OPERATIONS RESEARCH
DOI 10.1287/opre.1100.0864ec

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Electronic Companion—"Risk Analysis of Collateralized Debt Obligations" by Kay Giesecke and Baeho Kim, *Operations Research*, DOI 10.1287/opre.1100.0864.

Risk Analysis of Collateralized Debt Obligations
–Online Appendix–

Kay Giesecke* and Baeho Kim†

March 16, 2010

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*Department of Management Science & Engineering, Stanford University, Stanford, CA 94305-4026, USA, Phone (650) 723 9265, Fax (650) 723 1614, email: `giesecke@stanford.edu`, web: `www.stanford.edu/~giesecke`.

†Korea University Business School, Anam-dong, Seongbuk-gu, Seoul 136-701 Korea, Phone +82 2 3290 2626, Fax +82 2 922 7220, email: `baehokim@korea.ac.kr`.

A Proofs

Proof of Proposition 2.1. Let (U_n^*) be a sequence of standard uniform random variables that are independent of one another and independent of the (T_n^*). Let $\pi(du, dt)$ be the random counting measure with mark space $[0,1]$ associated with the marked point process (T_n^*, U_n^*). Note that $N_t^* = \int_0^t \int_0^1 \pi(du, ds)$, and that $\pi(du, dt)$ has intensity $\lambda^\pi(u, t) = \lambda_{t-}^*$ for $u \in [0,1]$, which we choose to be predictable. This means that the process defined by $\int_0^t \int_0^1 (\pi(du, ds) - \lambda^\pi(u, s) du ds)$ is a local martingale. Now for $u \in [0,1]$ define the process $I(u, \cdot)$ by $I(u, t) = 1_{\{u \leq Z_t^*\}}$. Then the counting process N generated by the selected event times can be written as

$$N_t = \sum_{n \geq 1} I(U_n^*, T_n^*) 1_{\{T_n^* \leq t\}} = \int_0^t \int_0^1 I(u, s) \pi(du, ds).$$

Since Z^* is predictable, $I(u, \cdot)$ is predictable for $u \in [0,1]$. The process $I(u, \cdot)$ is also bounded for $u \in [0,1]$. Therefore, a local martingale M is defined by

$$M_t = N_t - \int_0^t \int_0^1 I(u, s) \lambda^\pi(u, s) du\, ds.$$

But the definitions of $I(u, s)$ and $\lambda^\pi(u, s)$ yield

$$M_t = N_t - \int_0^t \int_0^{Z_s^*} \lambda_{s-}^*\, du\, ds = N_t - \int_0^t Z_s^* \lambda_s^*\, ds,$$

which means that N has intensity $Z^* \lambda^*$. $\square$

Proof of Proposition 2.2. Proposition 2.1 implies that the compensator $A = \int_0^\cdot \lambda_s ds$ to N is absolutely continuous with respect to the compensator A^* to N^*, with density Z^*. Because $Z_t^* > 0$ almost surely, A^* is also absolutely continuous with respect to A, with density $1/Z^*$. It follows that A^* can be written as

$$A_t^* = \int_0^t \frac{1}{Z_s^*}\, dA_s = \int_0^t \frac{1}{Z_s^*}\, \lambda_s\, ds.$$

The time change theorem of Meyer (1971) implies the result, since A^* is continuous and increases to ∞, almost surely. $\square$

Proof of Proposition 3.1. It suffices to show that $\int_0^t \lambda_s ds < \infty$ almost surely for each $t > 0$. We assume that $\gamma = 0$, without loss of generality. For $s \geq 0$ and $h(0) > 0$, let

$$h(s) = ch(0) + (1 - c)h(0) \exp(-\kappa h(0) s).$$

The function h describes the behavior of the intensity between events. For $T_n \leq t < T_{n+1}$ and $h(0) = \lambda_{T_n}$, we have that $\lambda_t = h(t - T_n)$. The inverse h^{-1} to h is given by

$$h^{-1}(u) = \frac{1}{\kappa h(0)} \log\left(\frac{h(0)(1-c)}{u - ch(0)}\right)$$

for $ch(0) < u \leq h(0)$. Consider the first hitting time

$$\varphi(h(0)) = \inf\{s \geq 0 : h(s) < h(0)/(1+\delta)\}.$$

For $c(1+\delta) < 1$, we have

$$\varphi(h(0)) = h^{-1}(h(0)/(1+\delta)) = \frac{1}{\kappa h(0)} \log\left(\frac{(1+\delta)(1-c)}{1-c(1+\delta)}\right).$$

Let $T_0 = 0$. The conditional probability given $\mathcal{F}_{T_n}$ of the intensity jumping at an event to a value that exceeds the value taken at the previous event is

$$\begin{aligned} P(\lambda_{T_{n+1}} > \lambda_{T_n} \,|\, \mathcal{F}_{T_n}) &= P(T_{n+1} - T_n < \varphi(\lambda_{T_n}) \,|\, \mathcal{F}_{T_n}) \\ &= 1 - \exp\left(- \int_0^{\varphi(\lambda_{T_n})} h(s)ds \right) \\ &= 1 - \exp\left(- \frac{c}{\kappa} \log\left(\frac{(1+\delta)(1-c)}{1-c(1+\delta)}\right) - \frac{\delta}{\kappa(1+\delta)} \right) =: M, \end{aligned}$$

where M is strictly less than 1 and independent of $n = 0, 1, 2, \ldots$ It follows that the unconditional probability $P(\lambda_{T_{n+1}} > \lambda_{T_n}) = M$, for any n. Further, an induction argument can be used to show that for any n and $k = 1, 2, \ldots$

$$P(\lambda_{T_{n+k}} > \lambda_{T_{n+k-1}} > \cdots > \lambda_{T_n}) = M^k.$$

Now letting

$$C_t^k = \{\omega : N_t(\omega) \geq k \text{ and } \lambda_{T_{n+1}}(\omega) > \lambda_{T_n}(\omega) \text{ for } n = N_t(\omega) - 1, \ldots, N_t(\omega) - k\}$$

for $t > 0$ and $k = 1, 2, \ldots$, we have that

$$\begin{aligned} P(C_t^k) &= P(N_t \geq k \text{ and } \lambda_{T_{N_t}} > \lambda_{T_{N_t - 1}} > \cdots > \lambda_{T_{N_t - k}}) \\ &\leq P(\lambda_{T_{N_t + k}} > \lambda_{T_{N_t + k - 1}} > \cdots > \lambda_{T_{N_t}}) \\ &= M^k \end{aligned}$$

independently of t. We then conclude that

$$\begin{aligned}
P\bigg(\int_0^t \lambda_s(\omega)ds < \infty\bigg) &\geq P\bigg(\sup_{s\in[0,t]} \lambda_s(\omega) < \infty\bigg) \\
&\geq P\bigg(\bigg\{\bigcap_{k=1}^{\infty} C_t^k\bigg\}^c\bigg) \qquad (1) \\
&= 1 - P\bigg(\bigcap_{k=1}^{\infty} C_t^k\bigg) \\
&= 1 - \lim_{k\to\infty} P(C_t^k) \qquad (2) \\
&\geq 1 - \lim_{k\to\infty} M^k \\
&= 1.
\end{aligned}$$

Equation (2) is due to the fact that $C_t^k \in \mathcal{F}_t$ for all $k = 1, 2, \ldots$ and $C_t^1 \supseteq C_t^2 \supseteq \cdots$. To justify the inequality (1), we argue as follows. For given $t > 0$ and $\omega \in \Omega$, define

$$A_t(\omega) = \big\{k \in \{0, 1, \ldots, N_t(\omega) - 1\} : \lambda_{T_{k+1}}(\omega) \geq \lambda_{T_k}(\omega)\big\}.$$

Let $\hat{\omega} \in \{\bigcap_{k=1}^{\infty} C_t^k\}^c$. Then, by definition of C_t^k, either (i) $N_t(\hat{\omega}) < \infty$, or (ii) there exists some $n < \infty$ such that $\lambda_{T_{n+1}}(\hat{\omega}) < \lambda_{T_n}(\hat{\omega})$ and $|A_t(\hat{\omega}) \setminus A_{T_n}(\hat{\omega})| < \infty$.

In case (i), thanks to the condition $|A_t(\hat{\omega})| \leq N_t(\hat{\omega}) < \infty$, each $\lambda_{T_{k+1}}(\hat{\omega}) - \lambda_{T_k}(\hat{\omega}) < \infty$ for all $k \in A_t(\hat{\omega})$. Thus, $\lambda_s(\hat{\omega}) < \infty$ for all $s \in [0, t]$. Hence, we conclude that $\hat{\omega} \in \{\omega : \sup_{s\in[0,t]} \lambda_s(\omega) < \infty\}$ in this case.

Now consider case (ii). Note that for any $s \in [0, t]$, the condition $c\lambda_{T_{N_s}}(\hat{\omega}) < \lambda_s(\hat{\omega})$ implies that $\lambda_s(\hat{\omega})$ remains at infinity once $\lambda_{T_{N_s}}(\hat{\omega})$ achieves infinity. Hence, $\lambda_{T_{n+1}}(\hat{\omega}) < \lambda_{T_n}(\hat{\omega})$ implies that $\lambda_{T_n}(\hat{\omega}) < \infty$, and we conclude that $\sup_{s\in[0,T_n]} \lambda_s(\hat{\omega}) < \infty$. Moreover, due to the condition $|A_t(\hat{\omega}) \setminus A_{T_n}(\hat{\omega})| < \infty$, we conclude that $\sup_{s\in[T_n,t]} \lambda_s(\hat{\omega}) < \infty$.

Therefore, $\{\bigcap_{k=1}^{\infty} C_t^k\}^c \subseteq \{\omega : \sup_{s\in[0,t]} \lambda_s(\omega) < \infty\}$ and (1) is justified. □

B Risk-neutral tranche loss distributions

To describe the estimation of the risk-neutral tranche loss distributions discussed in Section 4, it is necessary to explain the arbitrage-free valuation of index and tranche swaps. These contracts are based on a portfolio of C credit swaps with common notional 1, common maturity date H and common quarterly premium payment dates (t_m).

In an index swap, the protection seller covers portfolio losses as they occur (default leg), and the protection buyer pays $Sa_m(C - N'_{t_m})$ at each premium date t_m, where S is the swap rate and a_m is the day count fraction for coupon period m (premium leg). The value at time $t \leq H$ of the default leg is given by $D_t(H, 0, 1)$, where

$$D_t(H, \underline{K}, \overline{K}) = E_Q \left(\int_t^H \exp\left(-\int_t^s r_u du \right) dU_s(\underline{K}, \overline{K}) \,\Big|\, \mathcal{F}_t \right). \tag{3}$$

Here, $E_Q(\cdot \mid \mathcal{F}_t)$ denotes the conditional expectation operator relative to a risk-neutral pricing measure Q. The value at time $t \leq H$ of the premium leg is given by

$$P_t(S) = S \sum_{t_m \geq t} a_m E_Q \left(\exp\left(-\int_t^{t_m} r_u du \right) (C - N'_{t_m}) \,\Big|\, \mathcal{F}_t \right). \tag{4}$$

The index rate at t is the solution $S = S_t(H)$ to the equation $D_t(H, 0, 1) = P_t(S)$.

In a tranche swap with upfront rate G and running rate S, the protection seller covers tranche losses (14) as they occur (default leg) and, for $\overline{K} < 1$, the protection buyer pays GKC at inception and $Sa_m(KC - U_{t_m})$ at each premium date t_m, where $K = \overline{K} - \underline{K}$ is the tranche width (premium leg). The value at $\tau \leq H$ of the default leg is given by (3). The value at time $t \leq H$ of the premium leg is given by

$$P_t(\underline{K}, \overline{K}, G, S) = GKC + S \sum_{t_m \geq t} a_m E_Q \left(\exp\left(-\int_t^{t_m} r_u du \right) (KC - U_{t_m}) \,\Big|\, \mathcal{F}_t \right). \tag{5}$$

For a fixed upfront rate G, the running rate S is the solution $S = S_t(H, \underline{K}, \overline{K}, G)$ to the equation $D_t(H, \underline{K}, \overline{K}) = P_t(\underline{K}, \overline{K}, G, S)$. For a fixed rate S, the upfront rate G is the solution $G = G_t(H, \underline{K}, \overline{K}, S)$ to the equation $D_t(H, \underline{K}, \overline{K}) = P_t(H, \underline{K}, \overline{K}, G, S)$.

The valuation relations are used to estimate the risk-neutral portfolio and tranche loss distributions from market rates of index and tranche swaps. First we formulate a parametric model for the risk-neutral dynamics of N and L, the portfolio default and loss processes with replacement. Our risk-neutral model parallels the model under P. Suppose that N has risk-neutral intensity λ^Q with Q-dynamics

$$d\lambda_t^Q = \kappa_t^Q (c_t^Q - \lambda_t^Q) dt + dJ_t \tag{6}$$

where $\lambda_0^Q > 0$, $\kappa_t^Q = \kappa^Q \lambda_{T_{N_t}}^Q$ is the decay rate, $c_t^Q = c^Q \lambda_{T_{N_t}}^Q$ is the reversion level, and J is a response jump process given by

$$J_t = \sum_{n \geq 1} \max(\gamma^Q, \delta^Q \lambda_{T_n^-}^Q) 1_{\{T_n \leq t\}}. \tag{7}$$

The quantities $\kappa^Q > 0$, $c^Q \in (0, 1)$, $\delta^Q > 0$ and $\gamma^Q \geq 0$ are parameters such that $c^Q(1 + \delta^Q) < 1$. We let $\theta^Q = (\kappa^Q, c^Q, \delta^Q, \gamma^Q, \lambda_0^Q)$.

Since the Q-dynamics of λ^Q mirror the P-dynamics of the P-intensity λ, we can apply the algorithms developed above to estimate the expectations (3)–(5) and to calculate the model index and tranche rates for the model (6). We first generate event times T_n of N by Algorithm 3, with λ and its parameters replaced by their risk-neutral counterparts,

Contract	MarketBid	MarketAsk	MarketMid	Model
Index	332.88	333.13	333.01	333.08
0-10%	84.50%	85.00%	84.75%	86.75%
10-15%	52.75%	53.75%	53.25%	48.19%
15-25%	393.00	403.00	398.00	398.88
25-35%	70.00	85.00	77.50	79.43
MinObj				33.77
AAPE				2.90%

Table 1: Market data from Morgan Stanley and fitting results for 5 year index and tranche swaps referenced on the CDX.HY6 on 3/27/2006. The index, $(15-25\%)$ and $(25-35\%)$ contracts are quoted in terms of a running rate S stated in basis points (10^{-4}). For these contracts the upfront rate G is zero. The $(0, 10\%)$ and $(10, 15\%)$ tranches are quoted in terms of an upfront rate G. For these contracts the running rate S is zero. The values in the column Model are fitted rates based on model (6) and 100K replications. We report the minimum value of the objective function MinObj and the average absolute percentage error AAPE relative to market mid quotes.

and with initial condition $(N_\tau, T_{N_\tau}, \lambda^Q_{T_{N_\tau}}) = (0, 0, \lambda^Q_\tau)$. Then we apply the replacement Algorithm 2 as stated to generate event times T'_n without replacement, and the corresponding paths of N' and L' required to estimate the expectations (3)–(5). In this last step, we implicitly assume that the thinning probability (7), the rating distributions (8) and (9), and the distribution μ_ℓ of the loss at an event are not adjusted when the measure is changed from P to Q. The risk-free interest rate r is assumed to be deterministic, and is estimated from Treasury yields for multiple maturities on 3/27/2006, obtained from the website of the Department of Treasury.

The risk-neutral parameter vector θ^Q is estimated from a set of market index and tranche rates by solving the nonlinear optimization problem

$$\min_{\theta^Q \in \Theta^Q} \sum_i \left(\frac{\text{MarketMid}(i) - \text{Model}(i, \theta^Q)}{\text{MarketAsk}(i) - \text{MarketBid}(i)} \right)^2 \tag{8}$$

where $\Theta^Q = (0, 2) \times (0, 1) \times (0, 2)^2 \times (0, 20)$ and the sum ranges over the data points. Here, MarketMid is the arithmetic average of the observed MarketAsk and MarketBid quotes. We address the problem (8) by adapted simulated annealing. The algorithm is initialized at a set of random parameter values, which are drawn from a uniform distribution on the parameter space Θ^Q. For each of 100 randomly chosen initial parameter sets, the algorithm converges to the optimal parameter values given in Table 2. The market data and fitting results for 5 year index and tranche swaps referenced on the CDX.HY6 on 3/27/2006 are reported in Table 1. The model fits the data, with an average absolute percentage error of 2.9%.

Parameter	κ^Q	c^Q	δ^Q	γ^Q	λ^Q_τ
Estimate	0.522	0.333	0.366	1.646	8.586

Table 2: Estimates of the parameters of the risk-neutral portfolio intensity λ^Q, obtained from market rates of 5 year index and tranche swaps referenced on the CDX.HY6 on 3/27/2006.

C Cash CDO prioritization schemes

This appendix describes the prioritization schemes for the cash CDOs analyzed in this paper. These schemes were introduced by Duffie & Garleanu (2001).

C.1 Uniform prioritization

The total interest income $W(m)$ from the reference bonds is sequentially distributed as follows. In period m, the debt tranches receive the coupon payments

$$\begin{aligned} Q_1(m) &= \min\{(1+c_1)A_1(m-1) + c_1F_1(m), W(m)\} \\ Q_2(m) &= \min\{(1+c_2)A_2(m-1) + c_2F_2(m), W(m) - Q_1(m)\}. \end{aligned}$$

Unpaid reductions in principal from default losses, $J_j(m)$, occur in reverse priority order, so that the residual equity tranche suffers the reduction

$$J_3(m) = \min\{F_3(m-1), \xi(m)\},$$

where

$$\xi(m) = \max\{0, L'_{t_m} - L'_{t_{m-1}} - [W(m) - Q_1(m) - Q_2(m)]\}$$

is the cumulative loss since the previous coupon date minus the collected and undistributed interest income. Then, the debt tranches are reduced in principal by

$$\begin{aligned} J_2(m) &= \min\{F_2(m-1), \xi(m) - J_3(m)\} \\ J_1(m) &= \min\{F_1(m-1), \xi(m) - J_3(m) - J_2(m)\}. \end{aligned}$$

With uniform prioritization there are no early payments of principal, so $P_1(m) = P_2(m) = 0$ for $m < M$. At maturity, principal and accrued interest are treated identically, while the remaining reserve is paid in priority order. The payments of principal at maturity are

$$\begin{aligned} P_1(M) &= \min\{F_1(M) + A_1(M), R(M) - Q_1(M)\}, \\ P_2(M) &= \min\{F_2(M) + A_2(M), R(M) - Q_1(M) - P_1(M) - Q_2(M)\} \end{aligned}$$

where $R(M)$ is the value of the reserve account at M. The residual equity tranche receives

$$D_3(M) = R(M) - Q_1(M) - P_1(M) - Q_2(M) - P_2(M).$$

C.2 Fast prioritization

The senior tranche collects interest and principal payments as quickly as possible until maturity or until its remaining principal becomes zero, whichever is first. In period m, the senior tranche receives interest and principal payments

$$\begin{aligned} Q_1(m) &= \min\{B(m), (1+c_1)A_1(m-1) + c_1 F_1(m)\} \\ P_1(m) &= \min\{F_1(m-1), B(m) - Q_1(m)\} \end{aligned}$$

As long as the senior tranche receives payments the junior tranche accrues coupons. After the senior tranche has been retired, the junior tranche is allocated interest and principal until maturity or until its principal is written down, whichever is first:

$$\begin{aligned} Q_2(k) &= \min\{(1+c_2)A_2(m-1) + c_2 F_1(m), B(m) - Q_1(m) - P_1(m)\} \\ P_2(k) &= \min\{F_2(m-1), B(m) - Q_1(m) - P_1(m) - Q_2(m)\}. \end{aligned}$$

The equity tranche receives any residual cash flows. There are no contractual reductions in principal.

D Cash CDO valuation

This appendix explains the valuation of the cash CDO reference bonds and debt tranches. At time $t \leq H$, the value of the reference portfolio is

$$O_t(v) = \sum_{t_m \geq t} E_Q\left(\exp\left(- \int_t^{t_m} r_u du \right) B(m,v) \,\middle|\, \mathcal{F}_t \right) \tag{9}$$

where the notation $B(m) = B(m,v)$ indicates the dependence of the cash flow $B(m)$ on the coupon rate v of a reference bond. The par coupon rate of a reference bond is the number $v = v^*$ such that $O_{t_0}(v) = C$ at the CDO inception date t_0.

The par coupon rates for the debt tranches are determined similarly. Let $O_{jt}(v^*, c_1, c_2)$ be the value at time $t \leq H$ of the bond representing tranche $j = 1, 2$ when the reference bonds accrue interest at their par coupon rate v^*, and when the coupon rates on the debt tranches are c_1 and c_2, respectively. Because of the complexity of the cash flow "waterfall," this quantity does not admit a simple analytic expression. The par coupon rates are given by the pair $(c_1, c_2) = (c_1^*, c_2^*)$ such that $O_{jt_0}(v^*, c_1, c_2) = p_j$ for $j = 1, 2$.

For the CDX.HY6 of reference bonds, assuming that the cash CDO is established on 3/27/06 and has a 5 year maturity, the par coupon rates are estimated as follows. We first generate a collection of 100K paths of N' and L' under the risk neutral measure, based on the risk-neutral intensity model (6) with calibrated parameters in Table 2, as described in online Appendix 2. Based on these paths, we can estimate $O_{t_0}(v)$ for fixed v, and then solve numerically for the par coupon rate v^*. The risk-free interest rate r is deterministic, and is estimated from Treasury yields for multiple maturities on 3/27/2006. Given v^*, c_1, c_2, we can calculate the tranche cash flows for a given path of (N', L') according to the specified prioritization scheme, and then estimate $O_{jt_0}(v^*, c_1, c_2)$. We then solve numerically a system of two equations for (c_1^*, c_2^*).

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