressions can be derived under additional assumptions on the asset process and the default scenario.

Example 4.5. Consider the no-default scenario $S_t = \emptyset$. Assuming that assets follow a Markov process with stationary increments, Proposition 4.4 implies that

$$F_t(T_1,\dots,T_n) = \int_A \bar{C}_t^D(1 - G_t^1(x_1),\dots,1 - G_t^n(x_n)) \times \mathrm{d}C^M_{T_1-t,\dots T_n-t}(H_1(x_1 - V_t^1, T_1 - t),\dots,H_n(x_n - V_t^n, T_n - t)),$$

where $T_i > t$ and where we integrate over $A = (-\infty, M_t^1) \times \cdots \times (-\infty, M_t^n)$.

If M_t^i has a density $h_i(x,t) = \frac{\partial}{\partial x} H_i(x,t)$ for $t > 0$, we obtain

$$F_t^i(T) = \int_{-\infty}^{V_t^i} (1 - G_t^i(x)) h_i(x - V_t^i, T - t)\,\mathrm{d}x, \quad t < T. \tag{11}$$

Let us now turn to the implied contagion effects. For this, we fix an arbitrary firm $i \in I$ and suppose that this firm has direct ties, in the form of trade credits for example, with a number of other firms $J \subseteq I - \{i\}$. If we imagine firm i to be a supplier having granted trade credits to buying firms $j \in J$, then we would expect the supplier's default threshold D_i to depend on the thresholds D_j of buyers $j \in J$. Indeed, the supplier has to fund the loans, which affects its liabilities and hence its default threshold. The buyers incur directly new liabilities, which affect their respective default thresholds. If one buyer defaults on its obligations, we would expect a sudden increase in the supplier's default probability $F_t^i(T)$, corresponding to a contagion effect.

Based on (11), consider the mapping $t \to F_t^i(T)$ for fixed T. Assuming that assets V^i follow some sufficiently irregular but continuous process (Brownian motion for example), $h_i(x,t)$ is continuous in t. But $G_t^i(x)$ will jump at the unpredictable default times $\tau_j < \tau_i$ of buyers $j \in J$. This is easily seen from Lemma 4.1, noting that there is a sudden change in the pertaining default scenario. Indeed, upon τ_j the default threshold of firm j becomes suddenly known to investors, who immediately update their a posteriori threshold belief G^i_t for the supplier i. This leads in turn to a jump in the estimated default probability $F^i_t(T)$, which corresponds to a re-assessment of the supplier's financial health by bond investors in light of a buyer's default. We will illustrate these effects in an explicit example in Section 5, where we also consider the sign and the size of the jumps.

The implied bond prices exhibit an analogous pattern. Consider a defaultable zero bond [5] with maturity date T and a recovery payment of $1 - \delta_i$ at T. We let $\delta_i \in [0, 1]$ be independent, with expected value $\bar{\delta}_i$. Furthermore, δ_i is $\mathscr{G}_{\tau_i}$-measurable, meaning that only at default the recovery becomes public knowledge. Let $d(t, T)$ denote the time-t price of a default-free zero bond maturing at T. Assuming that defaults are independent of riskless interest rates, the defaultable zero bond has at time $t < \tau_i$ a price of

$$p_i(t, T) = d(t, T) - d(t, T)\bar{\delta}_i F^i_t(T), \quad t < T, \tag{12}$$

which is the value of a riskless zero bond less the expected default loss. From this it is clear that jumps in the default probability translate directly to jumps in the supplier's bond price upon defaults of buyers.

4.3. Default dependence structure

4.3.1. Default copula

It is obvious from Proposition 4.4 and Example 4.5 that joint default probabilities F_t depend critically on the nature of both historical asset low copula C^M and the threshold copula C^D. In analogy to Proposition 3.3, we have the following result, which relates these two copulas to the default (time) dependence structure C^τ.

Proposition 4.6. *Assume that the unconditional default probability F^i_0 is continuous and let J_i denote its generalized inverse. The copula C^τ of the default times is for all $u_i \in [0, 1]$ given by*

$$C^\tau(u_1, \ldots, u_n) = E[\bar{C}^D(1 - G^1(M^1_{J_1(u_1)}), \ldots, 1 - G^n(M^n_{J_n(u_n)}))].$$

The default dependence structure is therefore a function of both asset and threshold dependence structure, represented by C^M and C^D:

[5] In our model, the capital structure of the firms is based on consol bonds having no fixed maturity and paying out a constant coupon to the bond investors. We can strip the consol coupon into a continuum of zero coupon bonds with recovery being pro-rata based on the default-free market value that the strips contribute to the consol. As for the valuation of the consol bond, it is therefore enough to consider the valuation of the zero bonds.

$$C^\tau(u_1,\dots,u_n) = \int_{\mathbb{R}^n_-} \bar{C}^D_0(1-G^1(x_1),\dots,1-G^n(x_n)) \times \mathrm{d}C^M_{J^1(u_1),\dots,J^n(u_n)}(H_1(x_1,J^1(u_1)),\dots,H_n(x_n,J^n(u_n))).$$

Under incomplete information, the joint default behavior of several firms and thus the level of aggregated credit risk, depends not only on the tail properties of the copula C^M but also critically on those of C^D. Given marginal default probabilities F^i_0, lower tail dependent copulas C^M implying low values in all marginals simultaneously lead to an increased likelihood of joint defaults, corresponding to a high level of aggregate risk. Analogously, upper tail dependent copulas $\bar{C}^D_0$ implying low absolute threshold values in all marginals simultaneously induce higher joint default probabilities. We will illustrate these effects in Section 5.

In various applications it is often required to simulate correlated default times. In our model, having the default time copula C^τ at hand, we can proceed as follows. First, we generate a sample $(W_1,\dots,W_n)$ from C^τ. For details on appropriate algorithms we refer to Embrechts et al. (2001). In a second step, we set $\sigma_i = J_i(W_i)$. It is easy to see that $(\sigma_1,\dots,\sigma_n)$ has joint distribution function F_0, as desired.

4.3.2. Measuring default dependence

Besides providing instructive insights into the structure of joint defaults, the copula C^τ given by Proposition 4.6 also proves to be a sophisticated *measure of default dependence*. Indeed, C^τ uniquely describes the complete non-linear default (time) dependence structure. [6] As a joint distribution, C^τ satisfies a version of the Fréchet-bounds inequality $W^-(u) \leqslant C^\tau(u) \leqslant W^+(u)$ for all $u \in [0,1]^n$. If $C^\tau(u) = W^-(u) = \max(u_1+\cdots+u_n-n+1,0)$, then the defaults are perfectly negatively correlated (we speak of countermonotone defaults). If $C^\tau(u) = W^+(u) = \min(u_1,\dots,u_n)$, then the defaults are perfectly positively related (and we speak of comonotone defaults). If $C^\tau = \Pi$, where $\Pi(u_1,\dots,u_n) = u_1,\dots,u_n$, then the defaults are independent (this is the case if and only if assets and thresholds are independent, i.e. if $C^D = C^M_{t_1,\dots,t_n} = \Pi$ for all $t_i \geqslant 0$). This suggests a partial order on the set of (default) copulas, which can allow a comparison of default time vectors.

Although a pairwise copula is easily constructed from the n-dimensional default copula by setting $C^\tau(1,\dots,1,u,1,\dots,1,v,1,\dots,1)$, a scalar-valued pairwise dependence measure can perhaps provide more intuition about the degree of pertaining default dependence. We can define, for example, Spearman's *rank default correlation* measuring the degree of monotonic dependence, by

$$\rho_S(\tau_i,\tau_j) = \rho(F^i_0(\tau_i), F^j_0(\tau_j)), \tag{13}$$

where ρ is the ordinary linear (product-moment) correlation. Thus ρ_S is the linear correlation of the copula C^τ. By using the definition of linear correlation,

[6] An alternative choice is the copula of the vector $(N^1_t,\dots,N^n_t)$, where $N^i_t = 1_{\{t\geqslant\tau_i\}}$ is the default indicator, which represents the complete default event dependence structure. But since each N^i_t is a discrete random variable, the associated copula is not unique anymore. A way around that is the unique copula-type representation of bivariate random vectors with Bernoulli marginals suggested by Tajar et al. (2001).

$$\rho_S(\tau_i, \tau_j) = 12 \int_0^1 \int_0^1 (C^\tau(u, v) - uv) \, du \, dv, \tag{14}$$

showing that Spearman's rank correlation depends on the copula only. Also, ρ_S is seen to be a scaled version of the signed volume enclosed by the copula C^τ and the product copula $\Pi(u, v) = uv$. Thus ρ_S is a measure of the average distance between the actual distribution of (τ_i, τ_j) and their distribution given independence. Besides being invariant under (strictly) increasing transformations, ρ_S has the following useful properties, which are easily verified using (14): $\rho_S \in [-1, 1]$, $\rho_S = 1$ if and only if $C^\tau = W^+$, $\rho_S = -1$ if and only if $C^\tau = W^-$, and $\rho_S = 0$ if $C^\tau = \Pi$.

The default copula itself and the default copula based rank default correlation measures suggested above capture the complete non-linear default dependence; they should be preferred over the existing default correlation measures based on *linear* correlation ρ. Although simple measures such as $\rho(1_{\{t \geqslant \tau_i\}}, 1_{\{t \geqslant \tau_j\}})$ or $\rho(\tau_i - t, \tau_j - t)$ are widely used in the literature (see, for example, Zhou, 2001; Hull and White, 2000b; Kealhofer, 1998; Lucas, 1995; Li, 2000), they are fundamentally flawed. As recently analyzed in detail by Embrechts et al. (2001), covariance is not the natural measure of dependence for non-elliptical random vectors such as $(1_{\{t \geqslant \tau_i\}}, 1_{\{t \geqslant \tau_j\}})$ and (τ_i, τ_j). Using linear correlation in this context has severe limitations, since only the linear part of the dependence structure is captured.

5. An explicit example

In the previous sections we have made no specific assumptions on the asset dynamics and the prior threshold belief of investors. In this section we illustrate the implications of our model in the two-firm case based on geometric Brownian motion and uniform priors. The explicit results extend straightforwardly to the higher-dimensional case.

5.1. Assets and prior

We consider a bond market where bonds of two firms, labelled 1 and 2, are traded. For concreteness, we think of Firm 2 as directly supplying goods to Firm 1, which enters in a trade credit with Firm 2. It is natural to consider these firms as positively dependent: if Firm 1 defaults on its obligation from the trade credit, this immediately reduces the cash flow of Firm 2, making it more default-risky (supposing the collateral value of the traded goods is sufficiently low). Investors agree that the total market value Z^i of each firm i follows a geometric Brownian motion, i.e.

$$Z_t^i = Z_0^i e^{\mu_i t + \sigma_i W_t^i}, \quad Z_0^i > 0, \tag{15}$$

where $\mu_i \in \mathbb{R}$ and $\sigma_i > 0$ are constant drift and volatility parameters, respectively, and where W^i is a standard Brownian motion. In order to focus on the contagion effects associated with the direct linkages between buyer and supplier, we suppose that the firm values itself are not correlated (cyclical default correlations are hence not present). From now on, we consider the log-firm value as the "asset process" V^i in the sense of the previous sections. That is, we set $V^i_t = \mu_i t + \sigma_i W^i_t$ for $i = 1, 2$, so that V^i is a Brownian motion starting at zero. Notice that the copula of the historical asset low satisfies $C^M_{t_1,t_2} = \Pi$ for all pairs $(t_1, t_2) \in \mathbb{R}^2_+$, due to the W^i being independent.

The incomplete information available to the secondary market is modeled by the filtration $(\mathcal{G}_t)_{t \geqslant 0}$, which is generated by the firm value processes and the default indicator processes. At time $t = 0$, investors agree on a marginal prior on firms' default thresholds with respect to the firm values Z^i, which we assume to be uniform [7] on $(0, Z^i_0)$. Besides the marginal prior which models the idiosyncratic threshold uncertainty, we suppose that investors also agree on the Clayton copula family

$$C(u, v; \theta) = (u^{-\theta} + v^{-\theta} - 1)^{-1/\theta}, \quad (u, v) \in [0, 1]^2, \ \ \theta > 0, \tag{16}$$

as a parametric description [8] of the dependence structure between firms' thresholds, including the value of the parameter $\theta > 0$. This parameter specifies the degree of threshold dependence associated with $C(u, v; \theta)$. We have $\lim_{\theta\to\infty} C(u, v; \theta) = W^+(u, v)$, reflecting perfect positive dependence, as well as $\lim_{\theta\to 0} C(u, v; \theta) = \Pi(u, v)$, corresponding to independence. As scalar-valued measure of threshold dependence in relation to the parameter θ we consider Kendall's rank correlation ρ_K, given by

$$\rho_K = 4 \int_0^1 \int_0^1 C(u, v; \theta)\, dC(u, v; \theta) - 1 = \frac{\theta}{\theta + 2}. \tag{17}$$

This shows that ρ_K is determined by the copula only and is thus invariant under strictly increasing transformations of the thresholds. Notice that for $\theta > 0$, we have $\rho_K \in [0, 1]$, reflecting the positive dependence between the firms. Under these assumptions, the prior with respect to the log-firm values V^i is as follows.

Lemma 5.1. *The prior on the thresholds $D = (D_1, D_2)$ is represented by the unique joint distribution function*

$$G(x, y; \theta) = (e^{-\theta x} + e^{-\theta y} - 1)^{-1/\theta}, \quad (x, y) \in \mathbb{R}^2_-.$$

5.2. Belief updating

Let us suppose that $\tau_1 < \tau_2$ almost surely. That is, the buyer defaults before the supplier, so that we can examine the adverse effects of this event (the contagion effects) on the supplier. Taking into account Lemma 5.1, the conditional joint law of D follows from Lemma 4.1 by straightforward calculations.

[7] We focus on the instructive case of equal marginal priors for both firms, but we stress that different marginal priors are possible as well.

[8] This choice is only led by analytical considerations. There are many more suitable copula families, and we refer to Nelsen (1999) and Joe (1997) for a large number of examples.

Corollary 5.2. *Under the current assumptions, the a posteriori belief of the bond investors is represented by the conditional distribution*

$$G_t(x,y;\theta) = \frac{1}{K_t(\theta)}(\mathrm{e}^{-\theta(x\wedge M_t^1)} + \mathrm{e}^{-\theta(y\wedge M_t^2)} - 1)^{-1/\theta}, \quad t < \tau_1,$$

where $K_t(\theta) = G(M_t^1, M_t^2;\theta)$ *and for* $x \in [d_1 = M_{\tau_1}^1, 0]$

$$G_t(x,y;\theta) = \left(\frac{\mathrm{e}^{-\theta d_1} + \mathrm{e}^{-\theta(y\wedge M_t^2)} - 1}{\mathrm{e}^{-\theta d_1} + \mathrm{e}^{-\theta M_t^2} - 1}\right)^{-(1/\theta+1)}, \quad t \geqslant \tau_1.$$

Let us consider the conditional copula C_t^D associated with the conditional distribution G_t, cf. (10). Using the inverse functions I_t^i of the marginals G_t^i derived in Corollary 5.2, for $t < \tau_1$ it is easy to verify that

$$C_t^D(u,v;\theta) = G_t(I_t^1(u), I_t^2(v);\theta) = C^D(u,v;\theta), \tag{18}$$

so that the threshold copula is invariant under conditioning if $t < \tau_1$. The intuitive reason for this is that before τ_1 only "inessential" information on the upper bound of D_1 becomes available. Only at τ_1 the threshold D_1 is revealed. This information leads to an update of the conditional threshold copula, as the following extreme but instructive example shows. Suppose that the thresholds are comonotone, $C^D(u,v;\theta) = u \wedge v$. With equal marginal priors for both firms, this implies $D_1 = D_2$ almost surely, see Embrechts et al. (2001). Then for $t \geqslant \tau_1$ the copula C_t^D must satisfy

$$G_t(x,y;\theta\to\infty) = 1_{\{x\geqslant d_1, y\geqslant d_1\}} = C_t^D(1_{\{x\geqslant d_1\}}, 1_{\{y\geqslant d_1\}};\theta\to\infty),$$

implying that $C_t^D(u,v;\theta\to\infty) = uv$ for all $t \geqslant \tau_1$. Again at an intuitive level, if D_1 becomes known at τ_1, then $D_2 = D_1$ is non-random any more, which corresponds to $\mathscr{G}_t$-conditional independence. In general, if $C^D \neq \Pi$ then the threshold copula will not be invariant under conditioning for $t \geqslant \tau_1$.

5.3. Joint default probabilities

In order to establish joint default probabilities, we will need the density $h_i(\cdot,t)$ of the historical asset low M_t^i of the Brownian motion V_t^i with drift μ_i and volatility σ_i (cf. Borodin and Salminen, 1996):

$$h_i(t,x) = \frac{1}{\sigma_i\sqrt{t}}\phi\left(\frac{\mu_i t - x}{\sigma_i\sqrt{t}}\right) + \mathrm{e}^{2\mu_i x/\sigma_i^2}\left[\frac{2\mu_i}{\sigma_i^2}\Phi\left(\frac{x+\mu_i t}{\sigma_i\sqrt{t}}\right) + \frac{1}{\sigma_i\sqrt{t}}\phi\left(\frac{x+\mu_i t}{\sigma_i\sqrt{t}}\right)\right],$$

where Φ (respectively, ϕ) is the standard normal distribution (respectively, density) function.

Corollary 5.3. *Under the current assumptions, if both firms operate the conditional joint default probability is for* $t < T_i$ *given by*

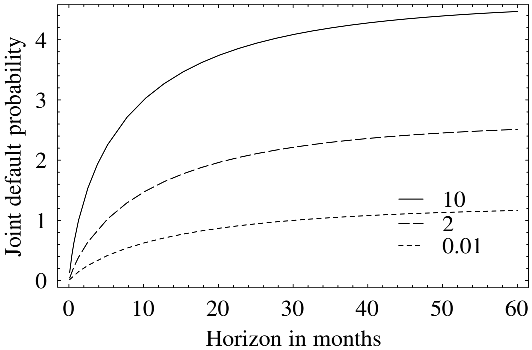

Fig. 1. The term structure of joint unconditional default probabilities in per cent, varying θ.

$$F_t(T_1, T_2; \theta) = \frac{1}{K_t(\theta)} \int_{-\infty}^{M_t^2} \int_{-\infty}^{M_t^1} (G(x,y) - G(x, M_t^2) - G(M_t^1, y) + 1) \times h_1(x - V_t^1, T_1 - t) h_2(y - V_t^2, T_2 - t)\, \mathrm{d}x\, \mathrm{d}y.$$

We emphasize that this result, along with the previous ones, can be straightforwardly extended to the general n-firm case with $n > 2$.

Fig. 1 graphs the term structure of joint unconditional default probabilities $F_0(T, T; \theta)$. The parameter for computations are as follows: $\mu_1 = \mu_2 = 6\%$ (the riskless rate), $\sigma_1 = 20\%$, $\sigma_2 = 30\%$, $M_t^1 = -0.2$, and $M_t^2 = -0.1$.

5.4. Contagion effects

We now turn to the implications of the direct relationships between buyer and supplier, corresponding to the threshold dependence structure $C^D(u, v; \theta)$. Fig. 2 displays the supplier's default probabilities $F_t^2(T; \theta)$ as a function of θ for various time

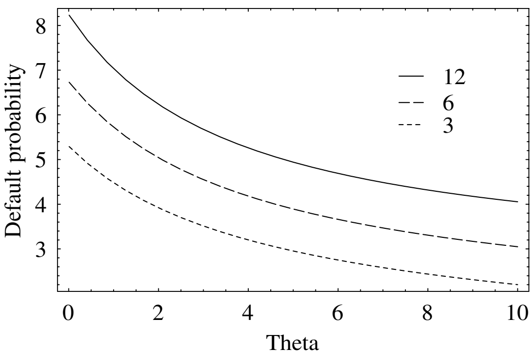

Fig. 2. Conditional default probability of the supplier in per cent, varying horizon T (both firms operate).

horizons T when both firms operate. The supplier's current asset value is set to $V_t^2 = -0.05$. Somewhat surprising at a first glance, $F_t^2(T;\theta)$ decreases in the degree of association θ. The intuition here is that, given a positive degree of monotonic association between the firms, the information that the buyer is still operating at time t and is hence honoring its obligations to the supplier, signals bond investors that the supplier is doing well (at least with respect to the relationship to the buyer). Basically, the stronger the association (the higher θ), the more convincing is the fact that the buyer operates and repays its debt to the supplier, leading to lower default probabilities of the supplier as perceived by bond investors.

Fig. 3 shows the supplier's conditional default probabilities as a function of θ if the buyer has already defaulted. We see that in this case $F_t^2(T;\theta)$ is increasing in θ. Here the intuition is as follows: with positive association between the firms, the default of the buyer lets investors "downgrade" the supplier, since the supplier typically suffers losses from this default. The stronger the dependence, the more affected is the supplier, and the more severe is the downgrade. The effect of a given change of θ on $F_t^2(T;\theta)$ is for all horizons higher if the buyer has defaulted, compared to the case where it still operates.

We now consider the jump in the supplier's conditional default probability upon the default of the buyer, in dependence of θ. For a time horizon of $T = 12$ months, Fig. 4 shows $F_t^2(T;\theta)$ when both firms still operate at t and when the buyer defaults at t. For a given θ, the difference between the curves is the jump that the supplier's default probability would experience upon the buyer's default. Quite intuitive, the jump size is increasing in θ. Clearly, if there is no relation between the firms, the default status of the buyer does not affect the suppliers default probability. Note that the jump effect is critically dependent on the time horizon T. Fairly intuitive, with $T \to 0$ the jump effect vanishes.

5.5. Portfolio default risk

We now consider a stylized portfolio composed of one zero-recovery zero-coupon bond issued by the buyer and one issued by the supplier. From a risk measurement

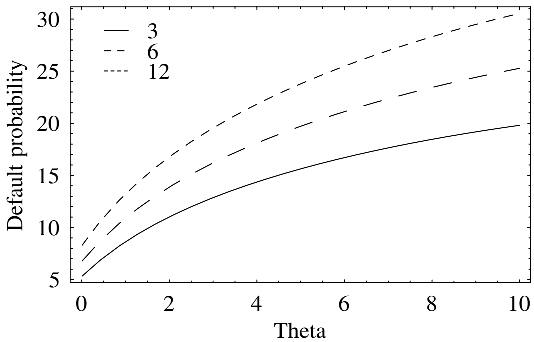

Fig. 3. Conditional default probability of the supplier in per cent, varying horizon T (the buyer has defaulted).

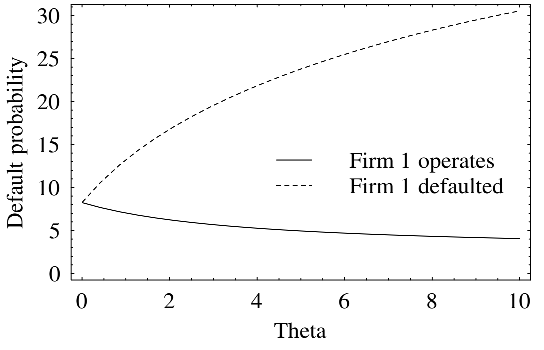

Fig. 4. Conditional default probabilities of supplier in per cent when both firms operate and when buyer has defaulted.

and management perspective we are interested in the distribution of aggregate losses $N_T = 1_{\{\tau_1 \leqslant T\}} + 1_{\{\tau_2 \leqslant T\}}$. As discussed in Section 4.3.1, the shape of this distribution will be closely related to the nature of the threshold copula C^D. Specifically, upper tail dependent C^D will imply higher aggregate loss fluctuations, corresponding to a higher risk of the portfolio. To study this issue, we consider the variance $\mathrm{Var}[N_T]$ for two different copula families. The first is the Clayton copula (16), which exhibits lower tail dependence (see Lindskog, 2000, for example). The second is the Gumbel copula

$$C(u, v; \theta) = \exp\left(-[(-\ln u)^\theta + (-\ln v)^\theta]^{1/\theta}\right), \quad (u, v) \in [0, 1]^2, \ \ \theta \geqslant 1,$$

which exhibits upper tail dependence for $\theta > 1$. Kendall's rank correlation is given by $\rho^{\mathrm{K}} = 1 - 1/\theta$.

We compare the loss variance $\mathrm{Var}_{\mathrm{Gu}}[N_T]$ that is obtained with the Gumbel copula with the variance $\mathrm{Var}_{\mathrm{Cl}}[N_T]$ obtained with the Clayton copula. For a horizon of $T = 12$ months, Fig. 5 graphs the ratio $\mathrm{Var}_{\mathrm{Gu}}[N_T]/\mathrm{Var}_{\mathrm{Cl}}[N_T]$ as a function of the rank threshold correlation ρ^{K}. For any given positive degree of monotonic threshold de-

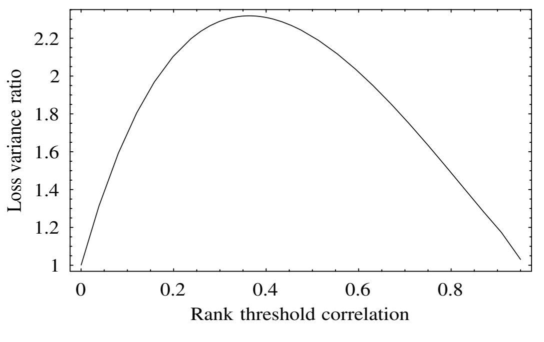

Fig. 5. Default number variance ratio for Gumbel and Clayton threshold copulas as a function of rank threshold correlation.

pendence, $\text{Var}_{\text{Gu}}[N_T]$ exceeds $\text{Var}_{\text{Cl}}[N_T]$ by a factor of at most 2.3. Even for fixed (i.e. calibrated) monotonic threshold dependence ρ^K, upper tail dependent copulas seems to induce fatter tails, i.e. higher loss fluctuations, than lower tail dependent ones. This points to the existence of *model risk* in choosing the "right" threshold copula; it parallels the finding of Frey and McNeil (2001) on asset copulas in latent variable models $(V_T^i, D_i)_{i\in I}$ with perfect information.

6. Conclusion

A thorough understanding of correlated multi-firm default mechanisms is of importance for corporate security valuation, design of credit derivatives, risk measurement and management in financial institutions, and the supervision of financial markets.

In this paper we distinguish between two correlation mechanisms: cyclical correlation and default contagion. Cyclical correlation between firms is due to their dependence on common smoothly varying macro-economic factors. Contagion between defaults of firms having direct ties to each other, for example through a parent–subsidiary relationship, refers to sudden jumps in credit spreads, which are correlated across several firms. If adverse contagion-induced spread jumps appear on a larger scale, they typically lead to sudden excessive losses in the credit portfolios of financial institutions, which need to be promptly covered through sufficient capital reserves. An explicit modeling of contagion effects is therefore particularly important, from both a risk management and a banking regulatory point of view.

We propose a structural model of correlated default, where firms are subject to both cyclical correlation and direct contagion processes. In our model contagion between defaults of related firms is due to public bond investors' complete information on firms' characteristics. Such incomplete information of investors was a major issue in the recent defaults of Enron or WorldCom. From observing the default status of the firms in the market, investors learn over time about the unknown characteristics. Specifically, an unexpected default leads to "surprise" updates of investors' prior beliefs, which imply jumps in default probabilities and credit spreads in accordance with stylized empirical facts. By accommodating such information-based contagion effects, in comparison with the asset value-based standard industry models KMV and CreditMetrics, our model leads to an improved estimate of correlated default risk.

Our results indicate that the disclosure of information matters a lot: increased transparency reduces the likelihood of contagion effects due to incomplete information of investors. Regulatory policy should thus support information disclosure of firms.

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Appendix A. Proofs

Proof of Proposition 3.3. As a corollary to Sklar's Theorem, for continuous marginals F_0^i we can construct the (unconditional) default time copula C^τ by means of the quantile functions J_i of the default times τ_i:

$$C^\tau(u_1,\dots,u_n) = F_0(J_i(u_1),\dots,J_n(u_n)).$$

The statement then follows immediately from (5) in case $t = 0$. □

Proof of Lemma 4.1. Since $M_t \in \mathscr{F}_t \subset \mathscr{G}_t$, from the structure of the σ-field $\sigma(\tau \wedge t)$ [9] and from Bayes' Theorem, on the set $\{S_t = s\}$ we have

$$\begin{aligned} P[D \in A \,|\, \mathscr{G}_t] &= P[D \in A \,|\, \sigma(\tau \wedge t) \vee \mathscr{F}_t] = P[D \in A \,|\, D \in B(M_t, s), \mathscr{F}_t] \\ &= \frac{P[D \in A \cap B(M_t, s) \,|\, \mathscr{F}_t]}{P[D \in B(M_t, s) \,|\, \mathscr{F}_t]}. \end{aligned}$$

This implies our assertion because D is independent of $\mathscr{F}_t$. □

Proof of Lemma 4.3. Fixing some $t \geqslant 0$, we have for $u_i = 1 - G_t^i(x_i) \in [0,1]$ and any $x_i \leqslant 0$ the equalities

$$\begin{aligned} \bar{C}_t^D(1-G_t^1(x_1),\dots,1-G_t^n(x_n)) &= P[D_1 > x_1,\dots,D_n > x_n \,|\, \mathscr{G}_t] \\ &= \sum_{i_1=1}^{2}\cdots\sum_{i_n=1}^{2}(-1)^{i_1+\cdots+i_n} P[D_1 \leqslant v_{1i_1},\dots,D_n \leqslant v_{ni_n} \,|\, \mathscr{G}_t] \\ &= \sum_{i_1=1}^{2}\cdots\sum_{i_n=1}^{2}(-1)^{i_1+\cdots+i_n} C_t^D(G_t^1(v_{1i_1}),\dots,G_t^n(v_{ni_n})), \end{aligned}$$

where $v_{j1} = x_j$ and $v_{j2} = 0$. The second equality can be verified using standard arguments. The third equality is a consequence of Sklar's Theorem. Since $G_t^i(0) = 1$ the claim follows. □

Proof of Proposition 4.4. By (2) and the law of iterated expectations,

$$\begin{aligned} F_t(T_1,\dots,T_n) &= P[D_1 \geqslant M_{T_1}^1,\dots,D_n \geqslant M_{T_n}^n \,|\, \mathscr{G}_t] \\ &= E\Big[P[D_1 \geqslant M_{T_1}^1,\dots,D_n \geqslant M_{T_n}^n \,|\, \mathscr{G}_t \vee \mathscr{F}_T] \,|\, \mathscr{G}_t\Big] \end{aligned}$$

for $T_i > t$ and $T = \max_i T_i$. Note that D is independent of $(\mathscr{F}_t)_{t \geqslant 0}$ and $M_{T_i}^i \in \mathscr{F}_T$. Our assertion follows from the fact that the survival threshold copula $\bar{C}_t^D$ satisfies

[9] The σ-field $\sigma(\tau \wedge t) \subseteq \mathscr{G}_t$ is generated by the events $\{\tau_i \leqslant u\} = \{M_u^i \leqslant D_i\}$ for $u \leqslant t$ and $i \in S_t$ as well as the atoms $\{\tau_i > t\} = \{M_t^i > D_i\}$ for $i \in I - S_t$.

$$P[D_1 > x_1, \dots, D_n > x_n \,|\, \mathscr{G}_t] = \bar{C}_t^D(P[D_1 > x_1 \,|\, \mathscr{G}_t], \dots, P[D_n > x_n \,|\, \mathscr{G}_t]) \\ = \bar{C}_t^D\big(1 - G_t^1(x_1), \dots, 1 - G_t^n(x_n)\big)$$

for any $x_i \leqslant 0$. □

Proof of Proposition 4.6. Noting that $F_0^i(T)$ is continuous in T, the copula representation of the multivariate distribution function $F_0(T_1, \dots, T_n)$ implies that $C^\tau(u_1, \dots, u_n) = F_0(J^1(u_1), \dots, J^n(u_n))$. The claim now follows directly from Proposition 4.4. □

Proof of Lemma 5.1. Let $\hat{D}_i$ denote the threshold with respect to the firm value Z^i. By assumption $\hat{D}_i$ is uniform on $(0, Z_0^i)$, and $(\hat{D}_1, \hat{D}_2)$ has copula C given by (16). In order to obtain the prior with respect to the log-firm value V^i on $(-\infty, 0)$, we introduce the transformation $D_i = \ln \hat{D}_i - \ln Z_0^i$, so that the marginal prior on D_i is represented by the marginal distribution function

$$G^i(x) = P[D_i \leqslant x] = P[\hat{D}_i \leqslant Z_0^i \mathrm{e}^x] = \mathrm{e}^x, \quad x \leqslant 0.$$

Notice that D_i is a strictly increasing transformation of $\hat{D}_i$. By the invariance property of copulas (cf. Nelsen, 1999), the copula C^D of the transformed threshold vector $D = (D_1, D_2)$ remains unchanged: $C^D(u, v; \theta) = C(u, v; \theta)$. Since the G^i are continuous, G is uniquely determined by C^D and the G^i. □

Proof of Corollary 5.3. The basis for the derivation in the Brownian case is Proposition 4.4 for $n = 2$, which implies

$$F_t(T_1, T_2; \theta) = \int_{-\infty}^{M_t^2} \int_{-\infty}^{M_t^1} \bar{C}_t^D\big(1 - G_t^1(x), 1 - G_t^2(y)\big)\, \mathrm{d}P\Big[M_{T_1}^1 \leqslant x, M_{T_2}^2 \leqslant y \,|\, \mathscr{G}_t\Big].$$

Using the fact that V^1 and V^2 are independent, and hence $P[M_{T_i}^i \in \mathrm{d}x \,|\, \mathscr{G}_t] = P[M_{T_i}^i \in \mathrm{d}x \,|\, V_t^i] = h_i(x - V_t^i, T_i - t)\mathrm{d}x$, the default distribution is

$$F_t(T_1, T_2; \theta) = \int_{-\infty}^{M_t^2} \int_{-\infty}^{M_t^1} \bar{C}_t^D(1 - G_t^1(x), 1 - G_t^2(y)) \\ \times h_1(x - V_t^1, T_1 - t) h_2(y - V_t^2, T_2 - t)\, \mathrm{d}x\, \mathrm{d}y.$$

The conditional copula $\bar{C}_t^D$ is given by (10). In the bivariate case $\bar{C}_t^D(u, v; \theta) = C_t^D(1 - u, 1 - v; \theta) + u + v - 1$. From (18), $C_t^D = C^D$ and by using (16)

$$\bar{C}_t^D(u, v; \theta) = ((1 - u)^{-\theta} + (1 - v)^{-\theta} - 1)^{-1/\theta} + u + v - 1 \quad \forall t \geqslant 0.$$

Substituting the marginals G_t^i, after simplification we get

$$\bar{C}_t^D(1 - G_t^1(x), 1 - G_t^2(y)) = G_t(x, y) - G_t^1(x) - G_t^2(y) + 1$$

and the result follows from Corollary 5.2. □

References

Allen, F., Gale, D., 2000. Financial contagion. Journal of Political Economy 108, 1–33.

Avellaneda, M., Wu, W., 2001. Credit contagion: Pricing cross-country risk in brady debt markets. International Journal of Theoretical and Applied Finance 4, 921–938.

Avellaneda, M., Zhu, J., 2001. Modeling the distance-to-default process of a firm. Working Paper, Courant Institute.

Black, F., Cox, J.C., 1976. Valuing corporate securities: Some effects of bond indenture provisions. Journal of Finance 31, 351–367.

Borodin, A., Salminen, P., 1996. Handbook of Brownian Motion: Facts and Formulae. Birkhäuser, Basel.

Collin-Dufresne, P., Goldstein, R., Helwege, J., 2002. Are jumps in corporate bond yields priced? Modeling contagion via the updating of beliefs. Working Paper, Carnegie Mellon University.

Davis, M., Lo, V., 2001. Infectious defaults. Quantitative Finance 1, 382–387.

Duffee, G., 1998. The relation between treasury yields and corporate yield spreads. Journal of Finance 53, 2225–2243.

Duffie, D., Garleanu, N., 2001. Risk and valuation of collateralized debt obligations. Financial Analysis Journal 57 (1), 41–59.

Duffie, D., Lando, D., 2001. Term structures of credit spreads with incomplete accounting information. Econometrica 69 (3), 633–664.

Duffie, D., Singleton, K.J., 1998. Simulating correlated defaults. Working Paper, GSB, Stanford University.

Embrechts, P., McNeil, A.J., Straumann, D., 2001. Correlation and dependence in risk management. In: Dempster, M. (Ed.), Risk Management: Value at Risk and Beyond. Cambridge University Press, Cambridge.

Frey, R., McNeil, A.J., 2001. Modeling dependent defaults. Working Paper, Department of Mathematics, ETH Zürich.

Frey, R., McNeil, A.J., 2002. Var and expected shortfall in portfolios of dependent credit risks: Conceptual and practical insights. Journal of Banking and Finance 26, 1317–1334.

Giesecke, K., 2001. Default and information. Working Paper, Cornell University.

Giesecke, K., Goldberg, L., 2003. Forecasting default in the face of uncertainty. Working Paper, Cornell University and Barra, Inc.

Giesecke, K., Weber, S., 2002. Credit contagion and aggregate losses. Working Paper, Cornell University and Technische Universität Berlin.

Gupton, G., Finger, C., Bhatia, M., 1997. Creditmetrics-technical Document. RiskMetrics Group, New York.

Huang, M., Huang, J., 2000. How much of the corporate treasury yield spread is due to credit risk? A new calibration approach. Working Paper, Stanford University.

Hull, J., White, A., 2000a. Valuing credit default swaps i: No counterparty default risk. Journal of Derivatives 8, 29–40.

Hull, J., White, A., 2000b. Valuing credit default swaps ii: Modeling default correlations. Journal of Derivatives 8, 897–907.

Iyengar, S., 1985. Hitting lines with two-dimensional Brownian motion. SIAM Journal of Applied Mathematics 45 (6), 983–989.

Jarrow, R.A., Yu, F., 2001. Counterparty risk and the pricing of defaultable securities. Journal of Finance 56 (5), 555–576.

Jarrow, R.A., Lando, D., Yu, F., 2000. Default risk and diversification: Theory and applications. Working Paper, Cornell University.

Joe, H., 1997. Multivariate Models and Dependence Concepts. Chapman and Hall, London.

Kealhofer, S., 1998. Portfolio management of default risk. Working Paper, KMV Corp.

Keenan, S., 2000. Historical default rates of corporate bond issuers 1920–1999. Moody's Investors Services.

Lambrecht, B., Perraudin, W., 1996. Creditor races and contingent claims. European Economic Review 40, 897–907.

Leland, H.E., 1994. Corporate debt value, bond covenants, and optimal capital structure. Journal of Finance 49 (4), 1213–1252.

Li, D.X., 2000. On default correlation: A copula function approach. Journal of Fixed Income 9, 43–54.

Lindskog, F., 2000. Modeling dependence with copulas. Dissertation, Department of Mathematics, ETH Zürich.

Lucas, D.J., 1995. Default correlation and credit analysis. Journal of Fixed Income 11, 76–87.

Nelsen, R., 1999. An Introduction to Copulas. Springer-Verlag, New York.

Rebholz, J., 1994. Planar diffusions with applications to mathematical finance. Ph.D. Dissertation, Department of Mathematics, University of California, Berkeley.

RiskMetrics Group, 2002. Creditgrades: Technical document.

Schönbucher, P., Schubert, D., 2001. Copula-dependent default risk in intensity models. Working Paper, Universität Bonn.

Tajar, A., Denuit, M., Lambert, P., 2001. Copula-type representation for random couples with bernoulli marginals. Working Paper, Institute de Statistique, Université Catholique de Louvain.

Yu, F., 2002. Accounting transparency and the term structure of credit spreads. Working Paper, University of California at Irvine.

Zhou, C., 2001. An analysis of default correlation and multiple defaults. Review of Financial Studies 14 (2), 555–576.