Online Appendix To

Exact Simulation of Point Processes With Stochastic Intensities

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Operations Research, Forthcoming

1 Recursive transform calculation

In the jump-diffusion setting of Section 5, the computation of the projected intensity requires the recursive calculation of the transform at an event time,

$$M_{T_{n+1}}(z,0) = J_{n+1}(z,0)\frac{\partial_v \exp(a(T_n,T_{n+1},v,L_{T_n}))M_{T_n}(b(T_n,T_{n+1},v,L_{T_n}),0)|_{v=z}}{\partial_v \exp(a(T_n,T_{n+1},v,L_{T_n}))M_{T_n}(b(T_n,T_{n+1},v,L_{T_n}),0)|_{v=0}}. \quad (1)$$

While analytical, this calculation tends to become more involved with every step. This appendix addresses this issue. We distinguish different cases according to the value of the parameter δ in the SDE (14), which controls the sensitivity of the state X to jumps in L.

1.1 State not sensitive to events

Suppose $\delta = 0$ so that $J_n(z,0) = 1$ in (47). In this case, the state X follows a diffusion process whose dynamics are not influenced by N. However, X does influence the point process intensity, and hence, the arrival dynamics. The intensity is also a function of L itself, so the point process retains the self-exciting property. Now, following an idea of Frey, Prosdocimi & Runggaldier (2007), with a suitable choice of the distribution of the initial value X_0, the transform (47) remains easy to compute even for large n.

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Proposition 1.1. *Suppose X_0 has a gamma distribution with parameters $(\frac{2\kappa c}{\sigma^2}, 1)$, i.e.*

$$M_0(z,0) = \mathbb{E}(\exp(-zX_0)) = (1+z)^{-\frac{2\kappa c}{\sigma^2}}, \quad z > 0. \tag{2}$$

Then there exists a polynomial $P_n(z)$ of degree $(n-1)$, such that

$$M_{T_n}(z,0) = C_n(zH_n + K_n)^{-\frac{2\kappa c}{\sigma^2}-n} P_n(z), \tag{3}$$

where $C_n = K_n^{\frac{2\kappa c}{\sigma^2}+n}/P_n(0)$, and H_n and K_n satisfy the recursions

$$\begin{cases} H_n = R_n H_{n-1} + U_n K_{n-1}, & H_0 = 1 \\ K_n = S_n H_{n-1} + V_n K_{n-1}, & K_0 = 1 \end{cases}$$

where

$$\begin{aligned} R_{n+1} &= \gamma + \kappa + e^{\gamma(T_{n+1}-T_n)}(\gamma - \kappa) \\ S_{n+1} &= 2(e^{\gamma(T_{n+1}-T_n)} - 1)\log(\theta + L_{T_n}) \\ U_{n+1} &= \sigma^2(e^{\gamma(T_{n+1}-T_n)} - 1) \\ V_{n+1} &= \gamma - \kappa + e^{\gamma(T_{n+1}-T_n)}(\gamma + \kappa). \end{aligned}$$

Let $c_n = -\frac{2\kappa c}{\sigma^2} - n$. The coefficients of $P_n(z) = \sum_{i=0}^{n-1} a_i^n z^i$ can be computed from the linear recursive relations

$$a_i^n = (c_n + i + 1)H_n U_n \bar{a}_{i-1} + \left[(c_n + i + 1)H_n V_n + K_n U_n (i+1)\right]\bar{a}_i + K_n V_n (i+1)\bar{a}_{i+1}$$

where

$$\bar{a}_k = \frac{1}{n^n}\sum_{i=0}^{n-2} a_i^{n-1} \sum_{l=\max(0,(k-i))}^{\min(k,(n-i-2))} \binom{i}{k-l}\binom{n-i-2}{l} R_n^{k-l} S_n^{i-k+l} U_n^l V_n^{n-i-2-l}$$

Proof. The claim is true for $n = 0$. For $W_{n+1} = 2\gamma e^{\frac{1}{2}(T_{n+1}-T_n)(\gamma+\kappa)}$, we can write

$$\begin{aligned} a(T_n, T_{n+1}, v, L_{T_n}) &= \frac{2\kappa c}{\sigma^2}\log\left(\frac{W_{n+1}}{vU_{n+1} + V_{n+1}}\right) \\ b(T_n, T_{n+1}, v, L_{T_n}) &= \frac{vR_{n+1} + S_{n+1}}{vU_{n+1} + V_{n+1}}. \end{aligned}$$

Using these expressions, for the case $n = 1$ we get

$$M_{T_1}(z,0) = \frac{\partial_v\{\left(\frac{W_1}{vU_1+V_1}\right)^{\frac{2\kappa c}{\sigma^2}} M_0\left(\frac{vR_1+S_1}{vU_1+V_1},0\right)\}|_{v=z}}{\partial_v\{\left(\frac{W_1}{vU_1+V_1}\right)^{\frac{2\kappa c}{\sigma^2}} M_0\left(\frac{vR_1+S_1}{vU_1+V_1},0\right)\}|_{v=0}}$$

$$= \frac{(zH_1+K_1)^{\frac{2\kappa c}{\sigma^2}-1}}{K_1^{\frac{2\kappa c}{\sigma^2}-1}}.$$

For the case $n \geq 2$, we can use a simple induction over n. Assume (3) holds for $n-1$. Let $\bar{P}_n(z) = \sum_{i=0}^{n-1} \bar{a}_i$. We have

$$(n-1)^{(n-1)} P_{n-1}\left(\frac{vR_n+S_n}{vU_n+V_n}\right) = n^n (vU_n+V_n)^{2-n}\bar{P}_n(v)$$

and

$$P_n(v) = U_n(vH_n+K_n)\bar{P}_n + \left(\frac{2\kappa c}{\sigma^2} - 1 - n\right) H_n(zU_n+V_n)\bar{P}_n(z)$$
$$+ (zH_n+K_n)(zU_n+V_n)\partial_z\bar{P}_n(z).$$

Now, by recursive updating of M_{T_n}, the induction assumption, and the above equations, it can be easily shown that (3) holds for n.

Moreover, we have introduced a scaling factor $1/n^n$ to bound the range of the coefficients. According to equation (3), $M_{T_n}(z,0)$ is proportional to the ratio of $P_n(z)$ and $P_n(0)$ so scaling the coefficients does not change $M_{T_n}(z,0)$. □

1.2 State weakly sensitive to events

Now suppose $\delta > 0$. In this case, the state jumps at event times of N. The point process is self-exciting because its intensity is a function of both X and the point process itself. Unfortunately, the formulation in the previous section does not apply. This is due to the exponential form of the $J_n(z,0)$ term. The terms of $M_{T_n}(z,0)$ for the gamma distribution case in Proposition 1.1 can be represented as polynomials growing linearly with n. This will not be the case in the presence of the exponential factor $J_n(z,0)$.

However, if $\delta\ell z \ll 1$ we can approximate

$$J_n(z,0) = \exp(-\delta\ell_n z) \approx 1 - \delta\ell_n z. \quad (4)$$

Define

$$\widetilde{M}_{T_{n+1}}(z,0) = (1-\delta\ell_{n+1}z)\frac{\partial_v \exp(a(T_n,T_{n+1},v,L_{T_n}))\widetilde{M}_{T_n}(b(T_n,T_{n+1},v,L_{T_n}),0)|_{v=z}}{\partial_v \exp(a(T_n,T_{n+1},v,L_{T_n}))\widetilde{M}_{T_n}(b(T_n,T_{n+1},v,L_{T_n}),0)|_{v=0}}$$

where $\widetilde{M}_{T_0}(z,0) = M_{T_0}(z,0)$. Then an argument analogous to that used in Proposition 1.1 can be used to show that under the assumption (2),

$$\widetilde{M}_{T_n}(z,0) = (1-\delta\ell_n z)C_n(zH_n+K_n)^{\frac{-2\kappa c}{\sigma^2}-n}P_n(z),$$

where $P_n(z)$ is a polynomial of order $n-1$.

Proposition 1.2. *Given $\delta \ll 1$, the error $|M_{T_n}(z,0) - \widetilde{M}_{T_n}(z,0)|$ due to the approximation (4) is of order $O(\delta^2)$.*

Proof. We have that $|M_{T_1}(z,0) - \widetilde{M}_{T_1}(z,0)|$ takes the form

$$|\exp(-\delta\ell_1 z) - (1-\delta\ell_1 z)| \frac{|\partial_v \exp(a(0,T_1,v,0))M_0(b(0,T_1,v,0),0)|_{v=z}|}{|\partial_v \exp(a(0,T_1,v,0))M_0(b(0,T_1,v,0),0)|_{v=0}|}.$$

Since

$$\begin{aligned} 0 <& |\partial_z \exp(a(0,T_1,z,0))M_0(b(0,T_1,z,0),0)| \\ &= \mathbb{E}(X_{T_1}\exp(-zX_{T_1})\,|\,\mathcal{G}_{T_1})\exp(a(0,T_1,0,0))M_0(b(0,T_1,0,0),0) < \infty \end{aligned}$$

the term $|\exp(-\delta\ell_1 z) - (1-\delta\ell_1 z)|$ is of order $O(\delta^2)$ and so is $|M_{T_1}(z,0) - \widetilde{M}_{T_1}(z,0)|$.

Next, for $n=2$, consider that

$$\begin{aligned} \partial_z \exp(a(T_1,T_2,&z,L_{T_1}))\widetilde{M}_{T_1}(b(T_1,T_2,z,L_{T_1}),0) \\ &= \partial_z \exp(a(T_1,T_2,z,L_{T_1}))\left[M_{T_1}(b(T_1,T_2,z,L_{T_1}),0) + O(\delta^2)\right] \\ &= \partial_z \exp(a(T_1,T_2,z,L_{T_1}))M_{T_1}(b(T_1,T_2,z,L_{T_1}),0) + O(\delta^2). \end{aligned}$$

Thus

$$\begin{aligned} \widetilde{M}_{T_2}(z,0) &= (1-\delta\ell_2 z)\frac{\partial_v \exp(a(T_1,T_2,v,L_{T_1}))M_{T_1}(b(T_1,T_2,v,L_{T_1}),0)|_{v=z} + O(\delta^2)}{\partial_v \exp(a(T_1,T_2,v,L_{T_1}))M_{T_1}(b(T_1,T_2,v,L_{T_1}),0)|_{v=0} + O(\delta^2)} \\ &= (1-\delta\ell_2 z)\left(\frac{\partial_v \exp(a(T_1,T_2,v,L_{T_1}))M_{T_1}(b(T_1,T_2,v,L_{T_1}),0)|_{v=z}}{\partial_v \exp(a(T_1,T_2,v,L_{T_1}))M_{T_1}(b(T_1,T_2,v,L_{T_1}),0)|_{v=0}} + O(\delta^2)\right) \\ &= (1-\delta\ell_2 z)\frac{\partial_v \exp(a(T_1,T_2,v,L_{T_1}))M_{T_1}(b(T_1,T_2,v,L_{T_1}),0)|_{v=z}}{\partial_v \exp(a(T_1,T_2,v,L_{T_1}))M_{T_1}(b(T_1,T_2,v,L_{T_1}),0)|_{v=0}} + O(\delta^2) \end{aligned}$$

and so $|\widetilde{M}_{T_2}(z,0) - M_{T_2}(z,0)|$ is given by

$$[\exp(-\delta\ell_2 z) - (1-\delta\ell_2 z)]\frac{|\partial_v \exp(a(T_1,T_2,v,L_{T_1}))M_{T_1}(b(T_1,T_2,v,L_{T_1}),0)|_{v=z}|}{|\partial_v \exp(a(T_1,T_2,v,L_{T_1}))M_{T_1}(b(T_1,T_2,v,L_{T_1}),0)|_{v=0}|} + O(\delta^2)$$

which is clearly of order $O(\delta^2)$.

It follows by extension that $|M_{T_n}(z,0) - \widetilde{M}_{T_n}(z,0)|$ is of order $O(\delta^2)$ for $n \geq 3$. □

References

Frey, Rüdiger, Cecilia Prosdocimi & Wolfgang Runggaldier (2007), Affine credit risk models under incomplete information, *in* J.Akahori, S.Ogawa & S.Watanabe, eds, 'Stochastic Processes and Applications to Mathematical Finance - Proceedings of the 6th Ritsumeikan International Symposium', World Scientific, pp. 97–113.