THE CORRELATION-NEUTRAL MEASURE FOR PORTFOLIO CREDIT

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Abstract

We derive a formula for a Fourier transform of a counting process that describes the arrival of unpredictable events, and we show how this transform facilitates an analytical treatment of a range of valuation, hedging and risk management problems that arise in single name and portfolio credit risk. Example applications include reduced form pricing of credit sensitive securities referenced on single or multiple issuers, hedging of constituent risks, model estimation, and credit portfolio risk measures. Our results cover situations with feedback, in which events have an impact on arrival rates (as with contagion) and risk-free interest rates (as with flights to quality). A complex-valued measure change neutralizes this feedback.

Key words: Counting process, point process, compensator, characteristic function, Fourier transform, Laplace transform, complex measure, credit derivative

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1 Introduction

The reduced form approach has become a standard tool for modeling dynamic credit risk. Its main advantage over alternatives is that it facilitates the specification of empirically plausible models for the term structure of single- and multi-issuer default risk that are analytically tractable. The computational tractability is due to a fundamental insight: in a reduced form model, the price of a credit sensitive security is equal to the risk-neutral expectation of the discounted promised cash flows. The discount rate is given by the sum of the risk-free interest rate and the intensity, or conditional default rate.[1] This observation allows the researcher to apply standard default-free term structure model formulations for the purpose of specifying an intensity based model of issuer default that leads to tractable valuation relations.

This article unifies and significantly extends the literature on the reduced form approach to credit risk by deriving a formula for a Fourier transform of a counting process N that describes the arrival of unpredictable events, and then showing that this transform facilitates an analytical treatment of a range of valuation, hedging and risk management problems that arise in single name and portfolio credit risk. The key insight of this article is that the counting process transform is analogous to the reduced form price of a defaultable security, and can therefore be calculated analytically for a broad class of counting process model specifications.

More specifically, let r be a stochastic discount rate process and Y a random variable that can represent a payoff at a horizon T. For real z, v and $t \leq T$, we consider the counting process transform

$$E_t \left[\exp\left(- \int_t^T r_s ds \right) e^{izY + iv(N_T - N_t)} \right] \tag{1}$$

where E_t denotes conditional expectation at time t. We express in terms of this transform the expected present value at t of the future cash flow $(a + bY)(c + dN_T)1_{\{N_T \leq K\}}$, where a, b, c, d and K are real numbers. If evaluated under risk-neutral probabilities, this expression leads directly to pricing relations for a variety of credit derivatives, including a "single name claim" that pays Y if the issuer survives to T and 0 otherwise, a "portfolio claim" that pays Y if the default count at T in a portfolio of credit sensitive securities exceeds K and 0 otherwise, an nth-to-default security that pays Y if the default count at T is equal to n, an option on the default count with strike K, a credit index or tranche on a portfolio of names, and other related securities. If evaluated under physical probabilities, the transform (1) leads to formulas for risk measures of corporate debt portfolios, such as value at risk and expected shortfall. Furthermore, the transform (1) can be used to implement random thinning, which attributes risk to portfolio constituents.[2] Random thinning facilitates applications such as capital allocation, hedging of constituent risks and estimation of portfolio risk from constituent data.

[1]See Duffie & Singleton (1999), Jarrow & Turnbull (1995) and Jarrow, Lando & Turnbull (1997).

[2]See Giesecke & Goldberg (2005) for an analysis of random thinning and applications to credit.

We show that the calculation of the counting process transform (1) reduces to the calculation of a Laplace transform of the payoff Y and the counting process compensator A that represents the cumulative intensity, or conditional arrival rate of events. More precisely, we show that the transform (1) is given by the formula

$$E_t^v \left[\exp \left(-\int_t^T r_s ds \right) e^{izY + (e^{iv}-1)(A_T - A_t)} \right] \tag{2}$$

where E_t^v denotes conditional expectation under an equivalent complex-valued measure P^v defined by the characteristic martingale that we associate to the counting process. This measure *neutralizes* the feedback from events to arrival and discount rates. It leads to an expression that is familiar in the defaultable term structure literature. If, as is common in applications, N is specified in terms of an intensity λ, then formula (2) is analogous to the price at t of a security that pays $\exp(izY)$ at T if the issuer survives to T and 0 otherwise, assuming the issuer defaults at intensity $(1-e^{iv})\lambda$. The calculation of this price is well understood for a wide range of parametric intensity specifications, including affine and quadratic models, since it is itself analogous to the calculation of the security price in a default-free economy in which the short-term risk-free rate of interest is $r + (1 - e^{iv})\lambda$. Formulae (1) and (2) thus extend the analytical tractability offered by extant default-free term structure model specifications to an intensity based counting process N and its applications discussed above.

Empirical observation dictates the level of generality we maintain. Our results are valid for any non-explosive counting process N whose arrivals are totally inaccessible or unpredictable, and whose compensator satisfies a mild growth condition. The probabilistic structure of arrivals is otherwise unconstrained. In particular, N need not be intensity based. This allows us to apply our results in situations where investors in defaultable securities have incomplete information. In these situations events are typically unpredictable but may not admit an intensity.[3] Furthermore, the dependence structure among the counting process, its compensator and the discount rate is unrestricted. This allows us to analyze situations with event feedback, in which events have an impact on arrival rates (as with contagion) and risk-free interest rates (as with flights to quality). Feedback phenomena occur when individual credit events send ripple effects through the economy.[4] They increase the volatility of default losses and fatten the tail of the loss distribution. Therefore, feedback phenomena are important for the risk management of credit portfolios, the valuation of tranche derivatives and contracts on the volatility of portfolio loss, and the hedging of constituent risks.

The remainder of this introduction discusses the related literature. Section 2 then introduces the complex-valued measure change and shows how to calculate the counting

[3]For examples in a single firm setting see Giesecke (2006); for an example with multiple firms see Giesecke (2004). For single firm examples that are intensity based see Duffie & Lando (2001).

[4]A case in point is the recent credit crisis, which was triggered by a string of mortgage defaults. See Jorion & Zhang (2007) for an empirical analysis of feedback in the credit market before that crisis.

process transform (1). Section 3 presents several applications, including credit derivatives valuation, portfolio credit risk measures, and random thinning. Section 4 develops a comprehensive affine example that illustrates our methodology. Section 5 summarizes our results and discusses additional applications in market micro structure analysis. The Appendix contains proofs and technical results not stated in the main body of the paper.

1.1 Related literature

The complex-valued measure change underlying formula (2) was developed by Carr & Wu (2004) for time-changed Lévy processes. They show that the characteristic function of a time-changed Lévy process is given by the Laplace transform of the time change evaluated at the characteristic exponent of the Lévy process. The Laplace transform is calculated under the complex measure defined by the time-changed Wald martingale associated with the Lévy process. The special case of formula (2) obtained by setting $r = z = t = 0$ is a consequence of Carr & Wu's (2004) result if the counting process N is realized as a time-changed Poisson process. Under this assumption, the compensator takes the role of the time change and the characteristic martingale coincides with the time-changed Wald martingale. Formula (2) extends beyond the special structure of a time-changed Poisson process and incorporates a stochastic discount factor as well as a random payoff.

Our results are also related to those of Collin-Dufresne, Goldstein & Hugonnier (2004), who derive an intensity based formula for the value of a "single name claim" that pays a random amount if the issuer survives to some future date T and a random recovery at a default before T. This formula requires a change of measure that puts zero mass on paths for which default occurs prior to T. The measure change of Collin-Dufresne et al. (2004) is absolutely continuous and generates a probability measure. Our measure change is equivalent and generates a complex-valued measure that is not a probability measure. The two measure changes are analogous in that they remove the feedback from an event on the intensity or the discount rate. The complex measure change goes beyond the first jump of the counting process that is addressed by the absolutely continuous measure change. It supports the valuation of derivatives on the full counting process, which may be claims referenced on one or multiple issuers.

Our transform based valuation results synthesize and extend a substantial literature on the reduced form pricing of single name and portfolio credit derivatives. First, by considering the counting process transform (1) we enlarge the set of portfolio derivative payoffs that can be tractably addressed. Second, formula (2) is equally applicable to both bottom up and top down multi-issuer model specifications. In a bottom up setting, the constituent intensities are the primitives and the intensity of the portfolio default count is the sum of the constituent intensities.[5] In a top down setting the intensity of

[5]See, for example, Das, Duffie, Kapadia & Saita (2007), Duffie, Eckner, Horel & Saita (2006), Duffie & Garleanu (2001), Eckner (2007), Jarrow & Yu (2001), Jarrow, Lando & Yu (2005), Mortensen (2006), Papageorgiou & Sircar (2007) and Yu (2007).

the default counting process is specified without reference to the constituents.[6] In both settings formula (2) provides a representation of the counting process transform which, depending on the concrete intensity specification, may lead to new pricing expressions for existing models. Since they support the use of well-established transform inversion methods, these new expressions may be more tractable than the original expressions, which in many cases require simulation methods for their evaluation. Third, our valuation relations are not subject to the "no-jump hypothesis" that restricts the applicability of alternative valuation formulas that are prominent in the literature.[7]

2 Counting process transforms

Uncertainty is modeled by a complete probability space $(\Omega, \mathcal{F}, P)$. A right-continuous and complete filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ models the information flow. Consider a sequence of totally inaccessible stopping times T^n that is strictly increasing and for which $T^1 > 0$ almost surely. These stopping times represent the ordered arrival times of events such as corporate defaults. Let N be the counting process given by

$$N_t = \sum_{n \geq 1} 1_{\{T^n \leq t\}}. \tag{3}$$

We assume that N_t is finite almost surely. The Doob-Meyer theorem guarantees the existence of a non-decreasing, continuous process A starting at 0 such that $M = N - A$ is a local martingale. The *compensator* A represents the cumulative conditional arrival rate of events. It is uniquely defined up to indistinguishability. Throughout, we fix a finite horizon $T > 0$ and suppose that $\exp(A_T)$ is square integrable.

The distribution of the compensator A determines the distribution of the counting process N. We are going to express a Fourier transform of N in terms of a Laplace transform of A. For real numbers v let $\Psi(v) = 1 - \exp(iv)$ be the characteristic exponent of the Poisson process, and consider the process $Z(v)$ given by the formula

$$Z_t(v) = \exp\big(ivN_t + \Psi(v)A_t\big). \tag{4}$$

Each $Z_t(v)$ is a complex-valued random variable whose modulus is strictly positive almost surely. In Proposition A.1, we show that $Z(v)$ is the stochastic exponential of the scaled local martingale $-\Psi(v)M$. Hence, $Z(v)$ is a local martingale that is complex-valued.[8] Below, we use $Z(v)$ to define an equivalent change of measure that is the key to the analysis of the characteristic function of the counting process N.

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[6]See, for example, Arnsdorf & Halperin (2007), Brigo, Pallavicini & Torresetti (2006), Davis & Lo (2001), Ding, Giesecke & Tomecek (2006), Errais, Giesecke & Goldberg (2006), Giesecke & Tomecek (2005), Longstaff & Rajan (2006), Lopatin & Misirpashaev (2007) and Tavella & Krekel (2006).

[7]The absolutely continuous change of probability measure developed by Collin-Dufresne et al. (2004) facilitates the removal of this hypothesis in a single-issuer and bottom up multi-issuer setting.

[8]A complex-valued local martingale is a process V of the form $V = X + iY$ where X and Y are real-valued local martingales.

2.1 Characteristic function

Let r be a non negative and bounded discount rate process. Theorem 2.1 gives a formula for the discounted conditional characteristic function of the counting process N.

Theorem 2.1. *The discounted conditional characteristic function of N is given by*

$$E\left[\exp\left(-\int_t^T r_s ds\right) e^{iv(N_T-N_t)} \,\Big|\, \mathcal{F}_t\right] = \mathcal{L}_t^v\big(\Psi(v), T\big) \tag{5}$$

where $t \leq T$ and $\mathcal{L}_t^v(u,T)$ is the discounted Laplace transform of the compensator under the equivalent measure P^v on $\mathcal{F}_T$ defined by the density $Z_T(v)$. Letting E^v denote P^v-expectation and $u \in \mathbb{C}_+$, the set of complex numbers with non negative real part,

$$\mathcal{L}_t^v\big(u, T\big) = E^v\left[\exp\left(-\int_t^T r_s ds\right) e^{-u(A_T-A_t)} \,\Big|\, \mathcal{F}_t\right]. \tag{6}$$

Theorem 2.1 asserts that the discounted characteristic function of a counting process N is given by the discounted Laplace transform of the compensator A evaluated at the characteristic exponent $\Psi(v)$ of the Poisson process. The discounted Laplace transform is taken under an equivalent measure P^v defined by the *characteristic martingale* $Z(v)$ associated to N by formula (4). Since $Z(v)$ is complex-valued so is P^v. Calculating the discounted characteristic function of N reduces to calculating the discounted Laplace transform of A under the complex measure P^v. Further technical details and a proof of this result are given in the appendix.

The characteristic function can be expressed in closed form if the Laplace transform is known in closed form. To calculate the Laplace transform, consider the specification of the counting process in terms of a non negative *intensity* process λ. This formulation is very common in applications. The intensity represents the conditional event arrival rate so the Laplace transform takes the form

$$\mathcal{L}_t^v\big(u, T\big) = E^v\left[\exp\left(-\int_t^T \big(r_s + u\lambda_s\big) ds\right) \,\Big|\, \mathcal{F}_t\right]. \tag{7}$$

Since the measure P^v underlying formula (7) is complex, we can interpret the expectation as an average over the complex-valued realizations $\exp(-\int_t^T (r_s + u\lambda_s)(\omega) ds)$ with respect to complex weights $dP^v(\omega)$ rather than real ones $dP(\omega)$. Formula (7) is a familiar expression in the defaultable term structure literature. Consider a zero coupon bond with unit face value maturing at time T, issued by a firm that defaults at an intensity given by $u\lambda$. Suppose the bond has zero recovery. Under technical conditions, the price of this bond at $t \leq T$ is given by formula (7) once we regard P^v as a pricing measure relative to a short-term interest rate r.[9] Along with Theorem 2.1, this observation has significant

[9]For a precise statement under different sets of assumptions see Duffie, Schroder & Skiadas (1996) and Collin-Dufresne et al. (2004).

implications for the specification of computationally tractable counting process models $(\mathbb{F}, N)$. It facilitates the adoption of a wide variety of parametric model formulations from an extensive bond pricing literature for the purpose of specifying a counting process whose characteristic function is analytically tractable.

Typically, the counting process N is specified under the reference probability P. The transform formula (5) calls however for the calculation of the Laplace transform (6) under the complex measure P^v. The computation of P^v-conditional expectations requires an understanding of the relationship between a P-local martingale and a P^v-local martingale, which we define as an adapted process X such that $XZ(v)$ is a P-local martingale, see Definition A.3. We extend the Girsanov-Meyer theorem to analyze this relationship.

Proposition 2.2. *Let V be a P-local martingale such that the quadratic covariation $[V,N]$ is locally of integrable variation. Denote by $\langle V,N\rangle$ the P-conditional covariation. Then, on the interval $[0,T]$ and for any real number v, a P^v-local martingale is given by*

$$V + \Psi(v)\langle V, N\rangle.$$

In particular, V is a P^v-local martingale if it does not have jumps in common with N.

Proposition 2.2 implies that a Brownian motion under P remains a Brownian motion under the measure P^v. The jumps of the local martingale $M = N - A$ coincide with the jumps of N so M does not remain a local martingale under P^v. This means the compensator of the counting process N must be adjusted if we change the measure.

Corollary 2.3. *On the interval $[0,T]$ the counting process N has P^v-compensator*

$$(1-\Psi(v))A = e^{iv}A.$$

Proposition 2.2 also implies that the measure change is redundant in the familiar doubly stochastic setting. The counting process N is doubly stochastic with respect to a right continuous and complete sub-filtration $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ of $\mathbb{F}$, if the compensator A is adapted to $\mathbb{G}$ and if, for all t and $T > t$, conditional on the sigma-field $\mathcal{F}_t \vee \mathcal{G}_T$ the variable $N_T - N_t$ has the Poisson distribution with parameter $A_T - A_t$. If the discount rate r is adapted to $\mathbb{G}$ as well, then by iterated expectations

$$E\left[\exp\left(-\int_t^T r_s ds\right) e^{iv(N_T-N_t)} \,\Big|\, \mathcal{F}_t\right] = \mathcal{L}_t\big(\Psi(v), T\big) \tag{8}$$

where $\mathcal{L}_t(u,T) = \mathcal{L}_t^0(u,T)$ is the discounted Laplace transform of the compensator under the reference measure P. If this Laplace transform has partial derivatives with respect to u of all orders at $u = 1$, then for $n \geq 0$

$$E\left[\exp\left(-\int_t^T r_s ds\right) 1_{\{N_T-N_t=n\}} \,\Big|\, \mathcal{F}_t\right] = \frac{1}{n!}\,(-1)^n\,\partial_u^n\,\mathcal{L}_t(u,T)\Big|_{u=1} \tag{9}$$

If N is doubly stochastic supported by a filtration $\mathbb{G}$, then the event $\{N_t = n\}$ is never contained in $\mathcal{G}_t$. Therefore, event feedback is ruled out: N cannot influence A or r because these processes are adapted to $\mathbb{G}$. In the general case, N does influence A and r and we must appeal to Theorem 2.1 in order to calculate the characteristic function. The complex measure change *neutralizes* the feedback from N to A and r and thereby restores the computationally convenient doubly stochastic setting. It always reduces the problem of calculating the discounted characteristic function to the simpler problem of calculating the discounted Laplace transform of the compensator.

2.2 Joint characteristic function

Anticipating applications to portfolio credit risk, we extend Theorem 2.1 to obtain a formula for the discounted joint characteristic function of N_T and a random variable $Y \in \mathcal{F}_T$ that can represent a cash flow at T. The proof of Theorem 2.4 below is based on the same argument as the proof of Theorem 2.1. We omit it.

Theorem 2.4. *The discounted joint characteristic function of N and $Y \in \mathcal{F}_T$ is given by*

$$E\left[\exp\left(-\int_t^T r_s ds\right) e^{izY+iv(N_T-N_t)} \,\Big|\, \mathcal{F}_t\right] = \mathcal{E}_t^v\big(\Psi(v), z, T\big) \tag{10}$$

where $t \leq T$, $z, v \in \mathbb{R}$ and $\mathcal{E}_t^v(u, z, T)$ is the extended discounted Laplace transform of the compensator under the equivalent measure P^v on $\mathcal{F}_T$ defined by the density $Z_T(v)$:

$$\mathcal{E}_t^v\big(u, z, T\big) = E^v\left[\exp\left(-\int_t^T r_s ds\right) e^{izY-u(A_T-A_t)} \,\Big|\, \mathcal{F}_t\right] \quad u \in \mathbb{C}_+. \tag{11}$$

Theorem 2.4 asserts that the calculation of the discounted joint characteristic function of N and Y reduces to the computation of the extended discounted Laplace transform of the compensator under the complex measure defined by the characteristic martingale associated to N by formula (4). If the counting process is specified by an intensity λ, then the extended Laplace transform takes the form

$$\mathcal{E}_t^v\big(u, z, T\big) = E^v\left[\exp\left(-\int_t^T \big(r_s + u\lambda_s\big) ds\right) e^{izY} \,\Big|\, \mathcal{F}_t\right]. \tag{12}$$

Again, formula (12) is a familiar expression in the defaultable term structure literature. Consider a security that pays $\exp(izY)$ at T if the issuer survives to T and 0 otherwise. Suppose the issuer defaults at an intensity given by $u\lambda$. Under technical conditions, the price of this security at $t \leq T$ is given by formula (12) once we regard P^v as a pricing measure relative to a short-term interest rate r.

3 Portfolio credit risk

We illustrate the application of our characteristic function formulae to portfolio credit risk. In this context, the process N counts the defaults in a portfolio of credit sensitive securities such as loans, bonds or credit swaps. The discount rate process r represents the short-term risk free rate of interest. A key quantity is the conditional expectation

$$G_t(x; a, b, c, d, T) = E\left[\exp\left(-\int_t^T r_s ds\right)(a + bY)(c + dN_T)1_{\{N_T \le x\}} \,\Big|\, \mathcal{F}_t\right] \tag{13}$$

which is defined for times $t \le T$, real valued a, b, c, d and x and a random variable $Y \in \mathcal{F}_T$ such that $|(a + bY)(c + dN_T)|$ is P-integrable. The "discounted conditional distribution function" of N_T is given by $G_t(x; 1, 0, 1, 0, T)$. The discounted expectation of N_T is gotten similarly, as is the expectation of YN_T. More generally, derivatives prices and risk measures can be expressed in terms of (13) as we illustrate below.

In order to calculate the expectation (13), we adopt an approach pioneered by Duffie, Pan & Singleton (2000) for affine jump diffusion processes and consider the Fourier-Stieltjes transform of $G_t(x)$. By integration by parts, for real v we get

$$\begin{aligned}\mathcal{G}_t(v; a, b, c, d, T) &= \int_{-\infty}^{\infty} e^{ivx} dG_t(x; a, b, c, d, T) \\ &= E\left[\exp\left(-\int_t^T r_s ds\right)(a + bY)(c + dN_T)e^{ivN_T} \,\Big|\, \mathcal{F}_t\right].\end{aligned}$$

Theorem 2.4 allows us to express this Fourier-Stieltjes transform in terms of the extended discounted Laplace transform of the compensator. We get

$$\begin{aligned}\mathcal{G}_t(v; a, b, c, d, T) = {} & ace^{ivN_t}\mathcal{E}_t^v(\Psi(v), 0, T) - bd\partial^2_{wz} e^{iwN_t}\mathcal{E}_t^w(\Psi(w), z, T)|_{w=v,z=0} \\ & - ibce^{ivN_t}\partial_z\mathcal{E}_t^v(\Psi(v), z, T)|_{z=0} - iad\partial_w e^{iwN_t}\mathcal{E}_t^w(\Psi(w), 0, T)|_{w=v}\end{aligned}$$

provided the partial derivatives of the extended Laplace transform exist. The expectation (13) can be obtained by Fourier inversion of the transform $\mathcal{G}_t$. Fixing t, a, b, c, d and T, note that $G_t(x)$ is almost surely an increasing right continuous function with left limits that is constant on the intervals $[n, n + 1)$ for n an integer, and vanishes for $x < 0$. Therefore, it suffices to recover the function at the jump points.

Proposition 3.1. *For all non negative integers n we have*

$$G_t(n; a, b, c, d, T) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{-ivn} - e^{iv}}{\Psi(v)}\,\mathcal{G}_t(v; a, b, c, d, T)\,dv.$$

Proposition 3.1 expresses the conditional expectation (13) in terms of the extended Laplace transform (11). As argued above, the Laplace transform is analogous to the price of a simple defaultable security and can be calculated explicitly for a wide variety of counting process specifications. In the following three sections we show how knowledge of the conditional expectation (13) facilitates an analytical treatment of a range of applications in that arise in portfolio credit risk, including credit derivatives valuation, risk measure calculation and random thinning.

3.1 Credit derivatives

We consider the valuation of portfolio credit derivatives, which are contingent claims on the default counting process N. We suppose the reference probability P is an equivalent martingale measure relative to the risk-free short rate process r. The processes N and r are specified directly under risk-neutral probabilities. Below we develop illustrative examples to portfolio derivatives valuation. We use the notation

$$F_t(x; a, b, c, d, T) = \mathcal{G}_t(0; a, b, c, d, T) - G_t(x; a, b, c, d, T).$$

3.1.1 Default-contingent claims

Consider a security that pays Y at time T if the portfolio default count at T exceeds $K \geq 0$ and 0 otherwise. At time $t \leq T$, this security has a value given by

$$E\left[\exp\left(-\int_t^T r_s ds\right) Y 1_{\{N_T > K\}} \,\middle|\, \mathcal{F}_t\right] = F_t(K; 0, 1, 1, 0, T).$$

Similarly, consider an nth-to-default security that pays Y at T if the default count at T is equal to an integer n and 0 otherwise. At time $t \leq T$, this security is priced at

$$E\left[\exp\left(-\int_t^T r_s ds\right) Y 1_{\{N_T = n\}} \,\middle|\, \mathcal{F}_t\right] = G_t(n; 0, 1, 1, 0, T) - G_t(n-1; 0, 1, 1, 0, T).$$

These security valuation formulae generalize the zero recovery formula in Theorem 1 of Collin-Dufresne et al. (2004) to the case where the security is referenced on a portfolio of firms rather than a single firm. We can specialize into a single firm setting by letting the firm's default time be the first jump time of N. The value at t of a security that pays Y if the firm survives to T and 0 otherwise is then given by $G_t(0; 0, 1, 1, 0, T)$.

The single name valuation formula derived by Collin-Dufresne et al. (2004) is not subject to the standard "no-jump hypothesis" that restricts the applicability of other valuation formulas derived in the defaultable term structure literature. Our single- and multi-name valuation formulae do not require this hypothesis either. In particular, this means the formulae apply in situations with feedback from events to arrival and interest rates. The "no-jump hypothesis" is typically violated in the presence of feedback.

3.1.2 Default count options

Consider a call option, struck at K with exercise date T, on the portfolio default count N. At time $t \leq T$, the option is priced at

$$\begin{aligned} C_t(K, T) &= E\left[\exp\left(-\int_t^T r_s ds\right) (N_T - K)^+ \,\middle|\, \mathcal{F}_t\right] \\ &= F_t(K; 1, 0, 0, 1, T) - F_t(K; K, 0, 1, 0, T). \end{aligned} \tag{14}$$

The price of a put option on the default count can be found by put-call parity. Options are the basic building block of many more complex portfolio credit derivatives such as credit indexes, tranches or options on these instruments.[10]

3.1.3 Credit tranches

A tranche swap on a portfolio with notional I is specified by a lower attachment point $\underline{K} \in [0, I]$, an upper attachment point $\overline{K} \in (\underline{K}, I]$ and a maturity date T. The tranche *protection seller* agrees to cover portfolio default losses as they occur, given that the cumulative losses are larger than $\underline{K}$ but do not exceed $\overline{K}$. Normalizing the loss at an event to 1, the cumulative payments U_t at t are

$$U_t = (N_t - \underline{K})^+ - (N_t - \overline{K})^+,$$

which is the payoff to a call spread on N with exercise date t. By integration by parts, the value at t of the protection seller's payments is given by

$$\begin{aligned} D_t &= E\left[\int_t^T \exp\left(-\int_t^s r_u du\right) dU_s \,\middle|\, \mathcal{F}_t\right] \\ &= C_t(\underline{K}, T) - C_t(\overline{K}, T) - U_t + \int_t^T \big(B_t(\underline{K}, s) - B_t(\overline{K}, s)\big) ds \end{aligned}$$

where $C_t(K, T)$ is given by formula (14) and $B_t(K, T)$ is given by

$$\begin{aligned} B_t(K, T) &= E\left[\exp\left(-\int_t^T r_s ds\right) r_T (N_T - K)^+ \,\middle|\, \mathcal{F}_t\right] \\ &= F_t(K; 0, 1, 0, 1, T) - F_t(K; K, 1, 1, 0, T), \end{aligned}$$

where F_t is evaluated for $Y = r_T$. The tranche *protection buyer* agrees to make premium payments at dates (t_m) that are proportional to the difference between the tranche notional $K = \overline{K} - \underline{K}$ and the tranche loss U. If S is the proportionality factor and c_m is the daycount fraction for the mth coupon period, then the value at t of these payments is

$$\begin{aligned} P_t(S) &= E\left[\sum_{t_m \geq t} \exp\left(-\int_t^{t_m} r_s ds\right) S c_m (K - U_{t_m}) \,\middle|\, \mathcal{F}_t\right] \\ &= S \sum_{t_m \geq t} c_m \Big\{\mathcal{G}_t(0; K, 0, 1, 0, t_m) - C_t(\underline{K}, t_m) + C_t(\overline{K}, t_m)\Big\}. \end{aligned}$$

The fair tranche spread at t is the solution $S = S_t$ to the equation $D_t = P_t(S)$. It can be explicitly expressed in terms of the function G_t. Similar calculations lead to the fair spread on a credit index.

[10] Bakshi & Madan (2000) show that the continuum (indexed by v) of discounted characteristic functions (5) and the continuum (indexed by K) of default count options (14) are equivalent classes of spanning securities in the space of integrable plus affine claims on N_T.

3.2 Risk measures

Our results are also useful for the computation of portfolio risk measures. We suppose the reference probability P is the physical measure so the counting process N is specified under historical probabilities. Normalizing the loss at an event to 1 and fixing a horizon T, the aggregate risk due to default is described by the distribution function $H(x;T) = P[N_T \leq x]$, which is given by $G_0(x;1,0,1,0,T)$ if G_0 is evaluated at $r=0$. A popular measure of portfolio risk is the value at risk $V(\alpha;T)$ at level $\alpha \in (0,1)$, defined as the α-quantile of the distribution $H(\cdot;T)$. Once that distribution is obtained from Proposition (3.1), $V(\alpha;T)$ can be calculated as the generalized inverse of $H(x;T)$ via

$$V(\alpha;T) = \inf\big\{x \geq 0 : H(x;T) \geq \alpha\big\}.$$

Value at risk suffers several shortcomings. It does not give any information about the size of the losses that occur with a probability of less than $(1-\alpha)$. Also, it is not in general sub-additive: the value at risk of a portfolio of positions is not necessarily bounded by the sum of the value at risks of the constituent positions. An alternative measure that addresses these caveats is average value at risk, given by

$$AV(\alpha;T) = \frac{1}{1-\alpha}\int_\alpha^1 V(x;T)dx$$

for some level $\alpha \in (0,1)$. Our results can be used to directly calculate the average value at risk by noting the alternative formula

$$AV(\alpha;T) = \frac{1}{1-\alpha}E\big[(N_T - V(\alpha;T))^+\big] + V(\alpha;T).$$

The expectation in this formula is equal to $C_0(V(\alpha;T),T)$, provided we set $r=0$.

3.3 Random thinning

Random thinning attributes credit risk to portfolio constituents. We illustrate how our results can be used to thin a portfolio default process N into its constituent default processes. The resulting constituent models facilitate the estimation of N from constituent data and the estimation of constituent hedges for a claim on N. A thinning process Z^k for constituent name $k = 1,2,\ldots,n$ is a predictable process whose value represents the conditional probability that name k defaults next, given default is imminent. The sum of the Z^k over k must equal 1 unless all names in the portfolio are in default, in which case each Z^k vanishes. Assuming that N has intensity λ, Proposition 3.5 in Giesecke & Goldberg (2005) states that the probability of name k defaulting in $(t,T]$ is

$$q_t^k(T) = \int_t^T E\big[Z_s^k \lambda_s \,|\, \mathcal{F}_t\big]\, ds.$$

Our results allow us to calculate this probability for the specification $Z^k = B^k 1_{\{N_- < n\}}$, where B^k is a non negative predictable process. Then,

$$q_t^k(T) = \int_t^T G_t(n-1; 0, 1, 1, 0, s)\, ds$$

where G_t is evaluated for $Y = B_s^k \lambda_s$ and $r = 0$. Ding et al. (2006) explicitly calculate G_t when B^k is deterministic between event times and N is a time-changed birth process.

4 Example: Affine point processes

We illustrate the complex-valued measure change and the explicit calculation of the Laplace transform for a family of self-exciting counting processes N whose intensities are driven by an affine jump diffusion state process. If N describes the defaults in a portfolio of credit sensitive securities, then this specification can incorporate event feedback and a dependence structure among default, recovery and risk-free interest rates. All the applications discussed in Section 3 are analytically tractable in this setting.

The right continuous and complete filtration $\mathbb{F}$ is generated by a Markov state process X that is a strong solution to the stochastic differential equation

$$dX_t = \mu(X_t)\, dt + \sigma(X_t)\, dW_t + \delta\, dL_t, \qquad X_0 \in \mathbb{R}. \tag{15}$$

Here W is a standard Brownian motion, $\mu(X)$ is the drift process, $\sigma(X)$ is the volatility process, $\delta \geq 0$ is a sensitivity parameter and

$$L_t = \sum_{n=0}^{N_t} \ell^n = \sum_{n \geq 1} \ell^n 1_{\{T^n \leq t\}}$$

is a point process whose jump times are those of the counting process N. The random jump sizes ℓ^n are independently drawn from a distribution ν on $\mathbb{R}_+$ that has no mass at zero. In credit applications, we interpret X as a risk factor process, N as the default counting process and L as a process that records financial loss due to default. The compensator A of N is specified in terms of an intensity λ by setting

$$A_t = \int_0^t \lambda_s ds.$$

We assume that the intensity is of the form $\lambda = \Lambda(X)$ for some function Λ on $\mathbb{R}$. The sensitivity parameter δ controls the feedback from L to X and λ. Both the timing and the loss at an event influence the intensity. The dependence structure between λ and the ℓ^k can capture the negative correlation among default and recovery rates that is often observed in practice, see Altman, Brady, Resti & Sironi (2005).

By Theorem 2.4, the calculation of the discounted joint characteristic function of N_T and a random variable $Y \in \mathcal{F}_T$ reduces to the calculation of the extended Laplace transform (12) of the compensator A. In order to obtain an explicit expression for this Laplace transform we impose additional structure on the state process X, the intensity λ, the discount rate process r and the variable Y. We assume that $r = R(X)$ for some function R on $\mathbb{R}$ and that $Y = Y(X)$ for some function Y on $\mathbb{R}$. Further,

$$R(x) = R_0 + R_1 x, \qquad Y(x) = Y_0 + Y_1 x$$

$$\mu(x) = K_0 + K_1 x, \quad \sigma(x)^2 = H_0 + H_1 x, \quad \Lambda(x) = \Lambda_0 + \Lambda_1 x,$$

for constant coefficients such that $\Lambda(X)$ and $R(X)$ are non negative and $\exp(\int_0^T \Lambda(X_s)ds)$ is square integrable for some horizon $T > 0$. Under these assumptions the processes N and L are *affine point processes* in the sense of Errais et al. (2006). Important special cases include the Poisson process ($H_0 = H_1 = \delta = 0$), the linear birth process ($K_0 = K_1 = H_0 = H_1 = 0$), and the Hawkes process ($H_0 = H_1 = 0$).

The discounted extended Laplace transform (12) of the compensator under the complex measure P^v takes the form

$$\mathcal{E}_t^v(u, z, T) = E^v \left[\exp\left(-\int_t^T [R + u\Lambda](X_s) ds \right) e^{izY(X_T)} \,\Big|\, \mathcal{F}_t \right]. \tag{16}$$

To calculate this conditional expectation, first observe that under the reference measure P the state process X is an affine jump diffusion in the sense of Duffie et al. (2000). The change of measure from P to the complex measure P^v affects the jump component L of X but not the Brownian motion W, which remains a Brownian motion under P^v by Proposition 2.2. The jumps of L arrive with P-intensity $\lambda = \Lambda(X)$ so Corollary 2.3 implies that the P^v-intensity of the jump times is given by

$$e^{iv}\Lambda(X) = e^{iv}\Lambda_0 + e^{iv}\Lambda_1 X.$$

Having determined the dynamics of the state process X under P^v we are in a position to apply Proposition 1 of Duffie et al. (2000), slightly extended to account for the fact that the variable u and the intensity are complex valued rather than real-valued, to obtain technical regularity conditions on the functions R, Y, μ, σ and Λ guaranteeing that

$$\mathcal{E}_t^v\big(u, z, T\big) = \exp\big(izY_0 + \alpha(t) + \beta(t)X_t\big) \tag{17}$$

for $t \leq T$. The coefficient functions $\beta(t) = \beta(u, z, v, t, T)$ and $\alpha(t) = \alpha(u, z, v, t, T)$ satisfy the ordinary differential equations

$$\partial_t \beta(t) = R_1 + u\Lambda_1 - K_1\beta(t) - \frac{1}{2}H_1\beta(t)^2 - e^{iv}\Lambda_1(\theta(\delta\beta(t)) - 1) \tag{18}$$

$$\partial_t \alpha(t) = R_0 + u\Lambda_0 - K_0\beta(t) - \frac{1}{2}H_0\beta(t)^2 - e^{iv}\Lambda_0(\theta(\delta\beta(t)) - 1) \tag{19}$$

with boundary conditions $\beta(T) = izY_1$ and $\alpha(T) = 0$ and jump transform

$$\theta(c) = \int_{\mathbb{R}_+} e^{cz} d\nu(z), \quad c \in \mathbb{C}. \tag{20}$$

Theorem 2.4 implies that for $t \le T$, the discounted conditional characteristic function of (N_T, Y_T) is given in terms of the solutions to the equations (18)–(19) as

$$\begin{aligned}\mathcal{E}_t^v\big(\Psi(v), z, T\big) &= E\left[\exp\left(-\int_t^T r_s ds\right) e^{izY + iv(N_T - N_t)} \,\Big|\, \mathcal{F}_t\right] \\ &= \exp\big(izY_0 + \alpha(\Psi(v), z, v, t, T) + \beta(\Psi(v), z, v, t, T)X_t\big).\end{aligned}$$

It is instructive to compare the P^v-Laplace transform (16) with the Laplace transform $\mathcal{E}_t(u, z, T) = \mathcal{E}_t^0(u, z, T)$ calculated under the reference measure P. The P-intensity of N is $\lambda = \Lambda(X)$ and we apply Proposition 1 in Duffie et al. (2000), again slightly extended to account for the fact that the variable u is complex valued, to obtain technical regularity conditions on the functions R, Y, μ, σ and Λ under which

$$\mathcal{E}_t\big(u, z, T\big) = \exp\big(izY_0 + a(t) + b(t)X_t\big) \tag{21}$$

for $t \le T$, where the coefficient functions $b(t) = b(u, z, t, s)$ and $a(t) = a(u, z, t, s)$ satisfy the ordinary differential equations

$$\partial_t b(t) = R_1 + u\Lambda_1 - K_1 b(t) - \frac{1}{2} H_1 b(t)^2 - \Lambda_1(\theta(\delta b(t)) - 1) \tag{22}$$

$$\partial_t a(t) = R_0 + u\Lambda_0 - K_0 b(t) - \frac{1}{2} H_0 b(t)^2 - \Lambda_0(\theta(\delta b(t)) - 1) \tag{23}$$

with boundary conditions $b(T) = izY_1$ and $a(T) = 0$ and jump transform given by formula (20). The difference between equations (18)–(19) and equations (22)–(23) lies in the jump terms $\Lambda_j(\theta(\delta b(t)) - 1)$ for $j = 0, 1$. Under P^v these terms are adjusted by the factor e^{iv}. The coefficient functions and hence the Laplace transforms $\mathcal{E}_t^v(u, z, T)$ and $\mathcal{E}_t(u, z, T)$ agree for all real v, z, complex $u \in \mathbb{C}_+$ and $t \le T$ if the sensitivity parameter $\delta = 0$. Under this condition, events do not feed back on arrival or interest rates.

For notational simplicity, we have considered an affine jump diffusion state variable X valued in $\mathbb{R}$ with time-independent coefficient functions and a single jump term. The results of Duffie et al. (2000) allow us to easily extend our calculations to the case where X is valued in $\mathbb{R}^d$ and has m jump terms, and the affine functions R, μ, σ, Λ and the jump transform θ are time-dependent. An extension to higher dimensions facilitates the specification of an intensity model with stochastic volatility, for example.

There are other, non-affine model specifications that lead to an analytically tractable Laplace transform. An example is the class of quadratic models, in which Λ and R are quadratic functions and X follows the diffusion process $dX_t = \mu(X_t)\,dt + \sigma\,dW_t$, for μ an affine function and σ a constant. In this case there is no feedback from N to A or r. Formula (8) applies, and we only need to calculate the Laplace transform under the reference measure P. The results of Leippold & Wu (2002) and Chen, Filipovic & Poor (2002) imply that this Laplace transform is an exponentially quadratic function of the state, with coefficient functions that satisfy a system of known ordinary differential equations.

5 Summary and other applications

We consider a counting process N that describes the arrival of unpredictable events. We derive a formula for a Fourier transform of N that includes as a special case the characteristic function of N. This formula reduces the calculation of the transform to the calculation of a Laplace transform of the counting process compensator, or cumulative intensity. The Laplace transform is a familiar expression in the defaultable term structure literature. It is analogous to the price of a simple defaultable security, which can be computed explicitly for a wide range of intensity specifications. Our transform formula extends this computational tractability to the counting process N and its applications.

We examine several applications in portfolio credit risk, where N counts the defaults in a portfolio of credit sensitive securities. Example applications include the valuation of credit derivatives such as credit indexes and tranches, the calculation of risk measures for corporate debt portfolios, and random thinning. Our transform formula facilitates an analytical treatment of these applications when current default and interest rates are correlated with past arrivals, i.e. in situations that involve event feedback and flights to quality. These phenomena are prominent in credit markets.

Our results have potential applications in several other areas. We mention the theoretical and empirical analysis of market microstructure problems. Engle & Russell (1998) and Engle (2000) pioneered an econometric approach in which security trade times form a counting process whose intensity can be estimated from high-frequency trade arrival data using maximum likelihood and other methods. Temporal clustering of arrivals due to event feedback is a classical feature of the data. Our results support the specification and testing of novel intensity specifications that incorporate event feedback and lead to closed from expressions for the characteristic function of the event count and other related quantities. The characteristic function supports prediction and hypothesis testing for an intensity specification. At present, these and other applications rely on computationally expensive Monte Carlo simulations. Moreover, tractability concerns often unnecessarily constrain the intensity specification.

A Technical results and proofs

We begin by establishing the martingale property of the complex-valued process $Z(v)$ defined in formula (4). Recall that $M = N - A$ is the compensated jump local martingale associated with the counting process N, and that $\Psi(v) = 1 - \exp(iv)$.

Proposition A.1. *For each real number v the process $Z(v)$ is a complex-valued local martingale that satisfies the exponential equation*

$$Z_t(v) = 1 - \Psi(v) \int_0^t Z_{s-}(v) dM_s.$$

Proof. Fix a real number v and let $S(v) = ivN + \Psi(v)A$. Suppress the dependence of $S(v)$ and $Z(v) = \exp(S(v))$ on v. The process S is a complex-valued semimartingale. By the complex version of Ito's formula stated as Theorem 36 in Chapter II of Protter (2004),

$$Z_t = 1 + \int_0^t Z_{s-} dS_s + \frac{1}{2}\int_0^t Z_{s-} d[S,S]_s^c + \sum_{0<s\leq t} (Z_s - Z_{s-} - Z_{s-}\Delta S_s)$$

where $[S,S]^c$ is the path-by-path continuous part of $[S,S]$, $S_{0-} = S_0 = 0$ and $Z_{0-} = Z_0 = 1$. Since N and A are of finite variation $[S,S]^c$ vanishes. Decomposing S into its real and imaginary parts, we see that

$$\int_0^t Z_{s-} dS_s = iv \int_0^t Z_{s-} dN_s + \Psi(v) \int_0^t Z_{s-} dA_s.$$

The Stieltjes integrals on the right hand side of this equation are well-defined because Z is of finite variation. Since the event times are totally inaccessible, the compensator A has continuous paths almost surely. Using the continuity of A and the fact that N is a pure jump process, we calculate that

$$\begin{aligned}
\sum_{0<s\leq t} (Z_s - Z_{s-}) &= \sum_{0<s\leq t} e^{\Psi(v)A_{s-}} (e^{ivN_s} - e^{ivN_{s-}}) \\
&= \sum_{0<s\leq t} e^{\Psi(v)A_{s-}} (e^{iv(N_{s-}+1)} - e^{ivN_{s-}}) \Delta N_s \\
&= \int_0^t Z_{s-}(e^{iv} - 1) dN_s \\
&= -\Psi(v) \int_0^t Z_{s-} dN_s.
\end{aligned}$$

Finally, by the definition of S and the continuity of A,

$$\sum_{0<s\leq t} Z_{s-}\Delta S_s = iv \sum_{0<s\leq t} Z_{s-}\Delta N_s = iv \int_0^t Z_{s-} dN_s.$$

It follows that Z satisfies the exponential equation

$$Z_t = 1 - \Psi(v) \int_0^t Z_{s-} dM_s$$

where $M = N - A$ is the compensated local jump martingale associated with N. Since Z_- is predictable and $|Z_-|$ is locally bounded, $\int Z_- dM$ is a local martingale and so is Z; see Theorem 29 in Chapter IV of Protter (2004). □

Lemma A.2. *Fix a finite horizon $T > 0$ and suppose $\exp(A_T)$ is square integrable. Then for all real v, the stopped process $Z^T(v) = (Z_{t\wedge T}(v))$ is a complex-valued martingale that is uniformly integrable.*

Proof. The stopped process $Z^T(v)$ is a local martingale by Proposition A.1. Since A is non decreasing, for all real v we have

$$\sup_{t\geq 0} |Z_t^T(v)| = \sup_{t\geq 0} \exp((1-\cos v)A_{t\wedge T}) \leq \exp(2A_T).$$

It follows that $\sup_{t\geq 0}|Z_t^T(v)|$ is integrable since $\exp(2A_T)$ is. Then, by Theorem 51 in Chapter I of Protter (2004), $Z^T(v)$ is a uniformly integrable martingale. □

Fix a finite horizon $T > 0$ and suppose $\exp(A_T)$ is square integrable. Lemma A.2 implies that $Z_T(v)$ is integrable for all real v. Define a measure P^v on $\mathcal{F}_T$ by

$$P^v(B) = E\big[Z_T(v)1_B\big], \quad B \in \mathcal{F}_T.$$

Since the density Z_T is complex-valued, P^v is a *complex measure* on $\mathcal{F}_T$. This means P^v is a complex-valued countably additive function on $\mathcal{F}_T$, see Rudin (1987, Definition 1.18 and Chapter 6). The total variation $|P^v|$ of the complex measure P^v is a positive measure. For $B \in \mathcal{F}_T$, it is given by

$$\begin{aligned} |P^v|(B) &= E\big[|Z_T(v)|1_B\big] \\ &= E\big[\exp((1-\cos v)A_T)1_B\big] \\ &\leq E\big[\exp(2A_T)\big]. \end{aligned}$$

It follows that $|P^v(B)| \leq |P^v|(B) < \infty$ for any $B \in \mathcal{F}_T$ and $v \in \mathbb{R}$. While P^v does not satisfy the monotonicity property, note that $P^v(\Omega) = E[Z_T(v)] = Z_0(v) = 1$ for all real v. Furthermore, since $|Z(v)| > 0$ P-almost surely, P^v and P are *equivalent* measures.

Definition A.3. *Let v be real. A P^v-*local martingale* is an adapted right continuous process $(X_t)_{t\leq T}$ with left limits such that the process $(X_t Z_t(v))_{t\leq T}$ is a P-local martingale.*

Let $X \in \mathcal{F}_T$ be a random variable such that $|XZ_T(v)|$ is P-integrable. Definition A.3 suggests to define the P^v-conditional expectation of X, denoted by $E^v[X \,|\, \mathcal{F}_t]$, as

$$E^v\big[X \,|\, \mathcal{F}_t\big] = \frac{1}{Z_t(v)} E\big[XZ_T(v) \,|\, \mathcal{F}_t\big], \quad t \leq T, \quad v \in \mathbb{R}.$$

Proof of Theorem 2.1. In view of the preceding discussion, for $0 \leq t \leq T$ and any real number v we calculate

$$\begin{aligned} E\left[\exp\left(-\int_t^T r_s ds\right) e^{iv(N_T-N_t)} \,\Big|\, \mathcal{F}_t\right] &= E\left[\exp\left(-\int_t^T r_s ds\right) \frac{Z_T(v)}{Z_t(v)} e^{-\Psi(v)(A_T-A_t)} \,\Big|\, \mathcal{F}_t\right] \\ &= E^v\left[\exp\left(-\int_t^T r_s ds\right) e^{-\Psi(v)(A_T-A_t)} \,\Big|\, \mathcal{F}_t\right] \\ &= \mathcal{L}_t^v\big(\Psi(v), T\big). \end{aligned}$$

The second equality follows from the definition of $Z(v)$ and the third equality uses the fact that $|e^{-\Psi(v)A_T}| \leq 1$ for all $v \in \mathbb{R}$. □

Proof of Proposition 2.2. Fix a real number v. We suppress the dependence of $Z(v)$ on v. In view of Definition A.3, we show that the process $Z(V + \Psi(v)\langle V, N\rangle)$ is a P-local martingale. Integration by parts yields

$$Z_t(V_t + \Psi(v)\langle V, N\rangle_t) = \int_0^t Z_{s-}dV_s - \Psi(v)\int_0^t Z_{s-}(V_{s-} + \Psi(v)\langle V, N\rangle_{s-})dM_s \\ + \Psi(v)\int_0^t Z_{s-}d\langle V, N\rangle_s + [Z, V + \Psi(v)\langle V, N\rangle]_t$$

where $M = N - A$ is the compensated local jump martingale associated with N. The first and second terms on the right side of this equation define P-local martingales. It remains to show that the sum of the two remaining terms define P-local martingales as well. Since $\langle V, N\rangle$ is the unique predictable process of finite variation such that $[V, N] - \langle V, N\rangle$ is a P-local martingale L, we can write

$$\Psi(v)\int_0^t Z_{s-}d\langle V, N\rangle_s = \Psi(v)\int_0^t Z_{s-}d[V, N]_s - \Psi(v)\int_0^t Z_{s-}dL_s. \tag{24}$$

Furthermore, using the representation of the characteristic martingale Z given in Proposition A.1, we have

$$[Z, V + \Psi(v)\langle V, N\rangle]_t = -\Psi(v)\int_0^t Z_{s-}d[V, N]_s + \Psi(v)[Z, \langle V, N\rangle]_t. \tag{25}$$

Since $\langle V, N\rangle$ is of finite variation, it is a quadratic pure jump semimartingale and

$$[Z, \langle V, N\rangle]_t = \sum_{0<s\leq t} \Delta Z_s \Delta\langle V, N\rangle_s.$$

But since the arrivals of N are totally inaccessible, the jumps of

$$[V, N]_t = \sum_{0<s\leq t} \Delta V_s \Delta N_s$$

can occur only at totally inaccessible times. It follows that $\langle V, N\rangle$ has continuous paths. This implies that $[Z, \langle V, N\rangle]$ vanishes. Adding equations (24) and (25) shows that

$$\Psi(v)\int_0^t Z_{s-}d\langle V, N\rangle_s + [Z, V + \Psi(v)\langle V, N\rangle]_t = -\Psi(v)\int_0^t Z_{s-}dL_s$$

defines a P-local martingale, which completes the proof. □

Proof of Corollary 2.3. Consider the P-local martingale $M = N - A$. We have $[M, N] = [N - A, N] = [N, N] = N$ so $\langle M, N\rangle = A$. By Proposition 2.2, $M + \Psi(v)\langle M, N\rangle = N - A + \Psi(v)A = N - (1 - \Psi(v))A$ is a P^v-local martingale on $[0, T]$ so $(1 - \Psi(v))A$ is the P^v compensator to N on $[0, T]$. □

Proof of Proposition 3.1. We fix t, a, b, c, d, T and suppress the dependence of $G(x)$ and $\mathcal{G}(v)$ on these parameters. Let $X_k = G(k) - G(k-1)$ for integers k. For real v we have by the inversion theorem

$$X_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikv} \, \mathcal{G}(v) \, dv.$$

Then, for any non negative integer n,

$$\begin{aligned} G(n) = \sum_{k=0}^{n} X_k &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{k=0}^{n} e^{-ikv} \, \mathcal{G}(v) \, dv \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - e^{iv(n+1)}}{1 - e^{iv}} e^{-ivn} \, \mathcal{G}(v) \, dv. \end{aligned}$$

Noting that $\Psi(v) = 1 - e^{iv}$ completes the proof. □

References

Altman, Edward, Brooks Brady, Andrea Resti & Andrea Sironi (2005), 'The link between default and recovery rates: Theory, empirical evidence and implications', *Journal of Business* **78**(6), 2203–2227.

Arnsdorf, Matthias & Igor Halperin (2007), BSLP: markovian bivariate spread-loss model for portfolio credit derivatives. Working Paper, Quantitative Research J.P. Morgan.

Bakshi, Gurdip & Dilip Madan (2000), 'Spanning and derivative-security valuation', *Journal of Financial Economics* **55**, 205–238.

Brigo, Damiano, Andrea Pallavicini & Roberto Torresetti (2006), Calibration of cdo tranches with the dynamical generalized-poisson loss model. Working Paper, Banca IMI.

Carr, Peter & Liuren Wu (2004), 'Time-changed Lévy processes and option pricing', *Journal of Financial Economics* **71**, 113–141.

Chen, Li, Damir Filipovic & Vincent Poor (2002), 'Quadratic term structure models for risk-free and defaultable rates', *Mathematical Finance* **14**, 515–536.

Collin-Dufresne, Pierre, Robert Goldstein & Julien Hugonnier (2004), 'A general formula for the valuation of defaultable securities', *Econometrica* **72**, 1377–1407.

Das, Sanjiv, Darrell Duffie, Nikunj Kapadia & Leandro Saita (2007), 'Common failings: How corporate defaults are correlated', *Journal of Finance* **62**, 93–117.

Davis, Mark & Violet Lo (2001), Modeling default correlation in bond portfolios, *in* C.Alexander, ed., 'Mastering Risk Volume 2: Applications', Prentice Hall, pp. 141–151.

Ding, Xiaowei, Kay Giesecke & Pascal Tomecek (2006), Time-changed birth processes and multi-name credit. Working Paper, Stanford University.

Duffie, Darrell, Andreas Eckner, Guillaume Horel & Leandro Saita (2006), Frailty correlated default. Working Paper, Stanford University.

Duffie, Darrell & David Lando (2001), 'Term structures of credit spreads with incomplete accounting information', *Econometrica* **69**(3), 633–664.

Duffie, Darrell, Jun Pan & Kenneth Singleton (2000), 'Transform analysis and asset pricing for affine jump-diffusions', *Econometrica* **68**, 1343–1376.

Duffie, Darrell & Kenneth J. Singleton (1999), 'Modeling term structures of defaultable bonds', *Review of Financial Studies* **12**, 687–720.

Duffie, Darrell, Mark Schroder & Costis Skiadas (1996), 'Recursive valuation of defaultable securities and the timing of resolution of uncertainty', *Annals of Applied Probability* **6**, 1075–1090.

Duffie, Darrell & Nicolae Garleanu (2001), 'Risk and valuation of collateralized debt obligations', *Financial Analysts Journal* **57**(1), 41–59.

Eckner, Andreas (2007), Computational techniques for basic affine models of portfolio credit risk. Working Paper, Stanford University.

Engle, Robert F. (2000), 'The econometrics of ultra-high-frequency data', *Econometrica* **68**, 1–22.

Engle, Robert F. & Jeffrey R. Russell (1998), 'Autoregressive conditional duration: A new model for irregularly spaced transaction data', *Econometrica* **66**, 1127–1162.

Errais, Eymen, Kay Giesecke & Lisa Goldberg (2006), Pricing credit from the top down with affine point processes. Working Paper, Stanford University.

Giesecke, Kay (2004), 'Correlated default with incomplete information', *Journal of Banking and Finance* **28**, 1521–1545.

Giesecke, Kay (2006), 'Default and information', *Jornal of Economic Dynamics and Control* **30**(11), 2281–2303.

Giesecke, Kay & Lisa Goldberg (2005), A top down approach to multi-name credit. Working Paper, Stanford University.

Giesecke, Kay & Pascal Tomecek (2005), Dependent events and changes of time. Working Paper, Stanford University.

Jarrow, Robert A., David Lando & Fan Yu (2005), 'Default risk and diversification: Theory and applications', *Mathematical Finance* **15**, 1–26.

Jarrow, Robert A., David Lando & Stuart M. Turnbull (1997), 'A markov model of the term structure of credit risk spreads', *Review of Financial Studies* **10**(2), 481–523.

Jarrow, Robert A. & Fan Yu (2001), 'Counterparty risk and the pricing of defaultable securities', *Journal of Finance* **56**(5), 555–576.

Jarrow, Robert A. & Stuart M. Turnbull (1995), 'Pricing derivatives on financial securities subject to credit risk', *Journal of Finance* **50**(1), 53–86.

Jorion, Philippe & Gaiyan Zhang (2007), 'Good and bad credit contagion: Evidence from credit default swaps', *Journal of Financial Economics* **84**(3), 860–883.

Leippold, Markus & Liuren Wu (2002), 'Asset pricing under the quadratic class', *Journal of Financial and Quantitative Analysis* **37**, 271–295.

Longstaff, Francis & Arvind Rajan (2006), An empirical analysis of collateralized debt obligations. Forthcoming, *Journal of Finance*.

Lopatin, Andrei & Timur Misirpashaev (2007), Two-dimensional Markovian model for dynamics of aggregate credit loss. Working Paper, Numerix.

Mortensen, Allan (2006), 'Semi-analytical valuation of basket credit derivatives in intensity-based models', *Journal of Derivatives* **13**, 8–26.

Papageorgiou, Evan & Ronnie Sircar (2007), Multiscale intensity models and name grouping for valuation of multi-name credit derivatives. Working Paper, Princeton University.

Protter, Philip (2004), *Stochastic Integration and Differential Equations*, Springer-Verlag, New York.

Rudin, Walter (1987), *Real and complex analysis*, McGraw-Hill, New York.

Tavella, Domingo & Martin Krekel (2006), Pricing nth-to-default credit derivatives in the PDE framework. Working Paper, Octanti Associates.

Yu, Fan (2007), 'Correlated defaults in intensity based models', *Mathematical Finance* **17**, 155–173.