The Market Price of Credit Risk: The Impact of Asymmetric Information

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July 7, 2008‡

Abstract

Risk-averse investors in credit sensitive securities such as equity and bonds require compensation for bearing exposure to non-diversifiable corporate default risk. One component of this compensation is an event premium for the abrupt changes in security prices that occur at default. While empirical research points to the significance of event premia in corporate bond and credit swap markets, the economic nature of the event premium is not fully understood. This paper uses a structural model of corporate default risk to show that informational asymmetries can induce an event premium. If public investors are unable to observe the threshold asset value at which firm management liquidates the firm, then they face instantaneous default risk as they cannot discern the firm's distance to default. Investors are taken by surprise when the firm reaches the default threshold, causing a sudden downward jump in the prices of securities issued by or referenced on the firm. The resulting event premium is governed by the degree of investors' aversion to the randomness in the location of the unobserved default threshold. Firm management has an incentive to improve the threshold transparency in order to reduce the credit premium required by investors, and therefore the cost to the firm of equity and debt financing.

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‡This paper was previously circulated under the title "The Market Price of Credit Risk." We thank Greg Anderson, Tim Backshall, Roveen Bhansali, Ursula Gritsch, Monique Jeanblanc, Alec Kercheval, Jyh-Huei Lee, Vijay Poduri, Philip Protter, Sergiy Terentyev and Yashan Wang for enlightening discussions, and seminar participants at Columbia University's Center for Applied Probability 2003 Workshop on Derivative Securities and Risk Management, the Erasmus Center for Financial Research, the Bank of Italy, the 2004 Daiwa International Workshop on Financial Engineering, the 2004 Global Derivatives Research Conference at JP Morgan, the Global Association of Risk Professionals 2004 Credit Summit and BNP Paribas for comments. We are deeply grateful to Steve Evans for introducing us to Jacod's theory of martingale representation and explaining its relevance to the proof of Proposition 5.2. We are also grateful to Robert Jarrow and Stefan Weber for thorough and insightful reviews of an earlier version of this article, and to Baeho Kim for research assistance.

1 Introduction

It is a standard economic principle that non-diversifiable risk commands a premium. Credit risk is no exception: risk-averse investors in credit sensitive securities such as bonds or equity must be compensated for assuming corporate default risk that cannot be diversified away. The corresponding credit premium is empirically well-documented.[1] Yet its economic determinants are not fully understood. Based on a first passage model of corporate default risk with incomplete information, this article argues that informational asymmetries can induce a *transparency premium* that is realized as compensation for the short-term uncertainty associated with the default event itself.

First passage models have a long tradition. According to these models, a firm defaults if its value falls below a barrier. If, as in Black & Cox (1976), Leland (1994), Longstaff & Schwartz (1995) and many others, the default barrier is deterministic, then credit risk is driven exclusively by uncertainty about the continuous firm value and the credit premium takes a familiar form. The required excess return on any credit sensitive security issued by or referenced on the firm is equal to its risk times the market price of that risk. Here "risk" is measured in terms of diffusive price volatility. The market price of risk is given by the excess return on the firm per unit of firm risk.

However, this representation of the credit premium neglects the short-term uncertainty surrounding the default event. If investors can discern the firm's distance to the default barrier, then failure is typically predictable in the sense that investors are warned when it is imminent. The existence of short-term uncertainty in corporate security markets is highlighted by the prevalence of positive short-term credit spreads and the precipitous drops in security prices that occur at default. The jumps in security prices may command an event risk premium, over and above the premium due to diffusive price volatility. Collin-Dufresne et al. (2002), Berndt et al. (2005), Driessen (2005), Eckner (2007) and Azizpour & Giesecke (2008) find that the event premium is a significant factor in bond, credit swap, and index and tranche credit markets.

How can a first passage model be reconciled with these empirical findings? The short-term uncertainty can be incorporated into a first passage model by relaxing the assumption that investors are completely informed. Following Giesecke (2006), we recognize that investors are typically unable to precisely measure the barrier asset value at which firm management liquidates the firm. Indeed, public financial statements are often of limited use to determine this barrier; balance sheet data may be outdated and noisy. Then, since the firm's distance to default cannot be observed, default is a totally inaccessible event; it comes unannounced. This property leads to model forecasts of positive short spreads

[1] Amato (2005) and Berndt, Douglas, Duffie, Ferguson & Schranz (2005) estimate the credit premium using data on credit swap spreads. Collin-Dufresne, Goldstein & Helwege (2002), Driessen (2005), Elton, Gruber, Agrawal & Mann (2001) and Fons (1987) analyze the premium using corporate bond price data. Liu, Longstaff & Mandell (2006) estimate the premium using interest rate swap spread data. Eckner (2007) and Azizpour & Giesecke (2008) infer premia from index and tranche swap rates.

and jumps in security prices at default that are consonant with empirical observation. In this paper we show that the information asymmetry also has significant implications for the structure of the credit risk premium demanded by investors.

The risk premium corresponds to a pricing measure, which can be represented by its density (likelihood ratio) with respect to the physical measure that describes the empirical likelihood of events. In our first passage model with incomplete information, the space of densities is parameterized by pairs of processes that are predictable in the information filtration that describes investors' information flow. Thus, the risk premium can always be decomposed into two economically meaningful components. The diffusive risk premium, which is realized as a change to the drift term in the price process of a security issued by or referenced on the firm, is proportional to the security's diffusive price volatility. The proportionality factor can be interpreted as a market price of diffusion risk in the firm value. With asymmetric information, there is also short-term uncertainty about the default event. This uncertainty induces a downward jump in security prices at default. Empirical observation shows that firm equity drops to near zero. Since firm debt is senior to equity, bondholders usually lose something but generally do not lose everything. Consequently, net firm value, which is equal to the sum of equity and debt values, also drops at default. Risk-averse investors require an event risk premium for bearing exposure to these jumps. The event premium prescribes the mapping between instantaneous default probabilities under the physical measure and the pricing measure.[2]

The event premium takes the form of a transparency premium. If the firm is fully transparent to investors and the default barrier is public information, then investors must bear only the volatility in the firm value representing the present value of the firm's future cash flows. Consequently, the credit premium has only a diffusion component. If firm management does not publicize the default barrier, then investors face an additional source of uncertainty, namely the location of the barrier. Investors form a distribution on the barrier that represents their imperfect prior information, which may be based on balance sheet data, for example. We show that there is an additional event premium whose value is a function of the historical low of the firm value process. This function is determined by investors' prior barrier distributions under the physical and pricing measures. The magnitude of the event premium is governed by the degree of investors' aversion to the randomness in the location of the unobserved default threshold. The premium vanishes when investors are neutral with respect to variation in that location, i.e. when they are indifferent about the transparency level of the firm.

[2]This economic picture is based on the explicit parametrization of the full space of martingales on the probability space that supports our incomplete information economy. Our analysis uses the powerful martingale representation results of Jacod (1977). Every uniformly integrable martingale can be represented in terms of a Brownian motion that drives the diffusion-type uncertainty in the firm value and the compensated default jump martingale, which represents the jump-type uncertainty in the firm value. The risk premia are proportional to the martingale coefficients in this representation. Our analysis extends a similar representation result of Kusuoka (1999), who considers the square integrable case. An analogous martingale representation theorem for a filtration generated by a Lévy process is in Kunita (2004).

The results of our analysis point to the significance of investor information for credit risk premia. In particular, they indicate that informational asymmetries can be a driver of the substantial event risk premia that have been measured in different credit markets by Collin-Dufresne et al. (2002), Berndt et al. (2005), Driessen (2005), Eckner (2007) and Azizpour & Giesecke (2008). Further, they indicate that firm management has an incentive to publicize its liquidation policy. Threshold transparency reduces credit premia, and therefore the cost of equity and debt financing. Our model predicts that corporate security investors may charge non-transparent firms an event premium.

Our results also have implications for the design of credit models. The credit premium is of central importance to financial practitioners and academicians since it connects the two main purposes of a credit model. First, a credit model is used to forecast the probability of default. As such, the model must reflect the historical default experience. However, a credit model is also a tool for pricing and hedging credit sensitive securities. In this context, it must fit market prices. In order to build a coherent model that serves both purposes, we need to understand the relationship between actual defaults and security prices. The credit premium maps the actual likelihood of default to the pricing or martingale likelihood of default that is used to price securities.

The remainder of this paper is organized as follows. Section 2 outlines the first passage model assumptions and some immediate consequences. The underlying probabilistic structure is discussed in Appendix A. Section 3 analyzes the space of pricing measures. Section 4 studies the risk premium decomposition and the structure of the event premium, and provides an illustrative numerical case study. Section 5 examines the valuation of credit sensitive securities subject to fractional recovery. Section 6 concludes. Technical arguments and proofs are in Appendix B.

2 The I^2 model

We analyze the credit premium in the context of I^2, a first passage model with incomplete information introduced in Giesecke (2006) and empirically tested in Giesecke & Goldberg (2004a). This section reviews the basic model assumptions and their implications.

2.1 Assumptions

The uncertainty in the economy is modeled with a complete probability space $(\Omega, \mathcal{G}, \pi)$. We consider a fixed firm and make the following assumptions. The probabilistic structure underlying these assumptions is discussed in more detail in Appendix A.

A1. Capital structure of the firm:
The firm is financed by equity and a zero coupon bond. Debt is senior to equity.

A2. Gross firm value:
The gross firm value X_t is the present value at time t of all future cash flows

generated by the firm. It follows a geometric Brownian motion under the measure π. This is described by the equation

$$\frac{dX_t}{X_t} = \mu^\pi dt + \sigma dW_t^\pi, \quad X_0 > 0, \tag{1}$$

where $\mu^\pi \in \mathbb{R}$ is a drift parameter, $\sigma > 0$ is a volatility parameter, and W^π is a standard Brownian motion relative to π with augmented filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$. The equation (1) has the unique strong solution $X_t = X_0 e^{V_t}$ where $V_t = mt + \sigma W_t$ is a Brownian motion with drift $m = \mu^\pi - \frac{1}{2}\sigma^2$.

A3. Default time:
We assume that the firm defaults if the gross firm value X falls to some barrier. This default threshold is modeled by a random variable $d \in (0, X_0)$ that is independent of $\mathcal{F}_\infty = \bigvee_{t\geq 0} \mathcal{F}_t$. Defining the normalized default threshold $D = \log(d/X_0) \in (-\infty, 0]$, we can write the random default time τ as

$$\tau = \inf\{t > 0 : V_t \leq D\}. \tag{2}$$

Associated to the default time τ is the indicator process N defined as $N_t = 1_{\{\tau \leq t\}}$, which is 0 before default and 1 afterwards.

A4. Information structure:
Investors observe the gross firm value and the default but not the level d. Therefore, their information is *incomplete*. This puts investors at a disadvantage relative to firm management. The value of d is assumed to be firm inside information. In mathematical terms, the public information flow is modeled by the augmented, right continuous[3] filtration $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ generated by $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(N_s : s \leq t)$.

A5. Default barrier prior:
Lacking definite knowledge of the default point, investors agree on a prior distribution function G^π on the normalized default threshold D with respect to π. We assume G^π has a strictly positive density function g^π.

A6. Credit sensitive securities:
The firm has issued credit sensitive claims including equity and a zero coupon bond. A general claim is characterized by its payoff $c_T \in L^1(\Omega, \mathcal{F}_T, \pi)$ at a horizon $T \leq \bar{T}$, where $\bar{T} > 0$ is a fixed finite horizon. The payoff is made if there was no default by T.[4] Mathematically, the time T payoff is

$$C_T = c_T 1_{\{\tau > T\}}. \tag{3}$$

[3] See Bélanger, Shreve & Wong (2004, Appendix A) for a proof of right continuity.

[4] The choice $c_T \in \mathcal{F}_T$ is without loss of generality: For any $\mathcal{G}_T$-measurable random variable c, we have $c1_{\{\tau>T\}} = \bar{c}1_{\{\tau>T\}}$, where $\bar{c}$ is $\mathcal{F}_T$-measurable.

If the firm defaults before T, a recovery payment is made. We follow the fractional recovery convention introduced by Duffie & Singleton (1999). Let R be an $\mathbb{F}$-predictable process with values in $[0,1]$. If the firm defaults at time t, a fraction R_t of the market value of the claim just prior to default is recovered. The fraction $1-R_t$ represents *bankruptcy costs*. Therefore, the payoff at the default time is

$$C_\tau = R_\tau C_{\tau-} 1_{\{\tau \leq T\}}. \tag{4}$$

A credit sensitive claim is characterized by the triple (T, c_T, R).

A7. Dynamics of credit sensitive claim prices:
The cum-dividend price dynamics of the credit sensitive claim (T, c_T, R) with respect to $(\pi, \mathbb{G})$ are described by the stochastic differential equation

$$\frac{dC_t}{C_{t-}} = d\mu_C^\pi(t) + \sigma_C(t)(1-N_t)dW_t^\pi - (1-R_t)dN_t, \quad t \leq T, \quad C_0 > 0. \tag{5}$$

The price jumps down at default. The jump frequency is driven by the Brownian motion W^π, see C3 below. In equation (5), $\mu_C^\pi = (\mu_C^\pi(t))_{t\geq 0}$ is a $\mathbb{G}$-adapted process that starts at zero and has continuous paths of finite variation. It describes the cumulative growth rate of C. Further, $\sigma_C = (\sigma_C(t))_{t\geq 0}$ is a strictly positive $\mathbb{F}$-predictable[5] process that describes the diffusive volatility of C. The processes μ_C^π and σ_C are chosen such that (5) is well-defined.

A8. Riskless bonds:
On the financial market investors can trade in riskless bonds. Given some constant riskless rate r, these are valued at e^{rt} at time t.

2.2 Consequences

The model assumptions stated in Section 2.1 have several consequences which are important in the sequel. Once again, Appendix A provides more details with respect to the underlying probabilistic structure.

C1. Observability of defaults:
Assumption A4 implies that the default time τ is a stopping time in the investor filtration $\mathbb{G}$. It means that at each point in time, the default status of the firm can be observed. Note that τ is not a stopping time in the firm value filtration $\mathbb{F}$.

C2. Unpredictability of defaults:
Assumptions A3–A5 imply that the default time τ is *totally inaccessible* in the investor filtration $\mathbb{G}$. In mathematical terms, $\pi[\tau = \sigma < \infty] = 0$ for all $\mathbb{G}$-predictable

[5]This assumption is without loss of generality: Jeulin & Yor (1978) show that for any $\mathbb{G}$-predictable process Y there exists an $\mathbb{F}$-predictable process $\bar{Y}$ such that $Y_t = \bar{Y}_t$ on the set $\{t \leq \tau\}$.

times σ. On an intuitive level, it means that default cannot be anticipated. This is economically reasonable. Since investors are not privileged to firm inside information, they do not know the true distance between gross firm value and the default barrier. The unpredictability of defaults is consistent with the sudden downward jumps in the market value of corporate securities at default, see A7.

C3. Default trend and compensator:
The default indicator N is a submartingale in the investor filtration $\mathbb{G}$. It admits a unique Doob-Meyer decomposition into the sum of a $(\pi, \mathbb{G})$-martingale H^π and a non-decreasing $\mathbb{G}$-predictable process called the compensator of N. Proposition 6.1 in Giesecke (2006) implies that the $(\pi, \mathbb{G})$-compensator is given by $A^\pi_{\cdot\wedge\tau}$, where

$$A_t^\pi = -\log G^\pi(M_t) \tag{6}$$

is the $\mathbb{F}$-adapted *trend* and where M is the historical low of log-firm values:

$$M_t = \min_{s \leq t} V_s.$$

The trend represents the cumulative default arrival rate prior to default. Since M is of finite variation and G^π has density g^π by A5, we get

$$dA_t^\pi = -\frac{g^\pi(M_t)}{G^\pi(M_t)} dM_t. \tag{7}$$

This equation shows that A^π increases only when $-M$ does, which is when $M_t = V_t$ and assets reach their historical low. The set of times $\{t \geq 0 : M_t = V_t\}$ has Lebesgue measure zero, and therefore the compensator does not admit an intensity. This means a positive process λ^π such that $A_t^\pi = \int_0^t \lambda_s^\pi ds$ does not exist.

C4. Gross firm value and survival information:
The $\mathbb{F}$-Brownian motion W^π is also a Brownian motion in investors' filtration $\mathbb{G}$. This is a consequence of the independence of W^π and the default barrier, see A3. A formal discussion is in Appendix A.

It follows that any $\mathbb{F}$-martingale is also a $\mathbb{G}$-martingale: a process that is fair with respect to the information described by the gross firm value filtration $\mathbb{F}$ remains fair in the investor filtration $\mathbb{G}$, which contains survival information in addition.

C5. Net firm value:
The net firm value $\mathcal{X}$ is the sum of the values of equity S and zero coupon debt B. Assumption A7 implies that the dynamics of $\mathcal{X}$ with respect to $(\pi, \mathbb{G})$ are of the form (5). They are described by the equation

$$\frac{d\mathcal{X}_t}{\mathcal{X}_{t-}} = d\mu_{\mathcal{X}}^\pi(t) + \sigma_{\mathcal{X}}(t)(1 - N_t)dW_t^\pi - J_t dN_t, \quad \mathcal{X}_0 > 0. \tag{8}$$

The cumulative growth rate $\mu^\pi_{\mathcal{X}}$ of the net firm value satisfies

$$d\mu_{\mathcal{X}}(t) = \frac{1}{\mathcal{X}_{t-}}\big(S_{t-}d\mu^\pi_S(t) + B_{t-}d\mu^\pi_B(t)\big)$$

where μ^π_S and μ^π_B are the cumulative growth rates of equity and bonds, respectively. Similarly, the diffusive volatility $\sigma_{\mathcal{X}}$ of the net firm value satisfies

$$\sigma_{\mathcal{X}}(t) = \frac{1}{\mathcal{X}_{t-}}\big(S_{t-}\sigma_S(t) + B_{t-}\sigma_B(t)\big)$$

on the no-default set $\{\tau > t\}$, where σ_S and σ_B are the diffusive volatilities of equity and bonds, respectively. At default, $\mathcal{X}$ jumps downwards, mirroring the losses of equity and bonds. If default were to occur at time t, noting that equity becomes worthless at default, the combined losses relative to $\mathcal{X}$ are

$$J_t = \frac{1}{\mathcal{X}_{t-}}\big(S_{t-} + (1 - R_t)B_{t-}\big),$$

where R denotes the recovery process of the zero coupon debt. The value $J_t\mathcal{X}_{t-}$ represents the costs of bankruptcy, see A6. This value is lost to third parties at default. Thus, the net firm value $\mathcal{X}_t$ differs from the gross firm value X_t, which represents the present value of the future cash flows generated by the firm (A2). It follows that the I^2 model assumptions A1-A8 are not consistent with the Modigliani-Miller theorem. See Giesecke & Goldberg (2004*b*) for a discussion.

3 Pricing measures

This section analyzes the set of pricing measures for I^2. The pricing measures encode the credit premium. Let $T \leq \bar{T}$, where $\bar{T} > 0$ is a finite horizon that we fix throughout. A pricing or equivalent martingale measure is characterized by two properties.

M1. Martingale property:
The discounted price process $(C_t e^{-rt})_{t\leq T}$ of any traded credit sensitive security (T, c_T, R) must be a $\mathbb{G}$-martingale with respect to the pricing measure.

M2. Equivalence:
The pricing measure and physical measure P belong to the same class. In other words, they agree on which sets in $\mathcal{G}_T$ have zero measure.

Let $\mathcal{P}$ denote the set of measures on $(\Omega, \mathcal{G}_T)$ satisfying M1 and M2. The mathematical conditions determining $\mathcal{P}$ arise from a fundamental economic result in Delbaen & Schachermayer (1997) that goes back to Harrison & Kreps (1979) and Harrison & Pliska (1981): Under broad assumptions, $\mathcal{P}$ is non-empty if and only if the security prices generated by the elements in $\mathcal{P}$ do not admit arbitrage opportunities. Further, $\mathcal{P}$ consists of a single measure if and only if markets are complete and every credit sensitive claim can be hedged perfectly.

Throughout this section, the reference measure π is the physical or "data-generating" measure P that represents the empirical likelihood of events. This means assumption A1-A8 hold under P. We examine the set $\mathcal{P}$ of martingale measures equivalent to P in more detail. Our analysis sheds light on the structure of the risk adjustment, which provides an economic link between P and the measures in $\mathcal{P}$.

The space $\mathcal{P}$ sits inside the set $\mathcal{E}$ of measures that are equivalent to the physical measure. Each $Q \in \mathcal{E}$ can be identified with a P-martingale $Z = Z(Q)$. In Theorem 3.1, we show that the space $\mathcal{E}$ is parameterized by a pair of $\mathbb{G}$-predictable processes α and β. The representation in Theorem 3.1 depends on the filtration $\mathbb{G}$ and the measure class of P but not on which securities are traded. The processes α and β define the components of the credit premium as shown in Section 4 below.

The next step is to express the Q-price processes of traded securities in terms of the P-martingale Z associated with Q. Direct analysis of the Theorem 3.1 representation for $Z = Z(\alpha, \beta)$ gives rise to necessary and sufficient conditions on α and β given in Theorem 3.2 for which the price processes are Q-martingales. These conditions provide a relationship among the risk premium processes α and β, and the cumulative drift μ_C^P and volatility σ_C of a credit sensitive security.

3.1 Equivalent measures

Let $L^1 = L^1(P)$ denote the P-integrable functions on $(\Omega, \mathcal{G}_T)$. The relationship between the physical measure P and any pricing measure $Q \in \mathcal{E}$ is expressed in terms of a strictly positive random variable $Z_T = dQ/dP \in L^1$ whose P-expected value is 1. The variable Z_T is called the Radon-Nikodym derivative of Q with respect to P. Let Z be the right-continuous version of the P-martingale defined by

$$Z_t = E^P[Z_T \,|\, \mathcal{G}_t], \quad t \leq T. \tag{9}$$

In Theorem 3.1 below we represent the density process Z in terms of the two martingales that generate the uncertainty in our model under the physical measure. These are the Brownian motion W^P that underlies the gross firm value process and the compensated jump martingale $H^P = N - A^P_{\cdot\wedge\tau}$ associated with the default process N. From C3, the trend $A^P = -\log G^P(M)$, where G^P is investors' prior default barrier distribution and M is the historical low of the log gross firm value. The martingale H^P is continuous except for a jump at the default time. It vanishes with complete information.

Theorem 3.1. *The density process Z satisfies*

$$Z_t = \exp\left(-\int_0^t \alpha_s dW_s^P - \frac{1}{2}\int_0^t \alpha_s^2 ds + \int_0^t \log(1+\beta_s) dN_s - \int_0^{t\wedge\tau} \beta_s dA_s^P\right) \tag{10}$$

where α and $\beta > -1$ are $\mathbb{G}$-predictable processes. For a sequence of $\mathbb{G}$-stopping times T_n that increase to T, these processes satisfy

$$E^P\left[\int_0^{T_n} \alpha_s^2 ds\right] < \infty \quad and \quad E^P\left[\int_0^{T_n} |\beta_s| dA_s^P\right] < \infty. \tag{11}$$

Conversely, $\mathbb{G}$-predictable processes α and β that satisfy (11) and for which $Z_T = Z_T(\alpha, \beta)$ defined by (10) satisfies $E^P[Z_T] = 1$, correspond to a measure $Q = Q(\alpha, \beta) \in \mathcal{E}$.

Kusuoka (1999) proves a result similar to Theorem 3.1 under the additional assumption that Z is square integrable.[6] Our result for I^2 is not subject to this restriction. Kunita (2004) proves an analogous result for a filtration generated by a Lévy process.

3.2 When are the price processes martingales?

The absence of arbitrage opportunities implies the existence of at least one martingale measure. Theorem 3.1 states that each measure $Q \in \mathcal{E}$ can be identified with a pair of processes (α, β). In Theorem 3.2 below we give necessary and sufficient conditions on (α, β) so that $Q(\alpha, \beta) \in \mathcal{P}$. These conditions are formulated in terms of the cumulative drift μ_C^P and the diffusive volatility σ_C of the value of a credit sensitive claim (T, c_T, R), the recovery process R, and the trend $A^P = -\log G^P(M)$.

Theorem 3.2. *The discounted price process of the credit sensitive claim (T, c_T, R) is a martingale under $Q(\alpha, \beta)$ with α and β satisfying conditions (11) if and only if*

$$\mu_C^P(t) = rt + \int_0^{t\wedge\tau} \sigma_C(s)\alpha_s ds + \int_0^{t\wedge\tau} (1+\beta_s)(1-R_s) dA_s^P, \quad 0 \le t \le T. \tag{12}$$

Equation (12) characterizes the cumulative drift μ_C^P of the security price process (5) under physical probabilities. This characterization is implied by the absence of arbitrage. It shows that the cumulative drift is an increasing process that has continuous sample paths and starts at zero, consonant with assumption A7. On the no-default set $\{\tau > t\}$, the cumulative drift is not absolutely continuous with respect to the Lebesgue measure. That is, there is no drift rate. This is due to the fact that an intensity does not exist (C3).

The base drift is given by the linear term rt, where r is the risk-free rate. The other two terms represent adjustments to the base drift that are constant after default. Thus, after default the price drifts at rate r, describing the risk-free growth of the claim's recovery value $C_{\tau-}R_\tau$ paid at default. The second term is real-valued, and vanishes only if α does. Its interpretation is discussed below. The third term compensates for the downward jump of the price at default. It is nonnegative, and vanishes if the claim is de facto default-risk free, i.e. if $R = 1$ and investors recover the full pre-default price at default.

[6] Kusuoka (1999) also assumes that the trend is of the form $A_t^P = \int_0^t \lambda_s^P ds$ for an intensity process λ^P. The proof of his representation Theorem 2.3 does however not require this assumption.

Thanks to a result of Jeulin & Yor (1978), the integrands of the second and third terms on the right hand side of equation (12) can be taken to be $\mathbb{F}$-predictable. Thus, the cumulative drift can be written as $\mu_C^P(t) = rt + h(t \wedge \tau)$, where h is an $\mathbb{F}$-predictable process. In other words, the two adjustments to the base drift rt depend only on the gross firm value process. The default time determines when these adjustments stop.

4 Credit risk premia

As in the previous section, the reference measure is the physical measure P. We fix $T \in (0, \bar{T}]$. Theorem 3.1 states that each pricing measure corresponds to a pair of predictable processes α and β satisfying conditions (11). We examine the relationship between these processes and the risk premia demanded by investors. We start with an observation that is a standard consequence of Girsanov's Theorem. We omit the proof.

Proposition 4.1. *Under the pricing measure $Q = Q(\alpha, \beta)$ with α and β satisfying conditions (11), the process $(W_t^Q)_{t \le T}$ given by $W_t^Q = W_t^P + \int_0^t \alpha_s ds$ is a $\mathbb{G}$-standard Brownian motion and the process $(H_t^Q)_{t \le T}$ given by $H_t^Q = N_t - \int_0^{t \wedge \tau} (1 + \beta_s) dA_s^P$ is a $\mathbb{G}$-martingale.*

The interpretation of α as a risk premium can be seen in the value dynamics of a credit sensitive claim. From A7, Theorem 3.2 and Proposition 4.1 we can calculate the value dynamics relative to $(Q, \mathbb{G})$ and identify the cumulative drift μ_C^Q as

$$\mu_C^Q(t) = rt + \int_0^{t \wedge \tau} (1 - R_s) dA_s^Q, \quad 0 \le t \le T.$$

In view of Theorem 3.2 and Proposition 4.1, we then get the excess cumulative drift p_C in the security value required by investors:

$$p_C(t) = \mu_C^P(t) - \mu_C^Q(t) = \int_0^{t \wedge \tau} \alpha_s \sigma_C(s) ds, \quad 0 \le t \le T.$$

Prior to default, the excess growth rate $dp_C(t)/dt$ is proportional to the diffusive price volatility $\sigma_C(t)$. The proportionality factor α_t equals the excess return per unit of diffusive volatility. It can thus be interpreted as the market price of Brownian motion driven diffusion-type risk in the value of the firm. The product $\alpha \sigma_C$ is the *diffusive risk premium.* After default, the recovery value paid at τ grows at the constant risk-free rate r under both measures so the excess growth rate $dp_C(t)/dt$ vanishes.

To understand the role of β, note that Proposition 4.1 implies that the compensator of N with respect to $(Q, \mathbb{G})$ is given by

$$A_{t \wedge \tau}^Q = \int_0^{t \wedge \tau} (1 + \beta_s) dA_s^P. \tag{13}$$

From equation (13) and the heuristic relation $dA^{\pi}_{t\wedge\tau} = E^{\pi}[dN_t \,|\, \mathcal{G}_t]$ implied by the $(\pi, \mathbb{G})$-martingale property of the compensated process $N - A^{\pi}_{\cdot\wedge\tau}$, we get that

$$Q[t < \tau \leq t + dt \,|\, \mathcal{G}_t] = (1 + \beta_t)P[t < \tau \leq t + dt \,|\, \mathcal{G}_t] \tag{14}$$

on the no-default set $\{\tau > t\}$. Hence, β_t provides the mapping between the instantaneous Q-default probability and the instantaneous P-default probability. This suggests the interpretation of β as the *default event risk premium* that is demanded by investors as compensation for bearing exposure to the downward jump in security prices at default. Note that this event premium vanishes in first passage models in which investors can observe the default barrier. This is because there is no short-term default risk with complete information. Prices do not jump at default so the credit premium has only a diffusive component that comes from the random movements in the gross firm value. Thus, traditional first passage models with complete information cannot account for the substantial event risk premia that have been measured by Collin-Dufresne et al. (2002), Berndt et al. (2005), Driessen (2005), Eckner (2007) and Azizpour & Giesecke (2008) in bond, credit swap, and index and tranche markets. Our analysis indicates that the informational asymmetries that public credit investors face play an important role for credit risk premia. In particular, incomplete information about the default barrier can induce the significant event premia found in credit markets.

The infinitesimal Q-default probability (14) can be interpreted as the undiscounted pre-default price of an insurance contract that pays one dollar if default occurs over the next infinitesimal period of time, and zero otherwise. If investors are risk-neutral with respect to default event risk, they value default insurance with the P-default probability, which represents the expected instantaneous default loss consistent with historical default experience. Here $\beta = 0$ and instantaneous default probabilities are equal under P and Q. If investors prefer event risk, then their value of the insurance contract is lower than the P-expected loss. Here $\beta \in (-1, 0)$ and the instantaneous Q-default probability is less than the P-probability. If investors are averse to event risk, then they pay a spread over the P-expected loss for the insurance contract. In this case $\beta > 0$ and the instantaneous Q-default probability exceeds the P-probability.

The size of the event premium β depends on investors' prior distribution of the unobserved default barrier. First note that the relationship (13) extends to the default trends under the physical and pricing measures:[7]

$$A^Q_t = \int_0^t (1 + \beta_s) dA^P_s. \tag{15}$$

[7] The random measures on $\mathbb{R}_+$ associated with the $\mathbb{F}$-predictable trends $A^P(\omega)$ and $A^Q(\omega)$ are both concentrated on the set $\{t \geq 0 : V_t(\omega) = M_t(\omega)\}$. Hence, for almost all $\omega \in \Omega$, they are absolutely continuous with respect to each other. Theorem 68 in Dellacherie & Meyer (1982, Chapter VI) states that the corresponding density process γ is $\mathbb{F}$-predictable. Equation (13) implies that $\gamma_t = 1 + \beta_t$ on the set $\{t \leq \tau\}$, where β is $\mathbb{F}$-predictable. Using an argument similar to that used in the proof of Proposition B.1, we can then show that $\gamma_t = 1 + \beta_t$ on $[0, \infty)$.

Using formula (7) for the increment dA^P, equation (15) can be rewritten in terms of the barrier distribution G^P and density g^P under the physical measure as

$$A_t^Q = -\int_0^t \frac{g^P(M_s)}{G^P(M_s)}(1+\beta_s)dM_s \tag{16}$$

where M is the historical low of the log-gross firm value. Comparing formula (16) with the formula for A^Q obtained from equations (6) and (7) applied under the pricing measure, we see that the event premium is determined by the relation

$$1+\beta_t = \frac{g^Q(M_t)/G^Q(M_t)}{g^P(M_t)/G^P(M_t)} \tag{17}$$

on the set of times $L = \{t : V_t = M_t\}$ at which the firm reaches its historical low. Equation (17) shows that $\beta_t = f(M_t)$ on L for a deterministic function f that is given in terms of the distributions of the default barrier under the physical and pricing measures. The typical case of event risk aversion $\beta \geq 0$ corresponds to the case where the conditional probability that a small increment in minimum firm value causes default, given that default has not yet occurred, is greater for the pricing measure than for the physical measure.

The magnitude of the event premium depends on the degree of investors' aversion to randomness in the location of the unobserved default barrier. In the special case where investors are neutral with respect to default barrier uncertainty, they do not risk-adjust their prior barrier distribution ($G^P = G^Q$), and formula (17) implies that the process β vanishes on the set L. Then, from formula (15), the default trends $A^Q = A^P$. In this case, investors do not require a premium for bearing exposure to event risk although there is short-term uncertainty about the default event. In other words, short-term uncertainty about the default event is not sufficient for the existence of event premia. It matters how investors appreciate their uncertainty about the default barrier location, i.e. how they risk-adjust their prior barrier distribution.

We numerically illustrate the relationship between the event premium and the risk-adjustment to the prior barrier distribution. Suppose the barrier d, scaled with the initial gross firm value X_0, has a Beta distribution on $(0,1)$ with parameters a^π and b^π under $\pi \in \{P, Q\}$. The mean is $a^\pi/(a^\pi + b^\pi)$ and the variance is $a^\pi b^\pi/[(a^\pi + b^\pi)^2(a^\pi + b^\pi + 1)]$. Giesecke & Goldberg (2004*a*) show how to calibrate the parameters from the market equity price and balance sheet data of an issuer. The normalized default barrier $D = \log(d/X_0)$ has prior distribution function under π given by

$$G^\pi(x) = \pi[d \leq X_0 e^x] = \frac{\Gamma(a^\pi + b^\pi)}{\Gamma(a^\pi)\Gamma(b^\pi)} \int_{-\infty}^x e^{ua^\pi}(1-e^u)^{b^\pi - 1} du \tag{18}$$

for $x \leq 0$, where Γ is the gamma function. Based on formula (18), we consider the event premium β_t for some fixed $t > 0$ at which the firm reaches its historical low M_t and β_t is determined by formula (17). The left panel of Figure 1 shows the premium as a function of the ratio $a = a^Q/a^P$ for fixed $b^Q = b^P = a^P = 2$. The right panel shows the premium

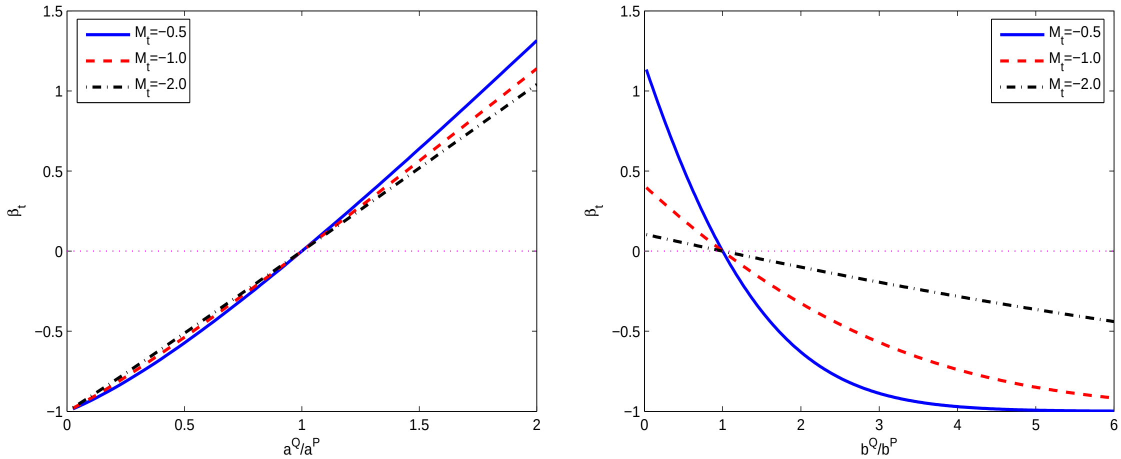

Figure 1: Event premium β_t when the scaled default barrier d/X_0 has a Beta distribution on $(0,1)$ with parameters a^π and b^π under $\pi \in \{P,Q\}$, where X_0 is the initial gross firm value. The left panel shows β_t as a function of the ratio a^Q/a^P for fixed $b^Q = b^P = a^P = 2$. The right panel shows β_t as a function of b^Q/b^P for fixed $a^Q = a^P = b^P = 2$.

as a function of the ratio b^Q/b^P for fixed $a^Q = a^P = b^P = 2$. The left panel of Figure 2 shows the Beta $(2,2)$ density function of the scaled barrier d/X_0 under P. The right panel shows two Beta densities that govern the scaled barrier distribution d/X_0 under Q: one is representative of values of $a < 1$, the other is representative of values of $a > 1$.

The event premium β_t is increasing in a, with a slope that depends on M_t, the historical low of the log gross firm value. We distinguish three regimes for the values of a. If $a \in (0,1)$, then investors prefer randomness in the default barrier location. The prior density of d/X_0 under the pricing measure (Figure 2, right panel, solid line) assigns more mass to low values of the barrier than the corresponding density under the physical measure (Figure 2, left panel). In particular, the Q-mean of d/X_0 is less than the P-mean. Ceteris paribus, for an observed gross firm value the firm is likely to be closer to the default barrier under P. Therefore, the instantaneous P-default probability (the instantaneous expected loss) exceeds the corresponding probability under Q, leading to a negative event premium via formula (14). The premium tends to its lower bound -1 with a or equivalently the instantaneous Q-default probability tending to 0. For $a = 1$ investors are neutral with respect to randomness in the default barrier location. The prior barrier distributions under P and Q agree (Figure 2, left panel) so the event premium vanishes. For $a > 1$ investors are averse to fluctuations in the default barrier location. The prior density of d/X_0 under the pricing measure (Figure 2, right panel, dashed line) assigns more mass to high values of the barrier than the corresponding density under the physical measure (Figure 2, left panel). In particular, the Q-mean of d/X_0 exceeds the P-mean, and the same relationship holds for the instantaneous default probabilities. Formula (14) requires a positive event premium β_t.

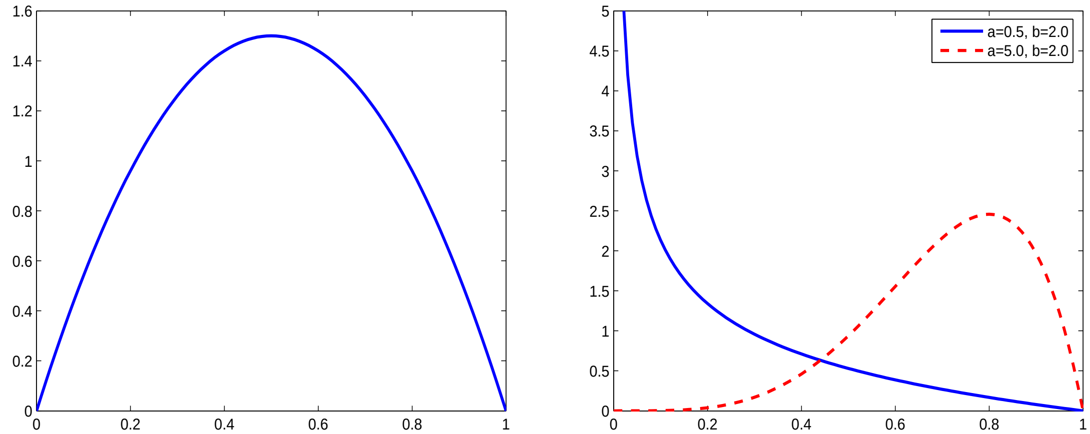

Figure 2: Prior Beta distributions underlying the event premium β_t shown in Figure 1. The left panel shows the Beta $(2,2)$ density function of the scaled barrier d/X_0 under the physical measure. The right panel shows two Beta densities that govern the scaled barrier d/X_0 under the pricing measure: the solid density is representative of values of a^Q/a^P less than 1, and the dashed density is representative of values of a^Q/a^P bigger than 1.

5 Valuing credit sensitive claims

The results of our credit premium analysis have implications for the design and estimation of corporate default models. To discuss these, we consider the valuation of a credit sensitive security, which is characterized by a triple (T, c_T, R). Here, $c_T \in \mathcal{F}_T$ is the payoff at the horizon $T \in (0, \bar{T}]$ if there was no default, and $1 - R$ is the fractional loss in the pre-default market value of the claim if the firm defaults before T. For $Q \in \mathcal{P}$, a no-arbitrage, cum-dividend price $C_t = C_t(Q)$ of this claim at time $t \leq T$ is given by

$$C_t = E^Q\big[e^{-r(T-t)}c_T 1_{\{\tau>T\}} + e^{-r(\tau-t)}R_\tau C_{\tau-}1_{\{\tau\leq T\}}\,|\,\mathcal{G}_t\big]. \qquad (19)$$

Formula (19) has the disadvantage of involving the default time τ explicitly. We provide an alternative reduced form formula that is based on the recovery-adjusted Q-trend $A^Q(R)$, which is defined in terms of the Q-trend A^Q by the formula

$$A_t^Q(R) = \int_0^t (1 - R_s)dA_s^Q. \qquad (20)$$

The Q-trend A^Q represents the risk-adjusted probability of default that is used for pricing. One approach to calculate A^Q is to reason as in the previous two sections. Suppose assumptions A1-A8 hold under the actual measure P. Then the P-trend A^P, the counterpart to A^Q that reflects the empirical likelihood of default, is given by $A^P = -\log G^P(M)$, see C3. Given the density $Z(\alpha,\beta)$ associated with a chosen pricing measure Q, we then calculate A^Q from A^P via formula (15). A second approach is to suppose A1-A8 hold under the pricing measure Q. In other words, the model is formulated directly under pricing probabilities. Then the Q-trend $A^Q = -\log G^Q(M)$ by C3. This approach to calculating A^Q is preferred if the risk premium is not of immediate interest, and the focus is on pricing and hedging credit sensitive securities.

Proposition 5.1. *Suppose the trend A^Q is continuous. If the process Y given by*

$$Y_t = e^{-r(T-t)} E^Q \left[c_T e^{A_t^Q(R) - A_T^Q(R)} \,|\, \mathcal{G}_t \right], \quad t \leq T, \tag{21}$$

has paths that are continuous at the default time, then the credit sensitive claim (T, c_T, R) admits an arbitrage-free value $C_t = Y_t(Q)$ on the no-default set $\{\tau > t\}$ at time $t \leq T$.

If assumptions A1-A8 hold under the actual measure P and the Q-trend A^Q is constructed from the P-trend $A^P = -\log G^P(M)$ and the risk premium processes α and β that determine the pricing measure Q, then the pricing formula (21) links the value of a credit sensitive claim and actual default experience, which governs A^P. Given A^P and a specification of the recovery process R, the pricing formula can be inverted to infer the risk premium processes α and β from traded security prices. This, via formula (17), can provide information about investors' risk adjustment to the prior barrier distribution G^P.

If assumptions A1-A8 hold under the pricing measure Q, then the pricing formula (21) can be inverted to calibrate the Q-model parameters from market prices of credit sensitive securities including equity and bonds, as in Giesecke & Goldberg (2004*a*). The list of parameters to be calibrated includes the parameters of the gross firm value process X, the prior barrier distribution G^Q and the recovery process R. Typically, the risk premium processes α and β cannot be inferred from price data alone.

The proof of Proposition 5.1 does not use the explicit formula for the trend A^Q that follows from assumptions A1–A8. In other words, the pricing formula (21) holds for any model in which the trend is continuous. Thus, Proposition 5.1 extends Corollary 5.5 in Giesecke (2006) to the case of non-trivial recovery. Proposition 5.1 is essentially Theorem 1 in Duffie & Singleton (1999) if the Q-trend is of the form $A_t^Q = \int_0^t \lambda_s^Q ds$ for an intensity process λ^Q. Recall from C3 that under assumptions A1-A8, the trend is continuous but does not admit an intensity λ^Q. Therefore, the additional generality provided by Proposition 5.1 is not vacuous. Bélanger et al. (2004) and Elliott, Jeanblanc & Yor (2000) use different sets of assumptions to derive pricing formulas similar to (21).

The process C in Proposition 5.1 uniquely defines the price of the claim only if markets are complete. In the incomplete case, the pricing measure is not unique so Proposition 5.1 leads to an interval of arbitrage-free prices for (T, c_T, R).

6 Conclusion

Based on a model of corporate default risk with asymmetric information, this paper argues that investors in credit sensitive securities issued by or referenced on a firm may demand a transparency premium for bearing any uncertainty associated with the firm's liquidation strategy. This conclusion has important consequences for a firm's information policy. In particular, firm management has an incentive to publicize the asset value at which it liquidates the firm. An improvement in transparency lowers the credit premium required by risk-averse investors for assuming non-diversifiable corporate default risk, and therefore the cost to the firm of equity and debt financing.

The transparency premium is realized as an event premium for short-term default event risk induced by incomplete information about the threshold asset value at which the firm defaults. If the firm is fully transparent, then investors must bear only the volatility in the firm value representing the present value of the firm's future cash flows. If firm management does not disclose the default barrier, then investors face an additional source of uncertainty, namely the location of the barrier. In this case, investors assume instantaneous default risk as they cannot discern the firm's distance to default. Default comes unannounced, and causes a downward jump in the prices of corporate securities. The resulting event premium is governed by the degree of investors' aversion to the randomness in the location of the unobserved default threshold. The premium vanishes when investors are neutral with respect to variation in that location, i.e. when they are indifferent about the transparency level of the firm. The substantial event premia that have been measured in several credit markets are consistent with the presence of information asymmetries that are economically important to investors.

A Probabilistic model structure

We detail the probabilistic structure underlying our model described in Section 2.

We introduce two probability spaces. The first is the filtered space $(\Omega_1, \mathcal{F}^1, (\mathcal{F}^1_t)_{t\geq 0}, \pi_1)$ supporting the standard Brownian motion $\tilde{W}$. The second is the space $(\Omega_2, \mathcal{F}^2, \pi_2)$ supporting the random variable $\tilde{d}$. Here we may set $\Omega_2 = (0, X_0)$ for some constant $X_0 > 0$ and $\tilde{d} = \omega_2$ for $\omega_2 \in \Omega_2$, and $\mathcal{F}^2 = \sigma(\tilde{d})$. Our reference probability space is

$$(\Omega, \mathcal{G}, \pi) = (\Omega_1 \times \Omega_2, \mathcal{F}^1 \otimes \mathcal{F}^2, \pi_1 \otimes \pi_2)$$

where the state of the world $\omega \in \Omega$ is the pair (ω_1, ω_2).

On this space, we define the standard Brownian motion $W^\pi(\omega) = \tilde{W}(\omega_1)$ and the random default barrier $d(\omega) = \tilde{d}(\omega_2)$ considered in assumptions A2 and A3, respectively. Notice that we do not observe ω_2, cf. assumption A5. Corresponding to assumption A3, we also introduce the random time τ by setting

$$\tau(\omega) = \inf\{t > 0 : V_t(\omega) \leq D(\omega)\},$$

where $V_t = mt + \sigma W_t$ for constants $m \in \mathbb{R}$ and $\sigma > 0$ and $D = \log(d/X_0)$ is the normalized default barrier. The measure π_2 induces a distribution function G^π of the normalized barrier D via $\pi_2(0, X_0 e^x) = G^\pi(x)$ for all $x \in (-\infty, 0)$. The distribution function G^π is often called "prior." Letting $M_t(\omega) = \min_{s\le t} V_s(\omega)$, we can write

$$\{\tau(\omega) > t\} = \{M_t(\omega) > D(\omega)\}.$$

Consider the standard filtration $\mathbb{F} = \mathcal{F}_{t\ge 0}$ generated by W^π on $(\Omega, \mathcal{G}, \pi)$. All the sets in $\mathbb{F}$ are of the form $F \times \Omega_2$ and $F \times \emptyset$ for $F \in \mathcal{F}^1_t$. Let $(\mathcal{S}_t)$ be the standard filtration generated by the indicator process $(1_{\{\tau\le t\}})$. Corresponding to assumption A5, we can now introduce the enlarged filtration $\mathbb{G}$ on the reference space by setting $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{S}_t$.

The $(\pi, \mathbb{F})$-Brownian motion W^π is also a Brownian motion in the enlarged filtration $\mathbb{G}$. Indeed, because W^π ignores ω_2,

$$E^\pi[W^\pi_t|\mathcal{G}_s] = E^\pi[W^\pi_t|\mathcal{F}_s] = W^\pi_s, \quad t \ge s,$$

proving the martingale property of W^π in $\mathbb{G}$. Since the quadratic variation $[W^\pi, W^\pi]_t = t$ does not depend on the filtration, the result follows by Lévy's theorem.

B Technical arguments and proofs

The proof of Theorem 3.1 is based on the following proposition.

Proposition B.1. *The density process Z can be expressed as a sum*

$$Z_t = 1 + \int_0^t a_s dW^P_s + \int_0^t b_s dH^P_s, \tag{22}$$

where a and b are $\mathbb{G}$-predictable processes. For a sequence of $\mathbb{G}$-stopping times T_n that increase to T, these processes satisfy

$$E^P\left[\int_0^{T_n} a_s^2 ds\right] < \infty \quad \textit{and} \quad E^P\left[\int_0^{T_n} |b_s| dA^P_s\right] < \infty. \tag{23}$$

Poof of Proposition B.1. For $Z_T \in L^2(\Omega, \mathcal{G}_T, P)$, the representation (22) holds under more stringent growth conditions on a and b: $E^P[\int_0^T a_s^2 ds] < \infty$ and $E^P[\int_0^T b_s^2 dA^P_s] < \infty$. See for example Kusuoka (1999). Here, we follow a different line of reasoning to obtain a broader result under weaker restrictions on the coefficients.

Let M_P be the collection of all martingales with respect to the stochastic basis $(\Omega, \mathcal{G}_T, (\mathcal{G}_t)_{0\le t\le T}, P)$ that take the form (22) and satisfy (23).

A special case of Jacod (1977, Theorem 2) is that if P is the unique measure on $(\Omega, \mathcal{G}_T, (\mathcal{G}_t)_{0\le t\le T})$ for which every element of M_P is a martingale, then every martingale on $(\Omega, \mathcal{G}_T, (\mathcal{G}_t)_{0\le t\le T}, P)$ is in M_P. We consider the subset $\mathcal{M}_P$ of M_P that consists of $(W_t, H_t, W_t^2 - t)_{0\le t\le \bar{T}}$. For readability we suppress the superscript P on W and H. It suffices to show that P is unique in the sense above with respect to the elements in $\mathcal{M}_P$.

Suppose P' is a measure on $(\Omega, \mathcal{G}_T, (\mathcal{G}_t)_{0\le t\le T})$ and $M_{P'} = M_P$. We show that $P = P'$. Since W and $(W_t^2 - t)$ are continuous martingales for both P and P' with respect to $\mathbb{G}$, Lévy's theorem implies that W is a Brownian motion under both measures. It follows that $P(W_{t_i} \in A_i; i = 1, 2, \ldots, n) = P'(W_{t_i} \in A_i; i = 1, 2, \ldots, n)$ for Borel sets A_i so that $P = P'$ on sets in $\mathcal{F}_T$.

Further, since H is a martingale for both P and P', the uniqueness of the Doob-Meyer decomposition implies that $A^P_{\cdot\wedge\tau}$ is the compensator of N for both P and P'. Let A^P and $A^{P'}$ be the $\mathbb{F}$-predictable trends of N under P and P' respectively. From the martingale property of $N - A^P_{\cdot\wedge\tau}$ and formula (6), $A^P_{t\wedge\tau} = A^{P'}_{t\wedge\tau}$ so that A^P and $A^{P'}$ agree for $t \leq \tau$. We show that they agree on $[0, \infty)$ almost surely. Let Γ be the infimum of all times at which the trends A^P and $A^{P'}$ disagree. Then Γ is an $\mathbb{F}$-stopping time which is an upper bound for τ. This means the running minimum log-firm value M_Γ is less than the default barrier D. But D is not observable in the filtration $\mathbb{F}$. It follows that $\Gamma = \infty$ almost surely and the trends A^P and $A^{P'}$ are indistinguishable.

Let $U \in \mathcal{F}_T$. Then

$$E^{P'}\left[1_U(1 - N_T)\right] = E^{P'}\left[1_U e^{-A^{P'}_T}\right] = E^{P'}\left[1_U e^{-A^P_T}\right] = E^P\left[1_U e^{-A^P_T}\right]$$

where the first equation follows from Bielecki & Rutkowski (2002, Corollary 5.1.1) and the third equation follows from the fact that the argument of the expectation is $\mathcal{F}_T$-measurable. Since every set in $\mathcal{G}_T$ can be arbitrarily well approximated by finite unions and complements of sets of the form $U \cap \{\tau \leq T\}$, it follows that P and P' agree on $\mathcal{G}_T$. Thus, P is the unique measure for which the processes in $\mathcal{M}_P$ are martingales.

Now, Jacod (1977, Theorem 2) implies that every martingale on $(\Omega, \mathcal{G}_T, (\mathcal{G}_t)_{0\leq t\leq T}, P)$ can be represented as in equation (22) with coefficients satisfying conditions (23). □

We now give the proof of Theorem 3.1.

Proof of Theorem 3.1. According to Jacod & Shiryaev (1987, Proposition 3.5a), $P[\inf_t Z_t > 0] = 1$. It follows that P-almost surely, $Z_{t_-} > 0$ for all t. Further, $1/Z_-$ is locally bounded. For if not, then the $\mathbb{G}$-stopping times $\Gamma_n = \inf_t(Z_t < 1/n)$ increase to a stopping time Γ that is strictly less than T on a set $U \in \mathcal{G}_T$ of positive measure. But then $E^P[Z_{\Gamma-}1_U] = 0$, contradicting the fact that P-almost surely, $Z_{t_-} > 0$ for all t.

Let $\alpha = a/Z_-$ and $\beta = b/Z_-$ where a and b are the $\mathbb{G}$-predictable processes defined in Proposition B.1. Since $1/Z_-$ is locally bounded, α and β satisfy (11) and thus, for positive $t \leq T$, the integrals $\int_0^t \alpha_s^2\, ds$ and $\int_0^t |\beta_s|\, dA^P_s$ are finite almost surely. For if there is a positive measure set $U \in \mathcal{G}$ on which one of these integrals diverges, we can choose a large n so that $U \cap \{T^n > t\}$ has positive measure and one of the integrals $E^P[\int_0^{T_n} \alpha_s^2 ds]$ and $E^P[\int_0^{T_n} |\beta_s| dA^P_s]$ will diverge as well.

Define a semimartingale Y by $Y_t = -\int_0^t \alpha_s dW^P_s + \int_0^t \beta_s dH^P_s$, for $t \leq T$. Equation (22) can be rewritten as

$$Z_t = 1 + \int_0^t Z_{s-} dY_s, \quad t \leq T \tag{24}$$

so Z is the stochastic exponential of Y. By Theorem 37 in Chapter II of Protter (2004),

$$Z_t = \exp\left(Y_t - \frac{1}{2}[Y,Y]^c_t\right)\prod_{s\le t}(1+\Delta Y_s)\exp(-\Delta Y_s) \tag{25}$$

where $[Y,Y]^c$ is the (path-by-path) continuous part of $[Y,Y]$, and $\Delta Y_t = Y_t - Y_{t-}$ is the jump of Y at time t. Since W^P is a P-Brownian motion and $A^P_{\cdot\wedge\tau}$ is of finite variation, $[Y,Y]_t = \int_0^t \alpha_s^2 ds + \int_0^t \beta_s^2 dN_s$ so $[Y,Y]^c_t = \int_0^t \alpha_s^2 ds$. Since τ is totally inaccessible, the compensator of N is continuous. It follows that H^P is continuous except for a jump of size 1 at τ. Therefore, $\Delta Y_t = \beta_\tau 1_{\{t=\tau\}}$. With these observations, formula (10) is a consequence of formula (25).

For the converse, suppose that there are $\mathbb{G}$-predictable processes α and β satisfying the conditions (11). Let $Z_T = Z_T(\alpha,\beta)$ be defined by (10). Then $Z_T > 0$ and if $E^P[Z_T]=1$, it is the density dQ/dP of an equivalent measure $Q = Q(\alpha,\beta)\in\mathcal{E}$ with respect to P. □

Proof of Theorem 3.2. For all bounded $Y \in \mathcal{G}_T$, the martingale Z satisfies the equation

$$Z_t E^Q[Y \mid \mathcal{G}_t] = E^P[Y Z_T \mid \mathcal{G}_t], \quad t \le T. \tag{26}$$

From (26), it is equivalent to require that $(\bar{C}_t Z_t)_{t\le T}$, where $\bar{C}_t = C_t e^{-rt}$, is a $(P,\mathbb{G})$-martingale. From (24),

$$dZ_t = Z_{t-}(-\alpha_t dW^P_t + \beta_t dH^P_t). \tag{27}$$

Noting assumption A7, by integration by parts

$$d\bar{C}_t = d(C_t e^{-rt}) = \bar{C}_{t-}(d\mu^P_C(t) - r dt + \sigma_C(t)(1-N_t)dW^P_t - (1-R_t)dN_t). \tag{28}$$

Since the cumulative growth rate μ^P_C has paths of finite variation (A7), the process defined by the Stieltjes integral $\int_0^t \bar{C}_{s-} d\mu^P_C(s)$ has paths of finite variation. Using this and Protter (2004, Chapter IV, Theorem 22), we get

$$[\bar{C}, Z]_t = \int_0^t \bar{C}_{s-} Z_{s-}(-\sigma_C(s)(1-N_s)\alpha_s ds - \beta_s(1-R_s)dN_s). \tag{29}$$

Integrating by parts, substituting equations (27), (28) and (29), and applying the Doob-Meyer decomposition $A^P_{\cdot\wedge\tau} + H^P = N$,

$$\begin{aligned} d(\bar{C}_t Z_t) = {} & \bar{C}_{t-} Z_{t-}\big[d\mu^P_C(t) - (r + \sigma_C(t)(1-N_t)\alpha_t)dt - (\beta_t+1)(1-R_t)dA^P_{t\wedge\tau} \\ & + (\sigma_C(t)(1-N_t) - \alpha_t)dW^P_t + (\beta_t R_t + R_t - 1)dH^P_t\big]. \end{aligned} \tag{30}$$

We see that $(\bar{C}_t Z_t)_{t\le T}$ is a $(P,\mathbb{G})$-martingale if and only if the drift in (30) vanishes. □

The following observation is required to prove Proposition 5.1.

Lemma B.2. *For each $Q \in \mathcal{P}$, a $(Q, \mathbb{G})$-martingale $H^Q(R)$ is defined by*

$$H_t^Q(R) = (1 - R_\tau)N_t - A_{t\wedge\tau}^Q(R), \quad t \le T.$$

Proof. The process $(1-R_\tau)N$ is zero before τ at which time it jumps to $(1-R_\tau) \in \mathcal{G}_\tau$ and stays there. Since R is $\mathbb{G}$-predictable, $(1 - R_\tau)N$ is $\mathbb{G}$-adapted. Since $A_{\cdot\wedge\tau}^Q(R)$ is clearly $\mathbb{G}$-adapted, so is the process $H^Q(R)$. We show that $H^Q(R)Z$ is a P-martingale. We have

$$[H^Q(R), Z]_t = \int_0^t Z_{s-}\beta_s(1 - R_s)dN_s.$$

Integration by parts together with (13) and (20) yields that

$$\begin{aligned} d(Z_t H_t^Q(R)) &= H_{t-}^Q(R)dZ_t + Z_{t-}((1 - R_\tau)dN_t - dA_{t\wedge\tau}^Q(R)) + Z_{t-}\beta_t(1 - R_t)dN_t \\ &= H_{t-}^Q(R)dZ_t + Z_{t-}(1 + \beta_t)((1 - R_\tau)dN_t - (1 - R_t)dA_{t\wedge\tau}^P) \\ &= H_{t-}^Q(R)dZ_t + Z_{t-}(1 + \beta_t)(1 - R_t)dH_t^P, \end{aligned}$$

where H^P is the compensated jump P-martingale. Since Z is also a P-martingale, $ZH^Q(R)$ is a P-martingale as well. This is equivalent to $H^Q(R)$ being a Q-martingale. □

Proof of Proposition 5.1. First note that if A^Q is continuous, then so is $A^Q(R)$. Let $K_t = e^{-rT}E^Q[c_T e^{-A_T^Q(R)} \,|\, \mathcal{G}_t]$ such that $Y_t = e^{A_t^Q(R)+rt}K_t$. Setting $\bar{Y}_t = Y_t e^{-rt} = e^{A_t^Q(R)}K_t$, by integration by parts we get

$$d\bar{Y}_t = e^{A_t^Q(R)}dK_t + \bar{Y}_{t-}dA_t^Q(R).$$

We now define the process U by

$$U_t = (1 - N_t)\bar{Y}_t + \int_0^t R_s\bar{Y}_{s-}dN_s;$$

our goal is to show that $(U_t)_{t\le T}$ is a Q-martingale. Since $\bar{Y}$ does not jump at τ by assumption, $\Delta\bar{Y}\Delta(1 - N) = 0$ and we have by integration by parts

$$\begin{aligned} dU_t &= (1 - N_{t-})e^{-A_t^Q(R)}dK_t - \bar{Y}_{t-}((1 - R_\tau)dN_t - dA_{t\wedge\tau}^Q(R)) \\ &= (1 - N_{t-})e^{-A_t^Q(R)}dK_t - \bar{Y}_{t-}dH_t^Q(R), \end{aligned}$$

for the Q-martingale $H^Q(R)$, cf. Lemma B.2. Since K is also a Q-martingale, $(U_t)_{t\le T}$ is as well a Q-martingale (all integrands are predictable and bounded for $T < \bar{T}$).

By the martingale property of U we get

$$(1 - N_t)Y_t e^{-rt} = E^Q\Big[(1 - N_T)Y_T e^{-rT} + \int_t^T e^{-rs}R_sY_{s-}dN_s \,\Big|\, \mathcal{G}_t\Big].$$

But this implies the valuation formula since $Y_T = c_T$. □

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