Operationalizing Kant:
Manifolds, Models, and Mathematics in Helmholtz's Theories of Perception

Cover Graphic: C.G. Theodor Reute, Ophtalmotrope
Source: Reute, C.G. Theodor. Ein neues Opthalmotrop. Leipzig: Otto Wigand, 1857.
See Hermann Helmholtz. Handbuch der physiologischen Optik. 2nd ed. Hamburg and Leipzig: Leopold Voss, 1896. 667

Tim Lenoir
Stanford University
History & Philosophy of Science

Although he considered himself a physicist, Hermann Helmholtz devoted more pages of his published work to physiological psychology and philosophical problems related to spatial perception than to any other subject. These writings constantly engage issues related to Kant's work on the perception of space, time, and the law of causality. Helmholtz's first published engagement with these issues was in his contribution to the 1855 centennial celebrating Kant's inaugural lecture in Königsberg in a lecture entitled *"Über das Sehen des Menschen."* There Helmholtz outlined the difficulty he saw in some contemporary theories of vision which fashioned themselves as Kantian in adopting a so-called "nativist" position which attributed spatial perception to the inborn neuroanatomical organization of the visual apparatus. He critiqued them as implicitly setting up two worlds, an objective physical world and a subjective world of intuition, somehow causally related to one another but existing as independent, parallel worlds relying on some unexplained pre-established harmony between perceptions and the real world as the basis for the objective reference of knowledge claims. Helmholtz, believing it was necessary to escape the subjectivity of idealist positions asked: But how is it that we escape from the world of the sensations of our own nervous system into the world of real things?"[1]

In addressing this question over a number of years Helmholtz consistently depicted himself as a Kantian; this in spite of the fact that he broke with Kant by defending the empirical origin of geometrical axioms. In many ways Helmholtz considered his work in epistemology not as contradicting Kant, but as updating Kant's views in light of new developments in experimental physiology which challenged the nativist approach. Although disputed in Helmholtz's own day, and I am sure utterly rejected by the collection of Kant scholars assembled here, Helmholtz thus considered himself to be more consistently Kantian than Kant had been himself. As I see it, Helmholtz pursued fairly consistently over many years a line of investigation centered around the questions, How do our mental representations of external objects get constructed? And how do those representations relate to the world of external objects (things in themselves)?

His general approach to these questions was treated in the 1855 lecture as a sign theory of perception, which over the next decade became central both to Helmholtz's theory of the visual field and to his epistemological concerns. Helmholtz's theory of signs is not usually the subject of attention, but I regard it as a focal point from which to consider a number of deeply interacting and intertwined threads in Helmholtz's experimental and theoretical researches in physiological optics, physiological acoustics, psychophysics, and his later investigations into the relations between the foundations of mechanics and of geometry. I have treated aspects of this network of investigations elsewhere.[2] In this paper I am primarily concerned with the interaction among Helmholtz's experimental researches in the theory of pigments, spectral colors, and color vision in the development of the sign theory. A further concern of mine is the positive contribution to Helmholtz's theories of sensation and perception made by a variety of new media technologies, particularly electric, photographic, and telegraphic inscription devices. The material characteristics of these media served Helmholtz as analogs and as models of the sensory processes he was investigating. These devices not only assisted Helmholtz in understanding the operation of the eye and ear through measurement; more to the point, Helmholtz conceived of the nervous system as a telegraph — and not just for purposes of popular presentation; he viewed its appendages — sensory organs — as media apparatus: the eye was a photometer; the ear a tuning fork interrupter with attached resonators. The output of these devices was encoded in the form of an n-dimensional manifold, a complex measure to which a sign, such as "*Rot*," *"Blau-grun,"* or "*u*" was attached. These materialities of communication were important not only because they enabled theoretical problems of vision and hearing to be translated, externalized, rendered concrete and manipulable in media technologies but furthermore because in this exteriorized form analogies could be drawn between devices; linkages could be made between different processes and between various aspects of the same process. Crucial to Helmholtz's theorizing were analogies between sound and color perception. Indeed, I will argue that a crucial step in his development of the trichromatic receptor theory of color sensation came through analogies between the technologies of sound and of color production. The juxtaposition of media enabled by the materialities-the exteriorized forms-of communication was a driving force in the construction of theory.

Helmholtz was engaged in developing an empiricist theory of vision in which the visual field is constructed from experience and stabilized through repetition. Helmholtz adopted the radical stance that nothing is given in the act of perception. He denied even the empirically plausible claim of his mentor, Johannes Müller, that single vision in a binocular field is determined by neuroanatomical linkages between geometrically corresponding points in the two retina and optical chiasma. For Helmholtz all such associations are learned. Even more significant for his theory of representation, Helmholtz's radical empiricist stance led him to question and ultimately abandon all "copy" theories of representation. For instance, Helmholtz rejected straightaway the notion that a camera obscura-like image on the retina (even if it could be assumed that a sharp image forms on the retina) is a copy of an external reality or that anything like a "copy" of the retinal image could be assumed to be conveyed via the nerves to the brain and reconstituted there as a virtual image. Even if there were somehow a copy of the image in the brain, in Helmholtz's view it would be totally inconsequential for a construction of the visual field, a sort of accidental byproduct of brain anatomy irrelevant to the actual process of vision:

> I maintain, therefore, that it cannot possibly make sense to speak about any truth of our perceptions other than practical truth. Our perceptions of things cannot be anything other than symbols, naturally given signs for things, which we have learned to use in order to control our motions and actions. When we have learned to read those signs in the proper manner, we are in a condition to use them to orient our actions such that they achieve their intended effect; that is to say, that new sensations arise in an expected manner. Any other comparison between perceptions and things exists neither in reality — on this point all schools of thought are in agreement — nor is any other sort of comparison conceivable and meaningful. This latter point is crucial for seeing how to get out of the labyrinth of contradictory opinions. To ask whether a perception which I have of a table, of its shape, solidity, color, weight, etc., is true and corresponds with the real thing or whether it is false and rests upon an illusion independently of the practical use to which I can put this perception has no more sense than to ask whether a certain sound is red, yellow, or blue. Perception and things perceived belong to two completely different worlds which admit of no more comparison to one another than colors and sounds or the letters of a book to the sound of the words which they signify.[3]

Further criticizing the notion that the truth of a perception depends on its being a copy of the object perceived, Helmholtz went on to question the sort of brain physiology required in order to guarantee some similarity between perceived object and its representation in the brain. What sort of similarity, for instance, should be imagined between the processes in the brain accompanying the perception of a table and the table itself? Should one assume that the shape of the table is traced by an electrical current in the brain? And when the percipient imagines that he is moving around the table, should a second person be electrically traced walking around the table in the brain? Even assuming such a fantasy could actually be the case, the problem of perception would not be resolved: "For the electrical copy of the table in the brain would now be a second object that would have to be perceived, but it would still not be a perception of the table."[4]

What was this symbolic language like? As the passage quoted indicates, and as Helmholtz elaborated elsewhere, the symbolic language must enable different modalities of sense to be linked within it. In addition, while the symbolic language does not provide copies of an external reality, Helmholtz argued that it is neither arbitrary nor independent of input from our sense organs. Principally the task of this symbolic language was to represent relationships between objects affecting one another and our sense organs. The structure of the relationships was the crucial aspect to be grasped in a representation. A copy of an object would in no way guarantee that the mind had grasped the relations between objects and sensations; and in fact a copy might be a very poor representation. A good representation was a symbol useful for organizing the practical activities in terms of which we interact with the external world through our senses:

> The perception of a single, individual table which I carry in me is correct and exact if I am able to deduce from it correctly and exactly which sensations I will have if I bring my eye and my hand into this and that position with respect to the table. What sort of similarity is supposed to exist between the perception and the object which I perceive by means of it, I am incapable of

> comprehending. The former is the mental symbol for the latter. The sort of symbol is not arbitrarily chosen by me, but rather is forced upon me by the nature of my sense organs and the nature of my mind. In this aspect the symbolic language [Zeichensprache] of our perceptions differs from the arbitrarily chosen phonetic and alphabetic symbols of our spoken and written language. A written expression is correct when the person who knows how to read it forms the appropriate perceptions in accordance with it; and the perception of a thing is correct for the person who knows how by means of it to predict which sensual impressions he will receive from the thing when he places himself in certain circumstances with respect to it. Moreover, of what sort these mental signs are is completely irrelevant as long as they form a sufficiently diverse and ordered system; just as it is irrelevant how words of a language sound as long as they are present in sufficient number and there are means for designating their grammatical relations to one another.[5]

In his series of lectures on the recent progress in the theory of vision held in Köln in 1868, Helmholtz refined this discussion of representation by drawing important distinctions between the notions of "sign" and "image." There he noted that in an image the representation "must be of the same kind" as the object represented. He went on to specify what he meant by "the same kind" as follows:

> A statue is an image of a man insofar as its form reproduces his; even if it is executed on a smaller scale, every dimension must be represented in proportion. A picture is an image, or representation of the original, first, because it represents the colors of the latter by similar colors, and secondly, because it represents a part of its relations in space (those which belong to perspective) by corresponding relations in space.
>
> The excitation of the nerves in the brain and the ideas in our consciousness can be considered images of processes in the external world insofar as the former parallel the latter, that is, insofar as they represent the similarity of objects by a similarity of signs and thus represent a lawful order by a lawful order.[6]

In this discussion Helmholtz treated images as relations among signs in which a correspondence of structural relation obtains between representation and the things represented. Such structural correspondence is not given but rather learned and constructed. Crucially for this theory of representation, while ideas or concepts are organized around structural correspondence with external affairs, sensations do not possess this property of structural correspondence with an external referent. They are signs or tokens for certain interactions between external bodies and our sense organs. The color "rot" is the sign of the interaction between the retina and light of 6878 millimeter wavelength. Indeed, the contribution from the side of the retina complicates things further, so that there is not a one-to-one correspondence of sign with external referent; for as we shall see, whether an object is seen as red or not depends on judgments concerning contrast and other factors. The properties attributed to light do not belong to it exclusively, but to the interaction with the eye: "Hence it is really meaningless to talk as if there were properties of light which belong to it absolutely, independent of all other objects, and which we may expect to find exhibited in the sensations of the human eye. The notion of such properties is a contradiction in itself. They cannot possibly exist, and therefore we cannot expect to find any correspondence of our sensations of color with qualities of light."[7] In Helmholtz's view, even at the level of sensation psychological factors involving experience and judgment are intermixed with physiological function and physical interaction with light.[8] Sensations considered as signs, Helmholtz tells us, do not possess the characteristics of constancy desirable in an ideal system of signs. At best, sensory signs have a relative constancy based on use.[9]

Essentially my claim is as follows: Helmholtz's model of representation was that of an abstract system of relations among sense data. Like Bernhard Riemann working independently at almost exactly the same time, Helmholtz treated the mental representation of sensations as n-dimensional manifolds. Different modalities of sense were characterized as manifolds obeying different metric relations. The sense data were organized into symbolic codes by a system of parameters due to the physical properties of each sense and adjusted by experience.[10] Helmholtz's ideas about the sign theory of representation were informed by a tradition of Kantian interpretation tracing its roots to the work of Johann Friedrich Herbart (1776-1841), particularly his *Einleitung in der Philosophie* and his *Psychologie als Wissenschaft neu gegrundet auf Erfahrung, Metaphysik, und Mathematik,* 2 vol. (1824-25). Herbart had been Kant's successor in Königsberg, and his collected works, issued in an excellent new six volume edition in 1850-51 by Gustav Hartenstein, had become part of a new movement "Back to Kant," stressing a "realist" interpretation of Kant's transcendental aesthetic. Herbart's work was taken up and extended by a number of philosophers, mathematicians and physicists in the early 1850s interested in developing a psychophysical approach to relating mental representations to the external world. Helmholtz, I will argue, was deeply sympathetic with this work, and in many ways his own research agenda addressed issues that had been posed within this Herbartian revival.

Helmholtz wasn't the sort of guy to lounge around reading works of philosophy and metaphysics that didn't somehow pertain to his experimental researches. If Helmholtz had not been aware of the ideas on representation circulating in what I've called the Back-to-Kant Herbartian revival, he certainly became aware of them through his encounter with Hermann Grassmann, whose enigmatic *Ausdehnungslehre* was a prize-winning if still unappreciated work in that movement. Helmholtz learned about Grassmann in connection with his work on color mixture. The second aspect of my claim, and perhaps its more substantive part, is that Helmholtz's encounter with Grassmann stimulated his own ideas about representation through a lively dialog with three lines of experimental investigation — color, sound, and electrotelegraphy — familiar to Helmholtz in the period 1850-63, the period spanning his work on the speed of nerve transmission, his work on physiological color mixing, and physiological acoustics, and culminating with the publication of Part II of the *Physiological Optics*. For my purposes, the most innovative character of Helmholtz's work derived from adaptation of a number of interrelated technical devices employed in telegraphy to the measurement of small intervals of time and the graphic recording of temporal events in sensory physiology. From as early as 1850 Helmholtz drew analogies between the electrical telegraph and the processes of perception. The telegraph began to serve as a generalized model for representing sensation and perception. In light of this telegraph analogy, Helmholtz, so I hypothesize, imagined the virtual image cast on the retina as dissolved into a set of electrical impulses, data to be represented by symbols as an "image" in the brain through a perceptual analog of Morse Code.[11] Throughout this period—between 1850-1855—Helmholtz was working intensively with the myograph and a variety of adaptations of electrical devices in the telegraph industry to measure the speed of nerve transmission and other features connected with nerve action and muscle contraction. Telegraphy was not only a useful model for representing and thinking about vision and hearing. Experiments involving those devices were also crucial in advancing his own program of sensory physiology. This role of telegraphic devices and a variety of imaging devices became particularly important in the period between 1855 and 1860. During this period, reacting to a critique of his theory of spectral color mixing by Hermann Grassmann, Helmholtz retracted his earlier (1852) rejection of the Young trichromacy theory of physiological color mixtures. Interestingly, Helmholtz suggests that he arrived at this view via a comparative analysis with hearing.[12] I will pursue this suggestion in depth. Helmholtz pursued a similar research strategy of representing tone production and reception in terms of a variety of components of electrical telegraphic circuitry combined with several techniques for graphic display of wave motion, particularly sound waves. These devices were crucial in his investigation of combination tones, the analog to forming color mixtures from primary colors. Helmholtz postulated retinal structures—three receptors sensitive primarily to wavelengths in the red, green and violet ranges respectively—analogous to the arches of Corti in the ear then provided the path back to accepting the Young trichromatic theory previously rejected. Once again, the new media technologies were crucial in this transition; for Helmholtz drew upon processes of photo-negative production in providing a physiological explanation of positive and negative after-images, crucial to refining the three-receptor hypothesis.

RIGID BODIES, MOTION, AND CAUSALITY: BACK TO HERBART

During the same months that Helmholtz was beginning to formulate the epistemological core of his program in physiological optics, the call to follow Herbart's lead in returning to Kant was moving into high gear. Throughout the late spring and summer of 1850 and into early 1851 Gustav Hartenstein published six volumes of Herbart's collected works.[13] If Helmholtz had not known about Herbart before, which is unlikely, the sensation aroused in Königsberg by the reappearance of the writings of the man who had been Kant's successor in Helmholtz's new home demanded attention, especially by a man who was beginning to fashion himself as the true intellectual heir to Kant. Even a cursory reading of Herbart would have alerted Helmholtz—as indeed it did Bernhard Riemann at about the same time—to the fact that here had been a kindred spirit; for Herbart had given profound consideration to the problems confronting Kant's theory of spatial intuition, and he had cast his own reformulation of the Kantian transcendental aesthetic in a form similar to the requirements of Helmholtz's own purposes.

Herbart wanted to reform elements of Kant's critical philosophy that had encouraged subjective idealism and foreground the realist elements of Kant's thought. A key part of this program was a fundamental revision of Kant's treatment of the relation between phenomena in space and time and the world of so-called "things in themselves." Herbart, like most philosophers of his generation, found Kant's treatment of that relationship enigmatic to say the least. For Kant space and time are the forms of the faculty of human sensuous intuition in which things are given to us as appearances, while causality is a form of judgement for estabishing links within these phenomena. In Kant's approach, therefore, causality only applies to the realm of possible experience, that is, to the realm of phenomena; it can never apply to the (noumenal) world of things in themselves.[14] On the one hand, while causality was supposed to apply only to relations within the realm of possible experience, Kant allowed that the Dinge an sich, which are not objects of possible experience for us, are yet somehow the *causes* of our sensations.[15]

Herbart's attempted solution to this conundrum was to rework the Kantian treatment of causality in order to allow a genuine causal connection between the objective, real world of things in themselves and the phenomenal world of our experience. His approach to this was to synthesize features of Kant's critical philosophy with the system of Leibniz and Wolff. In effect he attempted to retrace much of the same ground Kant himself had originally covered in his "pre-critical" period as he moved toward the construction of his early critical philosophy in the 1780s. As Michael Friedman has shown, from Kant's earliest publication, *Thoughts on the True Estimation of Living Forces* in 1747, through his treatise, *Physical Monadology* in 1756, and even through his *Innaugural Dissertation* of 1770, Kant attempted to refashion the Leibnizean-Wolffian tradition he inherited to harmonize metaphysics with advances in mathematics and Newtonian natural philosophy.[16]

In the Leibnizean-Wolffian system reality consists of an infinity of non-spatial, non-temporal, unextended simple substances or monads. These simple substances or monads do not interact with one another. The evolution of the states of each is completely determined by a purely internal principle of active force, and the appearance of interaction is explained by a pre-established harmony between the states of the diverse substances established originally at the creation by God. Since metaphysical reality is thus essentially non-relational, neither space nor time is metaphysically real: both are ideal phenomena representing the pre-established harmony—the mirroring of the universe from various points of view—between the really non-interacting simple substances.

In his early career Kant tried to revise aspects of this Leibnizean monadology by allowing that while (as in the monadology) reality consists of non-spatial, non-temporal, unextended simple substances; space, time and motion are phenomena derivative from this underlying monadic realm. Kant broke with the Leibnizean monadology by allowing interaction between the monads in terms of active (Newtonian) force where one substance exerts an action on another substance, thereby changing the internal state of the second, rather than Leibnizean notion of active force as an internal and hence non-relational principle. Space for Kant is thus phenomenal and not ideal as it was for the Leibnizean-Wolffian tradition. For Leibniz and Wolff space is ideal because relations between substances are ideal: each substance mirrors the entire universe internally due to its own inner principle, and space is an ideal representation of the underlying order of monads expressed in the pre-established harmony. Since each simple substance already expresses completely the order of the entire universe, nothing but the mere existence of substances is necessary to constitute phenomenal space. But Kant wanted to rid this system of the notion of pre-established harmony and conceptions leading to the ideality of space. For Kant, the committed Newtonian, relations of interaction between substances are in no way ideal. Thus, in his *Physical Monadology* Kant treated space as an external phenomenon derived from or constituted by the underlying non-spacial reality of simple substances (monads).

Kant encountered insurmountable obstacles in carrying this program through, and as Michael Friedman has shown, ultimately abandoned it.[17] The problems centered around Kant's initial conception of the nature of space as grounded in the law of interaction governing the external relations of non-spatial, unextended monads. In particular, it was unclear how to derive properties of space, such as its three dimensionality and infinite divisibility, from the underlying monadic realm. Friedman shows that although Kant did not frame his problem explicitly in terms of the limitations posed by the logical tools at his disposal — essentially those of monadic predicate logic — he nonetheless clearly stated the problem of representation encountered by a monadic predicate logic, and he took counter measures in order to preserve the truth of Euclidean geometry.[18] In particular, Kant knew that the syllogistic (monadic) logic he worked with prevented him from developing proofs that involved generating an infinity of objects, such as Euclidean demonstrations that require an infinity of points. Unable to capture the idea of an infinity of points through an iterative deductive logical procedure alone, Kant represented the idea of infinity intuitively through the iterative process of spatial construction with Euclidean procedures based on straight-edge and compass:

> I cannot represent to myself a line, however small, without drawing it in thought, that is gradually generating all its parts from a point. Only in this way can the intuition be obtained….The mathematics of extension (geometry), together with its axioms, is based upon this successive synthesis of the productive imagination in the generation of figures.[19]

Forced to abandon his original attempt to generate space from interactions among the elements of a non-spatial realm, Kant preserved both the particular character of spatial intuition and the generality of geometric proof by locating them in subjective human cognitive faculties; namely, a pure, *a priori* faculty of representation, on the one hand, and on the other hand a faculty of understanding that provided rules, schemata, or general constructive procedures for the synthesis of objects in senusous intuition. In this fashion Kant restricted the applicability of pure concepts of the understanding to the phenomenal world in space and time, and he excluded all access to an underlying monadic realm, whether that realm be conceived as constituting the phenomenal world or as in some sense the reality behind its appearance.

From a modern perspective, Kant's solution would appear too restrictive. And given a richer set of logical tools, it is possible to address these issues differently. Since the work of Frege, Pasch, Dedekind, Hilbert and others who established modern polyadic logic with universal quantification, existential quantification and quantifier-dependence based on a theory of order, it has been possible to overcome the problems Kant encountered and give an axiomatic characterization of a system of abstract relations from which the properties of Euclidean space can be derived logically. In contrast to Kant this late nineteenth century approach does not need to begin with intrinsically spatial elements. Nor does it need to construct its objects. Rather the elements of this abstract system of relations are completely undetermined, allowing for a distinction between pure and applied geometry, where pure geometry is analytic and based on concepts, while applied geometry is its model or spatial interpretation, a distinction Kant resisted.[20]

Herbart was no better off than Kant in having the logical resources to construct an axiomatic pure geometry, but he was keen on synthesizing certain elements of Leibniz's monadology with Kant's work on space and time. Herbart made two essential proposals. The first was to remove Kant's restriction on the applicability of the category of causality to the phenomenal realm alone. By treating causality as part of the real structure of the intelligible world [the world of things in themselves] rather than just a subjective category, Herbart saw the path toward removing the cleft between subjective and objective space. The notion of a rigid body was key to his solution of the problem.

Herbart's second proposal was to construct an abstract science of relations as a basis for various applied geometries, among which Euclidean geometry would be just one. For Herbart, intelligible space is an abstract science of consistent logical relations. "Geometry assumes space as given; and it makes its constituents, lines and angles, through construction. But for the simple essences (and natural philosophy must be reduced to them in order to find the solid ground of the real) no space is given. It together with all its determinations must be produced. The standpoint of geometry is too low for metaphysics. Metaphysics must first make clear the possibility and the validity of geometry before she can make use of it. This transpires in the construction of intelligible space."[21] In Herbart's view the space assumed by the geometer as given is the space borrowed from the world of the senses, and that is Kant's world of spatial intuitions. The geometry of intelligible space is a higher form of geometry which does not borrow its concepts from our sensuous experience but rather constructs them on the basis of certain elementary notions.[22] It starts from primitive notions such as "position," "between," "inside," "outside," and with rules for various logical relations. The concepts of "straight," and "rigid line" were the key concepts of the investigation.

Herbart's discussion of lines and planes exemplifies his approach and also relates directly to one of the central issues Helmholtz confronted in relating psychological and objective space.[23] I have pointed out above that Helmholtz regarded the key problem of a psychophysical account of vision to be the mechanism by which the input from the neurophysiological apparatus of vision is processed in the sensorium into the richly colored 3D world of visual experience. This last set of steps had to be a psychological process and could not be a simple replication of the physiology, because somewhere along the way physiology had to give way to a system of meaning and intention capable of action. Helmholtz's sign theory was intended to address this problem. But the construction and operation of these signs required an abstract system capable of creating the schematisms that would translate into action on the physiological side of the psycho/physical divide. Herbart's approach to geometry provided the blueprint.

Herbart believed that an abstract science of relations would provide tools for solving two different but analogous problems. On the one hand, as we have seen, he saw it as relevant to solving the fundamental metaphysical problem that had confounded Kant of relating the non-spatial world of intelligible essences to the phenomenal world. But there was an analogous problem in empirical psychology of relating the world of externally caused sense data to the world of our 3D visual experience. Dissenting from the views of some who thought that future physiological research would disclose that the nerves in the retina provide a one-to-one mapping of retinal data along neural pathways eventually terminating in the brain, Herbart argued that even if this nativist thesis should turn out to be true, a great divide persists between sense data and visual experience. Speaking directly to these concerns, Herbart had written that irrespective of the future advances physiology might make in establishing that images of external things are undistorted projections of real objects onto the retina, the entire "image" disolves into an undifferentiated chaos as soon as the perceived object emerges in the sensorium: "The soul must now generate from the ground up the completely destroyed spatial relationships [within the retinal image]. And it has to do this without distorting its perceptions in the slightest....But the perception of something spatial must have a certain similarity to the spatial thing itself, otherwise the perceived object resulting from this act of perception might be anything but something spatial."[24] The mistake of such views was to treat space as something real, and the phenomenal space of empirical psychology as "given" by the wiring of the biological substrate. Herbart, however, was insistent that from the viewpoint of empirical psychology, space is not something real, a single container in which things are placed. Rather it is a tool for symbolizing and representing the various modes of interaction with the world through our senses; and its measure is not given intrinsically, but rather "like all concepts of magnitude, must be considered merely as an aid to thought which has to be bent and shaped in accordance with the nature of the objects to which it is being applied, and never mistakenly conceived as delivering up their real predicates."[25]

In his discussion of lines and planes, Herbart introduced the notion of lines as magnitudes with direction. He used this concept to define a straight line operationally as the shortest path between two points. To get to this result he began by operationally defining, in terms of compound directions, a perpendicular between a line and a point not on it. He then treated any line as the hypotenuse of a triangle; accordingly, such a line could be considered as composed of compound directions by methods similar to constructing a parallelogram of motions.[26] An important consequence Herbart drew from this approach was that in principle there is no limit to the number of spatial dimensions in an abstract "philosophical" geometry. The same procedures deployed in constructing magnitudes of two or three dimensions could in principle be continued in the construction of spatial magnitudes of four, five, or any number of dimensions.[27] But such a construction, while logically possible, would be merely playing with concepts and rules of construction. "Intelligible space just like sensuous space," concluded Herbart, "can only have three dimensions."[28]

Herbart applied the same operational approach for generating other key concepts. Just as he treated straight lines as the path between two points requiring the least expenditure of motion, Herbart defined matter as a collection of points, lines, and surfaces having internal connections which are conserved [*Selbsterhaltung*] in all motions of the body.[29] Herbart dissented from definitions of matter as "collections of atoms," as "cohesion," or as relationships between repulsive and attractive forces giving rise to impenetrability—the sort of strategy found in Kant's *Metaphysical Foundations of Natural Science*.[30] The way we go about determining a rigid body Herbart argued, is by providing a consistent constant rule for preserving the same systematic relationship between points, lines, and surfaces of an object as it undergoes all forms of motion, namely, translations and rotations.[31] Phenomena that do not satisfy this criterion are not rigid bodies or characterize (variable) relationships between more than one body.

Herbart applied these ideas on geometry as one among many applications of an abstract science of relations to the problem of constructing sensory space in his *Psychologie als Wissenschaft, neu gegründet auf Erfahrung, Metaphysik und Mathematik,* originally published in 1824-1825. In order to provide a positive alternative to Kant's notion that space is given in intuition, Herbart, building upon the work of Locke and the associationists, aimed to show that psychological space could be constructed from simple sensations empirically. An essential requirement placed on such a construction was that it had to be capable of accounting for the apparent immediacy of visual experience. The task seemed self-contradictory; somehow visual space had to be empircally constructed in time and yet possess all the characteristics of being immediately given.[32] Herbart's attack on these problems stressed the following points: 1) the strength of associations between different perceptions or ideas should be distinguishable; 2) connections between perceptions often repeated in conjunction should be represented as flowing from one another with the strength of a mechanically determined necessity once a particular temporal threshold of association had been achieved; 3) Once this degree of association between perceptions had been attained a characteristic function was assigned to represent the path connecting them. The function would be retained in memory. When activated by the proper cue, the sequence of steps represented in the function would be run through extremely rapidly; and the more often the function was activated the more rapidly the cascade of associations would be completed, approaching the limits of an immediate perception. After the appropriate learning period, such a mechanism serves as the foundation of our perception of visual space as immediately given.[33]

The assumption that there are certain simple, pure sensations, such as principle colors, pure tones, etc., provided the starting point for Herbart's psychophysics. But in order for these to be perceived they required a contrasting sensation with some sensible difference between them. Already at the level of perception, according to Herbart, there is an active psychological process of comparison and measurement occuring, and it must be constantly reproduced. Thus constant exposure to the tone C without variation would result in its not being heard; staring at a patch of blue while keeping the eyes immobile would lead to its gradual disappearance.[34] Perceptions in this scheme are treated as forces counterbalancing one another in varying degrees of intensity.[35] By describing sensations as forces, Herbart drew a direct analogy to the resistance or pressure felt in the sense of touch, and also to the intensity of the force experienced as pleasure or pain. He treated the sensation itself as the result of a force relation between an external agent and the physiological apparatus which interacts with it. The subject experiences this relationship as what Herbart termed a *Hemmung*.

The treatment of perceptions as forces was essential to Herbart's goal of reducing them to the measure of his new psychophysics. The phenomenal world was to be constructed out of the dyamical and statical relations between elementary sensations and combinations of them in experience. At the center of this new theory of the faculty of sensibility was a general set of descriptive relations forming a mathematical calculus which he called *Reihenformen*, the objective of which was to express the relations between sensations in terms of their degree of intensity, quantity and quality. Herbart's aim was to apply the formalism of this general calculus to each sensory modality. By representing them in terms of a common logical calculus it would be possible to combine sensations of different modalities into a common, unified experience. The same dynamical laws in more or less restricted form governed the construction of each modality of perception and their synthesis into higher ordered unities, which he referred to as "complexions". For example, a single color, such as blue, can vary continuously in intensity and saturation. Without working out the details quantitatively, Herbart argued that such a manifold of sensations could be represented in terms of a linear series of sensations of increasing intensity. Furthermore, according to Herbart any color in the visual spectrum can be constructed from three primary colors, red, blue, yellow.[36] Herbart's scheme was to represent each color in the visual field as a point on a color surface, with one of the primary colors, blue for instance, at the center of coordinates, and units of red and yellow as the abscissas and ordinates.

Herbart did not demonstrate how to carry out the construction of his color surfaces and tone lines in terms of the calculus of *Reihenformen.* The objectives of his program and the rationale he offered for constructing it, however, offered an exciting blueprint for a constructivist theory of sensation. His rationale for arguing that each of the external senses is susceptible of a spatial representation was that each sense, such as hearing, color vision, visual acuity, or the tactile sense are capable of comparative discrimination. Although themselves simple, colors, tones, etc., do not have their own absolute measures; there is no least perceptible unit of "blue", for instance, out of which all other shades of blue are constructed. Thus it is not possible to assign a quantity of blue to "royal blue" which matches some direct physical measurement. Nevertheless, it is possible to assign magnitudes to a manifold of sensations, to construct a psychometric, because of the fact that we make contrasts between sensations. We can discriminate whether one color is brighter or darker than a comparison color of the same hue. Between three pitches,P_1, P_2, P_3, we can say whether the interval (P_1, P_2) is greater than, equal to, or less than (P_1, P_3). The capability of making such contrasts between pairs of sensations in a manifold gives rise to the possibility of providing a psychometric coordination between numbers and sensations and of the possibility of representing the manifold spatially.

Within the modality of color vision, for instance, Herbart suggested that a comparison between three arbitrary colors taken as primary could serve as the basis for constructing a metric in terms of which quantitative relationships between all other colors could be determined. Finally, and most interestingly in my view, when conjoined with certain other conditions (such as requiring that sensory magnitudes obey the associative, commutative and distributive laws of arithmetic) the possibility of assigning a measure to contrasting sensations could be carried further and given a geometrical representation as color spaces, tone lines, and finally visual space. The general calculus of *Reihenformen*, which Herbart conceptualized as differing degrees of restraints between perceptions, themselves treated as oriented magnitudes, meant that by following methods analogous to the combination of motions in the parallelogram of motions which we have discussed above in connection with Herbart's treatment of geometry, each external sense could be given a geometrical representation. Each "sensory-space" would have its own restrictions, of course, determined by the physiological conditions of its functioning. Thus, the fact that, beginning from any one color as origin, any other color could be constructed as a mixture of two other colors meant that the spatial representation of colors would be a closed two-dimensional surface, such as a color triangle [Farbendreieck].[37] The tonal system could be similarly constructed from three variables, pitch, intensity, and timbre. Similarly to the color system, by beginning at any particular tone, it would be possible to arrive at any other tone through a continuous series of changes in the three variables. But, as Herbart noted, there would be important differences. In the case of the color system, for example, it is possible to join the endpoints red and violet in a line via a continuous sequence of purples. This is not the case for the tone system, however. The lowest discernible tone, C_{-2} and the highest discernible tone D_8 are further separated from one another than any two other tones, and there is no continous transition through different tones between them. Hence the system of tones would have to be represented geometrically as a straight line.

Perhaps the most far reaching suggestion in Herbart's work was the notion that the dynamical relations expressed in the *Reihenformen* were a general set of operations performed by the mind in each of its activities of assembling sensations into perceptions and of relating perceptions in concepts—no matter what the sensory modality. An additional powerful idea in this work is the notion that space is not something given, but is rather a tool for symbolizing our different sensory modes of interacting with the world and synthesizing them in a common experience:

> Sensory space, to be exact, is not originally a single space. Rather the eyes, and the sense of feeling or touch independently from one another initiate the production of space; afterward both are melted together [*verschmolzen*] and further developed. We cannot warn often enough against the prejudice that there exists only one space, namely phenomenal space. There exists no such thing as space; but there do exist motivations [*Veranlassungen*] for generating a system of perceptions by fusing them through a network [*Gewebe*] of laws of reproduction, whose perceived object is something spatial, namely for the perceiver. There are numerous motivations for undertaking such constructions, and they are not all equally successful; for many attempts to construct space remain incomplete and in the dark [e.g., in optical illusions, TL].[38]

In elaborating upon this passage, Herbart explained that his objective was to link what he termed different manifolds of sense data:

$$\{ a + b + c + \ldots + n\} + \{A + B + C + \ldots + N\} + \{P + P' + P'' + \ldots + P^{n'} \} + \{\ldots\}$$

Thus, in the case of the construction of binocular visual space, the different manifolds would include data from each retina, such as color and spatial localization relationships, data from the innervations associated with the muscle tensions on each of the six pairs of eye muscles for the corresponding positions of the visual axes, data on accommodation, proprioceptive data from the neck muscles, etc. In the sensorium all of these different "spaces" would be reduced—or "fused" as Herbart expressed it—in terms of the generalized calculus of *Reihenformen* into visual space. In order to accomplish this he developed numerous types of *Reihenformen* which could form secondary series to primary series, "sidechain" series [*Seitenreihen*] which would take as its starting point a term within a series, and various other types of logarithmic series which were intended as structures for linking different series in a network. It is tempting to characterize Herbart's objective as analogous to seeking a master equation which would express the themes, variations, tempos, voices each separately present but harmoniously interwoven in a fuge.

Herbart was seeking a reliable set of operations for establishing that sensory manifolds constituted from data transmitted from the retina to the sensorium correspond in an adequate fashion to the structure of the external object assumed to be producing those sense data. A key requirement he argued was that sensory manifolds must be bound and ordered, and the stability of objects in the world required that at every instant the order among elements in the manifold must be conserved in a constant structure of relations.[39] His proposal for generating a procedural rule was to follow the same course outlined in his treatment of the connection between subjective and intelligible space; namely, to determine whether the perceived object meets the mathematical criteria for a rigid body. Pursuing this strategy, Herbart sketched a rule for carrying out the determination: any perception of a singular object must retain the same ordering of points from one instant to the next, no matter which point is taken as the starting point for determining the interconnections between all other points. If the object is a rigid body then a system of determinations (equations) exists between all points taken pairwise such that given one point, two others are determined; each of these two then determines two others, etc. When the gaze is shifted from a point, a, on the surface—a corner for instance— to another point b, a new set of relation pairs is determined. If the object is a rigid body at rest, then some constant relation ought to exist between the two systems of relation pairs. The warrant of success in the construction was to be provided by an experiment which would establish the conservation of a constant structure of relations during the act of perception. According to Herbart, the eyes are constantly moving back and forth in small excursions over the surface of objects taken to be at rest, checking the constancy of these structural relationships:

> When it wants to grasp an object the eye moves, not in a straight line but back and forth. In every forward motion a set [*Menge*] of reproduction rules is generated; in every return motion they are re-activated through the renewed sight of the object seen previously.[40]

The final determination of whether a set of sense data correspond to a single object, as well as the relation of that object to the viewer and to surrounding objects, is a determination requiring processing of various other forms of information. The information is not just given as fully constituted in the act of vision, as the nativist hypothesis argued; rather it must be sought after and evaluated: "Behind this entire consideration lies a physiological presupposition; namely, that the eye moves in accordance with the impulse [*Antrieb*] given by the act of perception."[41] The eye, therefore, is the tool employed by the sensorium to gather information and check the accuracy of the perceptions it constructs on the basis of the sense data it receives. In order to assemble this data in a self-consistent perception, the sensorium steers the eye in terms of several levels of conceptual input: "There are mainly four: enclosure of shape; the overlay of color against a background; the activity of the eye inside the contours of the field; and what is most important, the motion of the whole ensemble relative to a background."[42] Consider this statement in light of Herbart's interest in the connections between intelligible and phenomenal space—that is our relations to things in themselves on the one hand and our representation of those relations as phenomenal space on the other. The ultimate warrant for assuming a causally determined correspondence between our visual perceptions and the things they represent, the Archimedian lever which would guide the whole visual process giving it unity and direction, would be the criteria provided by the laws of motion applied to rigid bodies.

This was the outline for a bold empiricist program. To transform it from a daring speculation to an empirically fruitful theory required a variety of experimental demonstrations which would make plausible the claim that visual perception is constructed and that various levels of conceptual information are employed in bringing it about. Herbart's work also suggested the path to proceed with this part of the program: If the process of perception is fundamentally always concerned with shaping a perception and cross checking it for consistency in light of various regulative criteria, then each level involves a decision, and that decision can be in error. By exploring the various classes of illusion that occur in visual perception it would be possible to illucidate these conceptual components involved in the construction of space.

To move beyond speculation the program for empirical psychology envisioned by Herbart would require foundations grounded in experiment. It also required a more secure mathematical apparatus linking the laws of thought, the principles of mechanics, and the foundations of geometry. Several persons took up this project independently from one another in the 1850s. Moritz Drobisch, a former student and friend of Herbart, explicitly took up the task of providing more adequate mathematical foundations for Herbart's psychophysics.[43] Bernhard Riemann was stimulated through his study of Herbart's works to pursue the aluring goal of examining the connection between the foundations of mechanics and geometry.[44] In many ways, Herbart's empirical psychology can also be seen as a blueprint for Helmholtz's own researches.[45] For Helmholtz too found himself launched upon a broad-ranging project of revising Kant's transcendental aesthetic. At the center of that project was the establishment of an empiricist theory of the origins of both the geometrical axioms and visual space. These epistemological and physiological researches were themselves related streams of investigation in a network of reflections on the relationship between the principles of mechanics and geometry.

MATHEMATICAL REPRESENTATIONS AND COLOR SPACES

The development of a method for representing the various modalities of sense data within the sensorium was one of the fundamental problems confronting an empiricist theory of the origins of visual space. Herbart, as we have seen, had proposed treating each external sense as a particular type of spatial representation, such as a color space or a tone line. The idea of representing color mixtures and tonal relations spatially was not original with him, of course; it can be found in the works of Kepler, Newton, and J.H. Lambert to name only a few. What was most interesting about Herbart's proposal was that it sought to treat these "spaces" as different applications of an abstract science of geometrical relations. Furthermore, as I have attempted to show, Herbart's conception of how one ought to proceed in constructing an abstract geometry consistent with psychological methods of construction was to focus on the concept of magnitude and to treat lines and points as directed magnitudes.

Herbart was not particularly concerned with mathematics per se but more especially with psychological processes involved in mathematical construction, and he cautioned that his speculations would not satisfy the mathematicians' requirements of rigor. A number of Herbart's colleagues were, however, interested precisely in the problem of constructing an abstract "higher mathematics", including his illustrious Göttingen colleague, Carl Friedrich Gauß. Now Gauß had made many powerful hints about how he might proceed to construct such a higher geometry,[46] and he had actually begun to lay the central groundwork for it in his theory of curvature, but he left the task suggestively unfinished.[47] Furthermore, these mathematical efforts were not motivated by the epistemological and psychological concerns that occupied Herbart, so there was no assurance that, when and if completed, the work of the mathematicians would meet the needs of empirical psychology, or for that matter even support its primary objectives. A further crucial requirement of the empiricist program was that the mathematical structures used in attacking the construction of psychological space had also to be consistent with the physiological mechanisms involved. In my view, Helmholtz and Riemann brought these two strands of investigation together in a fruitful manner.

Gauß himself may not have been directly interested in this enterprise, but several of his friends and colleagues in Göttingen and Leipzig definitely were. In their view Leibniz held the key to this new domain of higher mathematics, sketched tantalizingly in references to what he had called analysis in situ and the logic of characteristic. The fact that Leibniz's monadology and mathematics were central to Herbart made Leibniz's views on this subject seem all the more relevant to their enterprise. Leibniz's brief but provocative comments on this subject became available for the first time through the publication of his correspondence with Christiaan Huygens, which appeared in the Ulyenbroek edition of Huygens's works in 1833.[48] Leibniz claimed to have discovered a new system "entirely entirely different from algebra and which has great advantages in representing to the mind, exactly and in a way faithful to its nature, even without figures, everything which depends on sense perception." [49] Whereas algebra expresses undetermined relationships between magnitudes and numbers, Lebniz's new logic of characteristic would also express situation, angles, and motion. The new system would combine the virtues of algebra and pictorial geometry, and with it one could mathematically treat mechanics as well as geometry. Leibniz noted that a chief value of his logic of characteristic would be to enable conclusions to be drawn by operations with its characters "which could not be expressed in figures, and still less in models, without multiplying these too greatly or without confusing them with too many points and lines in the course of the many futile attempts one is forced to make."[50] Leibniz's new calculus would provide methods of abridgment to lighten the burden of imagination in geometry and mechanics.

The history of attempts to reconstruct Leibniz's logic of characteristic is typically read as an episode in the history of mathematics and logic, particularly in connection with the history of vector analysis. But for my purposes here, it is also interesting to read it in the context of problems occupying mathematically oriented physiological psychologists inspired by Herbart, particularly Helmholtz. I have argued above that in his empirical psychology Herbart sought a generalized formal calculus that could be applied to external sense data rendering them as different forms of spatial representation. Moreover, I have argued that Herbart required—but never provided—methods for operationalizing the constuction of sensory space without resorting to the Kantian theory that space is simply given in intuition. The key site for Herbart was the productive imagination and its connection to the "higher geometry" articulated as a generalized science of relations. For visual space the notion a rigid body and forms for quickly calculating identical mappings of sense data through translations and rotations were crucial. In this context Leibniz's logic of characteristic, by linking algebra, pictorial geometry and mechanics would, I hypothesize, serve as an extremely suggestive stimulus to working out a concrete theory of how the mind interacts with the world through the construction of space.

Reconstructing Leibniz's logic of characteristic became a hot issue among several of Herbart's followers in the mid-1840s. Moritz Drobisch and August Ferdinand Möbius, both from Leipzig, took an interest in the problem in 1844. In early 1844, Drobisch proposed to the Fürstlich Jablonskischen Gesellschaft der Wissenschaften zu Leipzig the formulation of a prize question designed to spark interest in reconstructing Leibniz's analysis in situ.[51] The proposal was accepted and the prize question was published in the *Leipziger Zeitung* on 9 March, 1844:

> There are a few fragments remaining of a geometrical characteristic which Leibniz invented (see *Christi. Hugenii aliorumque seculi XVII. virorum celebrium exercitationes mathematicae et philosophicae.* Ed. Uylenbroek. Hagae comitum 1833. fasc. II, p. 6), in which, without taking recourse to the magnitudes of lines and angles, the relative situation of position is immediately represented by means of simple symbols and determined through connections between them, and therefore differs completely from our algebraic and analytical geometry. The question is whether this calculus can be reconstructed and further developed, or whether a calculus similar to it can be provided, which by no means seems impossible (cf. *Göttinger gelehrte Anzeigen*, 1834, p. 1940).[52]

There was no winner in 1845, and the prize question was repeated in 1846, this time doubling the value of the prize from 24 to 48 gold ducats. Shortly after the original prize question appeared in 1844, Hermann Grassmann published his *Ausdehnungslehre*, and he sent a copy of the work to Möbius, who had himself been working on a similar system of geometrical analysis, which he called the barycentric calculus. Möbius found the work interesting but not altogether felicitous in its presentation. The novelty of his methods, the peculiarity of his terminology, and the difficulties in his style of argumentation were persistent obstacles to the reception Grassmann's work among mathematicians such as Möbius.[53] Nevertheless, when the prize question was reissued in 1845, Möbius wrote to Grassmann, encouraging him to submit an entry. Grassmann's treatise, *Geometrische Analyse geknüpft an die von Leibniz erfundene geometrische Charakteristik*, Leipzig; Weidmann, 1847, was awarded the prize by a committee consisting, among others, of Wilhelm Weber, Ernst Heinrich Weber, Gustav Fechner, Möbius, and Drobisch, and the prize was given to Grassmann on July 1, 1846 by Drobisch at the opening ceremonies of the newly founded Königliche Gesellschaft der Wissenschaften zu Leipzig.

In spite of Grassmann's success the full merits of his work were not appreciated. In private correspondence in which they exchanged their evaluations of Grassmann's prize entry as well as the *Ausdehnungslehre*, Möbius and Drobisch both resisted the conclusion that he had fully succeeded in establishing the sort of mathematics Leibniz had in mind, but they did regard his approach as an outstanding first attempt nonetheless. Möbius, in particular, complained about the abstract constructivist character of the work. Möbius wanted to see the calculus worked out and applied to problems with more content than Grassmann had yet attempted. But his main criticism concerned "...the foreign manner in which the author attempts to ground his methods of calculation. The foundation of geometry is intuition [*Anschauung*]; and even if analytical geometry has often been criticized for loosing sight of this, nonetheless, the formulae from which the calculator begins are founded in geometrical intuition and he remains conscious of the fact that the symbols with which he calculates signify lines, surfaces, or bodies. Things are quite different in the present [Grassmann's] work, where in accordance with a certain analogy to arithmetic, objects are treated as magnitudes, which in themselves are not magnitudes at all and which are totaly incapable of being imagined."[54] Möbius was by no means clear about where he stood on the issue of Anschaulichkeit in relation to higher geometry, however. In 1846, for example, he had received a letter from, Ernst Freiedrich Apelt, then professor of philosophy in Jena, and a strict Kantian of the old school, asking Möbius if he had read Grassmann's *Ausdehnungslehre*: "It seems to me that a false philosophy of mathematics lies at the basis of it. *Anschaulichkeit*, the essential character of mathematical knowledge, appears to be banished in this approach. Such an abstract science of extensions as he attempts can only be derived from concepts. But the source of mathematical knowledge lies not in concepts; it is in intuition."[55] Möbius responded that until recently he had not studied Grassmann's book carefully, because "...as you have remarked yourself, the book rejects the essential character of mathematical knowledge, which is *Anschaulichkeit*. In glancing through the work recently, however, I have been struck by many things, on the expansion of concepts, on generalizations, or whatever you want to call them, which could be extremely influential for mathematics itself and for the systematic presentation of its elements in particular. In this context belongs the addition and multiplication of lines, if one considers not just their length but also their directions."[56]

Grassmann did not discuss at length the relationship between his mathematics and the philosophical issues surrounding the discussion of Kant's transcendental aesthetic, but his work was seen by his contemporaries as taking sides in that debate. On which side he stood was made clear in the forward and introduction to the *Ausdehnungslehre*. Like Leibniz's references to the goals of the logic of characteristic, Grassmann noted that his aim was to construct a system of pure forms of thought [*Denkformen*] abstracted from the contents of any particular mathematical discipline, such as Euclidean geometry. Geometry and mechanics, he noted were empirical disciplines based in an intuition of an objective reality, namely physical space, time, and motion. The princples of mechanics and geometry could not be deduced from pure thought; nor could they be derived from experience. Rather they were based in what he called a *Grundanschauung* in which the forms of pure thought were applied [*angewendet*] to what is given in intuition. Grassmann's goal was to develop forms of symbolic representation for creating mathematical concepts and theorems that embody the operations of the mind. He thought this would facilitate the development of empirical disciplines. Perhaps equally importantly his new form of analysis would collapse the distinction between the analytic and synthetic treatments of geometry.

Helmholtz's First Analysis of Spectral Colors and his Encounter with Grassmann

Helmholtz's controversy with Hermann Grassmann on the proper representation of spectral colors was crucial to the development of Helmholtz's views on mental representations and their physiological correlates. Helmholtz first encountered Grassmann's work in 1852.[57] As part of the procedure connected with his appointment at Königsberg, Helmholtz chose as the subject for his Habilatationschrift a critique of David Brewster's theory of color, which was based on the view that the spectral colors are mixtures of three elementary colors, red, yellow and blue. Grassmann found Helmholtz's paper wanting in certain respects, but his primary interest lay in the opportunity it offered for overcoming the steadfast resistance to his ideas, partially outlined above, by illustrating once again the general applicability of the methods of the *Ausdehnungslehre*. The exchange through papers in *Poggendorff's Annalen* between 1852-1855 not only led Helmholtz to revise his early ideas but brought him into direct contact with some exciting mathematical concepts for solving the problem of spatial representation in physiological optics.[58]

Of particular interest was Grassmann's development of the concept of a manifold, which consisted of systems of elements. Leaving the concept of "element" unspecified in content, Grassmann defined extensions as the collection of elements produced by continuous change of an original generating element. Different levels [*Stuffen*] of elements and extensions were generated by specifying a different rule of change [*Änderungsweise*] for each level. An m-level system [*System m-ter Stuffe*] of elements was defined in terms of m independent "modes of change." By interpreting elements as points and modes of change as directional coordinates, Grassmann defined Euclidean space as a geometrical interpretation of a 3-level system. Thus, he described points in a line as a 1-level system; a 2-level system would be interpretable as the elements of a plane, etc.[59] Interpreted geometrically, a line segment was a system of elements generated by motion in the same direction; parallel lines were two systems of elements with the same directional coordinates, etc. Having defined line segments as directed magnitudes, Grassmann interpreted addition geometrically as the resultant of joining the first and last elements in the collection of elements from which an extension is generated:

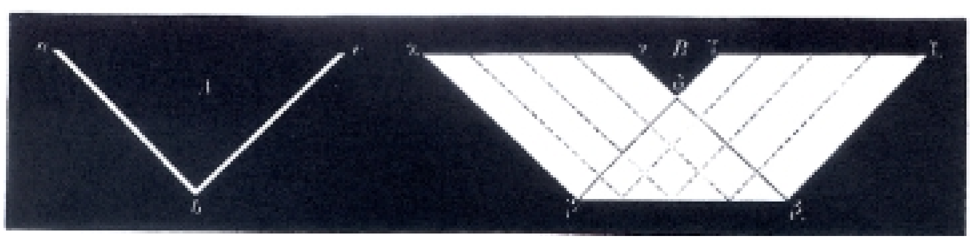
> ...Since the same laws for conjoining the m original modes of change also obey the laws of addition and subtraction, we can summarize the results of the preceding discussion in the following extremely simple statement:
> If [ab] and [bc] represent arbitrary modes of change, then
>
> [ac] = [ab] + [bc].

In a footnote to this passage accompanied by an illustration of the parallelogram rule for addition, Grassmann wrote:

> I cannot recommend strongly enough that one represent this result, which is one of the most difficult to grasp in this entire science, by means of the following geometrical construction.[60]

The parallelogram rule for combining forces thus served as the basis for the most fundamental conceptual operation in Grassmann's new science of abstract forms. He obviously sensed that unless he could convince mathematicians and physicists of the power of this elementary aspect of the new methods of representation he was developing, the entire mathematical edifice he was constructing would remain unappreciated. The critique of Helmholtz's paper on color mixtures offered an occasion to illustrate this point to an audience that had failed to take notice of the *Ausdehnungslehre* for nearly a decade.

Helmholtz's papers on color mixtures featured the adaptation and refinement of existing instruments and experimental practice which characterized all his early papers and aroused the admiration of his contemporaries.[61] In his first paper of 1852 the arrangement for mixing pure spectral colors consisted of two slits forming a V in a black blind.[62] The slits were inclined by 45° to the horizontal and were at right angles to one another. Different spectral colors passed through each slit were combined at the point of intersection. To generate the possible combinations of these two colors, a flintglass prism was placed vertically in front of the objective lens of a telescope, which was focused on the intersection of the slits, both prism and telescope at a distance of twelve feet from the V-slit blind. Helmholtz noted that a similar arrangement with a single vertical slit instead of a V-slit produces a rectangular spectrum in which the different color bands and Frauenhofer lines run vertically parallel to one another. The spectrum of an inclined slit is a parallelogram with two parallel horizontal sides and two sides parallel to the slit. In this inclined situation, the color bands and Frauenhofer lines run parallel in the direction of the slit. In the case of Helmholtz's V-slit the two spectra overlapped, with the two sets of color bands and Frauenhofer lines running in the directions of the slits. When viewed through the telescope, the area of overlap of the two spectra was a triangle, and within the triangle all the combination colors resulting from the mixture were visible. Beyond the edges of the triangle in the remaining portion of each parallelogram the spectral color admitted through each slit was visible. Helmholtz fixed cross-hair lines in his telescope which he oriented at 45° so that they ran parallel to the Frauenhofer lines. This enabled him to estimate the relative distances from the dark lines in the spectra of the color bands entering the specific mixture. In this first approach to the problem, Helmholtz did not attempt to provide a quantitative determination of the wavelengths of his color mixtures. In order to compare the relative intensities of the two colors entering a particular mixture, Helmholtz noted that by rotating the prism about the axis of the telescope the surface area of the illuminated parallelogram changed, being greatest when the slit and prism were parallel. In that position the illuminated area was a rectangle. By rotating the prism relative to the slit, therefore, the same quantity of light would illuminate a larger or smaller surface area, and appear correspondingly less or more intense. In the original position of the V-slit, the intensities of the two spectral colors were equal. By rotating the prism all combinations of relative intensities of the two spectra could be achieved.



[Figure 1: Helmholtz's V-Slit for Color Mixtures]

Using this experimental design Helmholtz arrived at several remarkable conclusions. The first of these was that color mixtures formed from pigments or powders differ markedly from color mixtures formed from pure spectral colors. In contrast to the experience of painters for a thousand years, Helmholtz wrote, the mixture of blue and yellow spectra, for example, does not yield green but rather a greenish shade of white.[63] The explanation, according to Helmholtz, is that the combination of spectral colors and the mixing of pigments rest on two different physical processes.[64] In the case of colored pigments and powders a portion of the light falling upon a colored body is reflected back as white light while of the remaining light which penetrates the body one portion is irregularly absorbed while another is reflected from the back surface and is taken by the observer as the color of the body. What transpires in the mixture of pigments is, Helmholtz concluded, the loss of different rays of colored light rather than the combination of the colors.

One of the aspects of the paper that attracted Grassmann's attention was the range of results achieved from mixing pairs of spectral colors. "The most striking of these results," Helmholtz wrote, " that among the colors of the spectrum only two combine to produce white, being therefore complementary colors. These are yellow and indigo blue, two colors which were previously almost always thought of as producing green."[65] It was because previous investigators had used mixtures of pigments rather than basing their theories on the mixture of pure spectral colors that this incredible error had been propagated and reinforced, according to Helmholtz.

Another surprising result of Helmholtz's investigation was his rejection of Thomas Young's hypothesis that all colors of the spectrum can be generated from three primary colors.[66] Helmholtz concluded by contrast that the least number of colors out of which the entire spectrum could be generated was five. He arrived at this result by trying to construct a color circle. As the best method of construction he favored Newton's procedure of producing each simple color by combining it from the neighboring colors on either side, but he restricted Newton's approach even further by adding that the distance between the two combining colors should not be too great. Otherwise, he said, the resulting intermediate shades would not match those of the spectrum. Proceeding in this manner, Helmholtz concluded that the minimal list of colors required to imitate the spectrum was red, yellow, green, blue, and violet.

Grassmann took issue with Helmholtz's claim that there is only one pair of complementary colors. Contrary to this assertion, Grassmann set out to show that Newton's view was indeed correct: Every color has a complement with which it combines to produce white light. Grassmann's demonstration of this claim was remarkable for the fact that he built his mathematical analysis directly upon certain structural features of the perception of color. He thus examined the purely phenomenal, mental side of color relations; here was exactly the analog of the problem of generating the spatial components of visual experience from the phenomena. First Grassmann observed that any compound color can be imitated by mixing a homogeneous color of a particular intensity with white light of a particular intensity. This observation was the basis for his central assumption; namely, that the eye is capable of making three types of comparison. It distinguishes hue, the intensity of the color, and the intensity of the white light mixed with the homogeneous color producing the sensation, i.e., the degree of saturation.[67]

To establish the proposition concerning complementary colors, Grassmann assumed as well-grounded in experience that (1) if one of the colors entering a mixture remains constant while the other undergoes a continuous change either in intensity, saturation, or hue, then the resulting mixture also changes color continuously. As a final preparatory step, he observed that (2) by a continuous transition in wavelength it is possible to traverse the entire series of color tones in the spectral series. To this it was also necessary to add that it is also possible to traverse a continuous series of purples from violet to red, thereby making it possible to complete an entire circuit of color perceptions. For his analysis Grassmann assumed that the transition from red, to orange, yellow, green, blue, violet, purple and back to red is positive, the reverse series negative. Three types of transition between two colors A and B were therefore possible: namely, either the hue of the color A takes on all the hues in the positive transition series between A and B; or it takes on all the hues in the negative series; or finally that the color undergoes one or more transitions through white light. With assumptions (1) and (2) Grassmann demonstrated the proposition that for every color there exists another homogeneous color with which it combines to produce white light.

The "proof" must have seemed strange to an experimentally oriented empiricist such as Helmholtz, for it proceeded as a reductio ad absurdam. Instead of experimentally demonstrating that every color has its complementary, Grassmann proceeded in an abstract fashion by attempting to show that if this were not the case our concept of continuity and our experience of the closure in the continuous transition of colors would be violated. Grassmann started with a given color of hue **a**, and he assumed that no other color could be mixed with it to form white. Next he chose an arbitrary color with color tone **x** and intensity **y**. He mixed the two colors keeping the color tone, **x**, constant while continuously varying the intensity, **y**. According to assumption (1), the result of mixing the two colors will be a continuously changing color, and a point will be reached where the intensity of the color a will vanish relative to the intensity **y**. Now one way in which the mixture of the two colors could change would be for them to merge into white. But since it was assumed at the outset that the mixture of the two colors could not produce white, the only other alternative is that the color of the mixture will have to change continuously in terms of the other variable, **x**; so that a will gradually merge into **x**. The transition of hues can proceed either positively or negatively depending on the hue **x**, but at some point a transition must occur in the series of mixtures from which point on **x** predominates. Since no transition between the two colors could take place through a band of white, it was required to establish the possibility of an abrupt transition between the two hues **x** and not-**x** (a). The proof strategy Grassmann pursued was to show that the assumption that the transition point was any color other than white led to a contradiction.

In order to show that his line of thinking was consistent with what was known experimentally to be the case, Grassmann proceeded to construct a color chart. He first showed that Newton's determinations of the beginning of each band of spectral color in terms of the mean determinations of the refractions and dispersion for his prism could be coordinated precisely with the Frauenhofer lines. To carry these considerations over into the construction of a color chart, Grassmann assumed in particular that Newton's red and violet fall together with the Frauenhofer **B** and **H** lines. This assumption determined that the colors would be represented as a circle. The ratios between Newton's color bands and the Frauenhofer lines permitted a placement of colors on the circle. For example, the transition line between red and orange lay between the **C** and **D** lines in the ratio of 7:6.

This placement compared well with Helmholtz's results. Helmholtz had found that mixtures of red with orange, yellow, and green produce the intermediate colors in the series; red mixed with green, for instance, gives a dull yellow, which by increased red goes over into orange and finally red. Similarly red mixed with violet, indigo, and blue gives all the intermediate colors. In particular, Grassmann declared, "Red mixed with blue produces a whitish violet, which goes over into rose and carmine as red begins to dominate the mixture. According to the theorem proved above the complementary color to red must lie between green and blue, therefore some tone of blue-green."[68] Helmholtz had noted that red mixed with the blue-green tones produces a series of flesh colored tones, but he had not described how it was possible for this transition to occur. Grassmann continued, "There is a gap here, therefore. Moreover, flesh tones are nothing other than red mixed with a great quantity of white, and no possible transition is imaginable other than that the red is diminished until it disappears relative to the white in the mixture, and then out of this white (or gray) the blue-green tones gradually emerge. In short, the usual transition through white occurs here."[69]

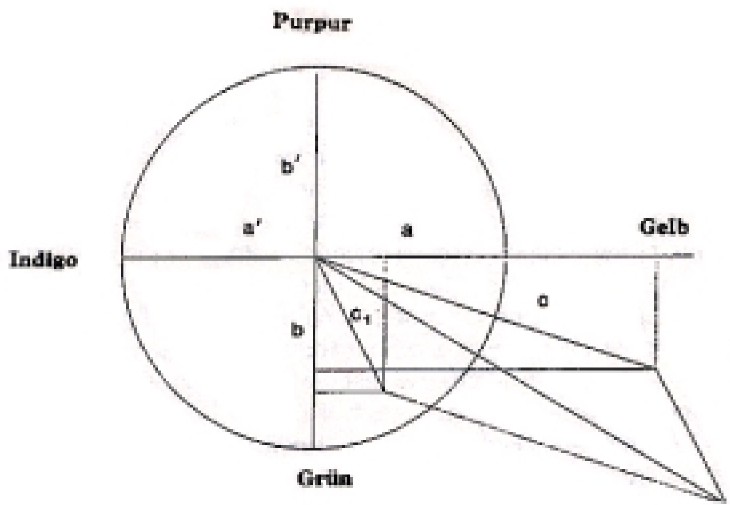
Grassmann concluded the paper with a discussion of the rule for combining colors. First he established a coordinate frame. He took an arbitrary homogeneous color a and its complement a', placing them in a line at equal and opposite directions from one another. Next he chose a color, **b**, which gave the same quantity of white when mixed with a and with **a'**. Its complement, **b'**, was chosen similarly, and the intensities of **b** and **b'** were chosen such that the intensity of the white produced from mixing them was equal to the intensity of the white produced from **a** and **a'**. Grassmann represented these conditions in terms of a right coordinate system with **aa'** bisected by **bb'**. In practical terms, if **a** is yellow, then **a'** is indigo, **b** a shade of green, and **b'** a purple. From mixtures of two of these four colors any other colors could be constructed as the geometrical sum using his parallelogram rule:

> Once this has been done, one can find the hue and color intensity for any mixture of colors through construction. One needs only to determine the extensions which represent the hue and color intensity of the colors being mixed and then to sum these geometrically, that is to say, combine them like forces. The geometrical sum (the resultant of those forces) represents the hue and intensity of the mixture. It follows immediately from this that the order in which one adds geometrically (combines the forces) is of no consequence for the result.[70]

To illustrate the procedure Grassmann constructed the arbitrary mixture of two colors $c + c_1$ in the green-blue quadrant of his color circle. c could be represented as the diagonal of the parallelogram with sides (βa and Φb, and c_1 could be represented as the diagonal of the parallelogram with sides β_1a and Φ_1b Thus the mixture of the two colors could be represented as:

$$(\beta \mathbf{a} + \Phi \mathbf{b}) + (\beta_1 \mathbf{a} + \Phi_1 \mathbf{b}) = (\beta + \beta_1)\mathbf{a} + (\Phi + \Phi_1)\mathbf{b}$$

This was the geometrical sum of the directed magnitudes representing the two colors entering into the mixture.



[Figure 2: Grassman's Method for Representing Color Mixtures]

In a footnote to this discussion, Grassmann indicated that a full exposition of the theory of geometrical sums was to be found in his *Ausdehnungslehre* of 1844 as well as in Mobius' *Mechanik des Himmels* (Leipzig, 1843).

Grassmann's discussion of color mixtures led Helmholtz to rethink his own approach to the subject. From Helmholtz's point of view there were several interesting features about the paper, but at the same time it contained several loose ends. For one thing, it would be necessary to re-examine the treatment he had given of complementary colors. The inadequacy of the instrumentation seemed to be the primary weak point in his work. Already in his first paper of 1852 Helmholtz had noted that a more refined instrumental arrangement for projecting the color mixtures onto a larger surface area and an improved method for measuring the distance of the color mixture from the nearest Frauenhofer line might lead to different results concerning the composition of the whitish hues.[71] But Grassmann's argument was totally inadequate. Among the community of measurement physicists Helmholtz respected, it was insufficient to establish a claim from some sort of abstract mathematical argument without also demonstrating the result empirically. The production of white light from complementary spectral colors had to be empirically demonstrated. Furthermore, Grassmann's reductio argument depended upon assumptions concerning the discriminating power of the eye as a measuring device. These were extremely interesting physiological assumptions. But having made them, Grassmann left the plane of physiological argumentation altogether. Now the mathematical methods he applied as consistent with his assumptions concerning the different parameters of sensitivity governing the perception of color had led Grassmann to the result that four colors would be required to perform the parallelogram construction. Yet even in Grassmann's ingenious construction, one of the colors, purple, was itself a mixture of red and violet. Perhaps three colors would indeed suffice. By wedding himself to a circle as the method for representing color spaces instead of a triangle, Grassmann had introduced some unnecessary assumptions. Furthermore, the representation in terms of a triangle would correspond more adequately to the physiological character of the argument. This too needed to be checked empirically. In any case, the entire issue of Young's theory was reopened and the theory of color mixtures given a decisively physiological dimension as a result of the encounter with Grassmann.

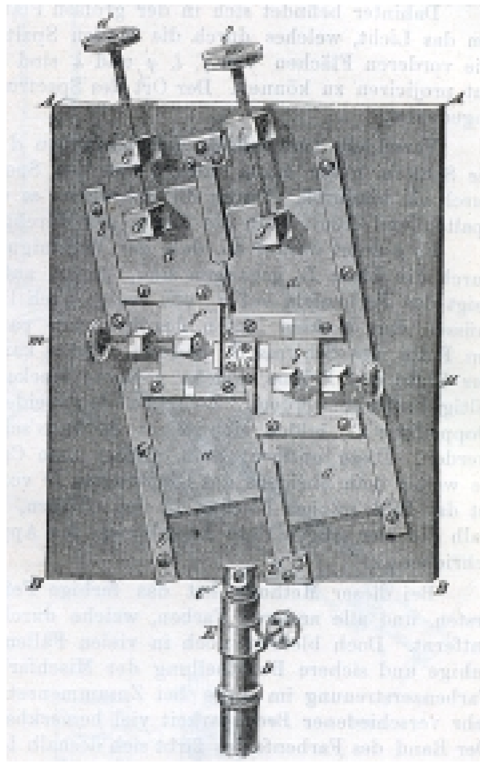
Helmholtz's Reconsideration of Color Mixtures

Helmholtz took up these issues in a paper entitled "Über die Zusammensetzung von Spectralfarben," in *Poggendorff's Annalen der Physik and Chemie*, in 1855. The data and conclusions of that paper were repeated in much expanded form in sections 19 thru 21 of the second part of the *Physiologische Optik*, published in 1860, and in what follows my discussion will refer to the treatment in both texts.

In the new approach to color mixtures Helmholtz refined each component of his apparatus for viewing and mixing spectral colors. He took his lead in this instance from the procedures developed by Foucault in a paper translated in *Poggendorffs Annalen* in 1853. First Helmholtz showed that in order to obtain the purest spectra it was necessary to use light from the sun or from the sky in the immediate vicinity of the sun. To this end he fixed a mirror on a heliostat in order to compensate for the sun's motion and to insure light of constant purity. The light from the heliostat was reflected through a slit in a blind and then through a vertically positioned prism. Light emerging from the prism passed through an objective lens, then through another rectangular slit in a diaphragm, **D**, and projected onto a second diaphragm, $\mathbf{S_1}$, placed at the focal length of the lens.

[Figure 3: Helmholtz's Experimental Arrangement for Color Mixtures]

In place of the V-slit arrangement used in his first set of experiments, Helmholtz used two parallel Gravesande slits capable of refined adjustment, making it possible to align the slits on any two spectral colors. The size and position of the slit was controlled by two pairs of sliding plates mounted on tracks, one pair of which moved in direction **bb** inclined to the horizontal and a second, **fΦ**, adjustable in a horizontal direction. By adjusting the screws **d** and **Δ** the two slits could be given any distance relative to one another within the range of the spectrum projected onto the diaphragm. The spectrum itself was projected onto a silvered rectangle in the center of the diaphragm.



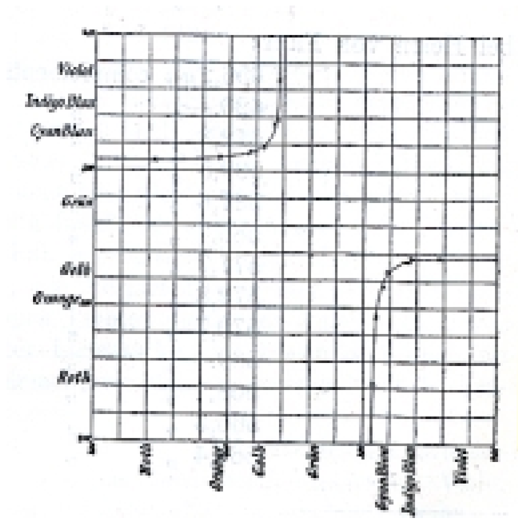
[Figure 4: Helmholtz's Adjustable Slit Diaphragm]

In his earlier experiments Helmholtz controlled the quantity of light entering the mixture by rotating the prism in front of the V-slit, and hence changing the surface area of the parallelograms projected to the telescope. This simple arrangement had worked beautifully for obtaining most mixtures, but it led to difficulties in the whiter hues. First because even when viewed through a telescope, the individual colors occupied a very small area, and, second, because the difficulty of discriminating the hue was increased by the number of other colors in the field of vision. This had been the source of the error in determining complementary colors. In the new apparatus the quantity of light was controlled directly by adjusting the width of the slit, by turning the screws **m** and **μ.** Hence both hue and intensity could be controlled more accurately in this arrangement.

After emerging from the slits the two bands of color now passed through a second objective lens, $\mathbf{L_2}$, of shorter focal length than $\mathbf{L_1}$, and were projected onto a white paper as a uniformly colored rectangle. By covering the slits one at a time the component colors of the mixture could be seen separately.

With this improved instrumentation Helmholtz determined anew the series of complementary spectral colors. He now was able to produce white from violet and greenish yellow, indigo blue and yellow, cyan blue and orange, greenish blue and red. He was, however, unable to produce white from mixing green with any other simple color, but only from mixing it with purple, "that is with at least two other colors, red and violet."[72]

In the next stage of his investigation Helmholtz determined the wavelengths of the complementary colors. He combined two spectral colors to produce white light and then removed the white screen $\mathbf{S_2}$ in the previous setup. About six feet away he placed a telescope and in front of its objective he positioned a glass containing a vertical thin wire diffraction grating and a horizontal millimeter scale to determine wavelengths. Helmholtz graphed the results in Figure 5 below.



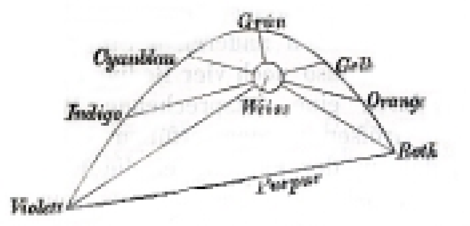
[Figure 5: Graph Relating Complementary Colors in Terms of Wavelengths]

From the graph it was immediately obvious why his V-slit arrangement could not have been expected to reveal colors complementary to red and violet: the transitions between the blue-green color bands proceed extremely rapidly, being represented nearly as a vertical line in the graph, and these colors form extremely narrow bands difficult to detect in that arrangement. Indigo blue and yellow, on the other hand, had the advantage of being relatively wide bands of color.

These results also had certain consequences for the geometrical representation of the color table. Helmholtz praised Newton's notion to represent the colors in a plane by means of the center of gravity method as one of the most ingenious of all his creative ideas.[73] But Newton himself had proposed the rule as a mere aid for summarizing the phenomena in a qualitative manner and had not defended its correctness as a quantitative explanation. Grassmann's contribution had been primarily to call attention to the mathematical assumptions underlying the center of gravity method, and Helmholtz was convinced by Grassmann's treatment that this was indeed the appropriate quantitative method to use. But he was not convinced that the color table should be represented as a circle. Grassmann's analysis simply duplicated Newton's assumptions. But in fact, the center of gravity method for mixing colors was compatible with many geometrical representations.[74] In order to determine which of those representations fit the causal picture most closely, it was necessary to interpret the parameters in Grassmann's model in terms of empirical measurements. Subjected to this requirement, the choice of a circle turned out no longer adequately to fit the refined data Helmholtz had derived. In terms of the center of gravity method for determining color mixtures, white can only be placed in the center of the line connecting the two colors when the "weights" on the ends are equal. In his treatment of this problem, Grassmann had assumed that complementary colors have equal intensities, which implied they should be placed on the circumference of a circle. This might seem like a natural assumption, since we tend to treat pure spectral colors as the most saturated colors, from which it would seem to follow that they ought to be placed on a scale at equal distances from white. But as Helmholtz demonstrated in a related set of experiments, that assumption can easily be shown to be false. Different degrees of color saturation had to be assigned to the colors entering a mixture of white light.[75]

The diaphragm and slit apparatus Helmholtz constructed allowed him to derive an approximate measure of the relative quantities of light entering the mixture of white, and from these measurements to determine the shape of the color space. By adjusting the screws **m** and **μ** (See Figure 4) the width of the slits and hence the quantity of light entering the mixture could be varied. After mixing white from a pair of complementary colors, Helmholtz measured the width of the slit of the brightest of the two colors with a micrometer. He then diminished the width of that slit until the two shadows cast by a wooden dowel appeared equally bright, and he measured the width of the slit once again. The ratio of the two widths of the slit was an approximation to the relative brightness of the two colors.

According to the physiological assumptions underlying Grassmann's construction of color space, quantities of colored light taken to be equal should be those quantities which, having a certain absolute intensity, *appear to the eye* as equally bright. In order to satisfy these requirements, the lengths of the lever-arms in Grassmann's construction had to be adjusted in accordance with the slit measurements discussed above. The result was a completely different shape for the color space. Instead of a circle or a circular lumen, it now turned out to be a hyperbola-like curve with violet, green, and red at the vertices.



[Figure 6: Helmholtz's Color Space for Objective (Spectral) Colors]

Grassmann had stimulated Helmholtz to revise fundamentally his approach to the theory of subjective colors. From Grassmann Helmholtz had also learned the power of abstract, structural mathematical approaches to these problems. But while acknowledging that Helmholtz had profited from the interaction, we should also not overlook the object lesson Helmholtz was prepared to give Grassmann in relating abstract mathematical structures to the requirements of physics and physiology. A point Helmholtz would state explicitly in the next stage of his researches in sensory physiology was that to be meaningful in reference to a physical problem, an abstract structure had to be embedded in measurements, and the internal logic of the abstract system had to be adapted to the requirements of the physical problem. That process of adaptation was achieved through a dialogue with the instruments. The result of the dialogue was a model of the physical system, in this case the physiological apparatus of the eye responsible for the production of color sensations. In his reference to Newton's approach, Helmholtz made clear that the proper mathematical description of a system was not one that served to represent usefully the phenomena but rather one which corresponded to the causal properties of the physical system. Helmholtz's color space was such a model of the physiological apparatus employed by the eye in measuring color. For indeed the causal structure consistent with the model represented in the color space — a causal structure which immediately leapt from the representation — was the production of color sensations from the excitation of three receptors sensitive to violet, red, and green.

Music to the Eye: Physiological Acoustics, Visualization Devices, and the Receptor Hypothesis

It is tempting to assume that Helmholtz reversed his stand on Young's theory of color vision in the course of writing up the new experiments central to the 1855 paper. However, the original paper says nothing about the three-receptor theory. Indeed, Helmholtz stated in a footnote for the version of the paper included in his *Wissenschaftliche Abhandlungen* that his first recorded support of the Young Hypothesis appeared in the second part of the *Physiological Optics*, published in 1862.[76] Nothing about the encounter with Grassmann would have forced a reversal of the position on the physiological bases of colors. The dispute with Grassmann had resulted in confirming Newton's center-of-gravity method for determining color mixtures but led to the conclusion that the form in which the color chart should be represented geometrically was a shape approaching a conic section rather than a circle as both Newton and Grassmann had assumed. Nothing concrete had been determined about the physiological causes of color mixtures. Helmholtz had shown that any three colors would suffice to generate the color chart, but those methods did not specify which three colors must be associated with the color receptors in the retina.

Helmholtz had established, furthermore, that the center-of-gravity method for representing color combinations was indeed useful, but he wanted to go further and establish that its underlying principles embodied a more general calculating apparatus for the representation of sensations; that is, he wanted to establish it as a psychophysical principle as well. What led Helmholtz to change his mind about the Young Hypothesis between 1855 and 1862? How did he arrive at red, green, and violet as the three primary physiological colors?

I want to suggest that work in physiological acoustics and a concerted effort to analogize the eye and ear provided the basis for Helmholtz's reevaluation of the Young Hypothesis.[77] Strong support for this suggestion comes from Helmholtz's popular lectures on the recent progress in the theory of vision, which he delivered in Köln in 1868. In that context Helmholtz explicitly pointed to the analogy with his work on the sensations of tone to convince his audience of the plausibility of the three-receptor hypothesis.[78] Moreover, the internal evidence of Helmholtz's papers in the period between 1855 and 1862 indicate that the receptor hypothesis was more strongly supported by his work in physiological acoustics before the line of investigation for establishing the receptor theory in physiological optics became clear. Indeed, the papers published in this period indicate that the juxtaposition of these two sensory modalities, the back-and-forth comparison of models in one domain with those in the other, guided Helmholtz toward the reversal of his earlier position on Young's trichromatic receptor hypothesis.

The analogy between physiological acoustics and color vision consisted in the assumption that just as in the ear a set of fundamental or primary tones is objectively based in the rods of Corti (discovered by Alfonso Corti in 1851), so in the eye a set of primary colors is based in specific nerve endings in the rods and cones. Neither assumption could be established in humans, although some evidence from comparative anatomy supported the analogy from the side of physiological acoustics. The analogy between eye and ear was a salient feature of Helmholtz's work on physiological acoustics. Comparisons between eye and ear are prominent in his first extended study in physiological acoustics in 1856,[79] and such comparisons abound in the first edition of Helmholtz's *Tonempfindung* (*Sensations of Tone*) published in 1863. But such analogies were by no means new, and Helmholtz was certainly familiar with similar comparisons made by other authors, particularly August Seebeck who had suggested an optical analog to acoustical resonance of spectral colors with vibrating molecules in groups of nerves in the retina as the mechanism for the sensation of brightness.[80]

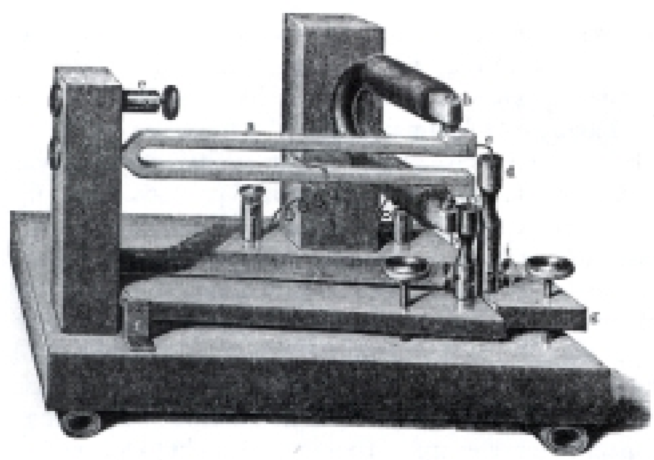
The objective Helmholtz set for his acoustical investigations was the solution of exactly the problem that a few months earlier had halted the further development of the Young three-receptor hypothesis: to determine that the mathematical form of the physical description of hearing had a material, physical basis in the physiology of the ear. Helmholtz wanted to show not just that Fourier analysis is a useful mathematical tool for representing the phenomena, but rather that the ear itself is a Fourier analyzer. Helmholtz's goal in the paper on combination tones was to establish that, like spectral colors, primary tones have an independent objective existence. He wanted furthermore to establish that combination tones have an objective existence independent of the ear, that they are not simply a psychological phenomenon:

> Is this means of analyzing forms of vibration which Fourier's theorem prescribes and renders possible, not merely a mathematical fiction, permissible for facilitating calculation, but not necessarily having any corresponding actual meaning in things themselves?... That …this [Fourier] analysis has a meaning in nature independently of theory, is rendered probable by the fact that the ear really effects the same analysis, and also by the circumstance already named, that this kind of analysis has been found so much more advantageous in mathematical investigations than any other. Those modes of regarding phenomena that correspond to the most intimate constitution of the matter under investigation are, of course, also always those which lead to the most suitable and evident theoretical treatment.[81]

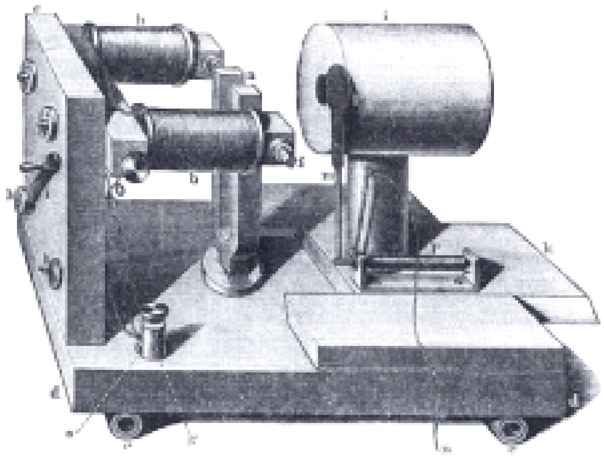
Expressing a metaphysical assumption persistent in all his work, Helmholtz regarded the best mathematical model to be one that mirrors the structure of both his measuring apparatus and the physical entities they are intended to represent. The reason Fourier analysis worked so well in capturing the phenomena was that the ear itself is a physical embodiment of a Fourier analyzer. And as we have seen, Helmholtz did not want simply to assume that the most elegant abstract mathematical model is best. Rather his procedure was to arrive at the mathematical model through the give-and-take triangulation of experiment, theory, and model building.

Helmholtz found it necessary to approach the problem in this way, because no adequate mathematical model existed. Work on the theory of vibrating strings provided a partial theoretical framework, but no complete mathematical model was in place to guide his efforts. As Helmholtz noted the "mathematical analysis of the motions of sound is not nearly far enough advanced to determine with certainty what upper partials will be present and what intensity they will possess."[82] A key feature of his work was in identifying the physical basis of tone quality, or timbre, the feature that distinguishes a violin from a flute or clarinet. A plausible assumption made by most physicists before Helmholtz was that tone quality was determined by the form of the sound wave, and that this corresponds to the form of water waves.[83] Helmholtz, too, embraced this assumption, but found that, "Unfortunately, the form of waves of sound, … can at present be assigned in only a few cases."[84] In order to make up for the defect in theory, he made use of a variety of devices for visualizing the form of a sound wave and mechanically analyzing it into its constituent primary tones.

Among the several experimental devices Helmholtz utilized were cleverly designed electromagnetic instruments for artificially producing and combining tones. Chief among these was the tuning-fork interrupter and tuning-fork resonator. In the tuning-fork interrupter a continuous direct current passing through a tuning fork was momentarily broken by electromagnets near the ends of the fork, which alternately attracted the ends of the fork, made contact and transmitted current-pulses at the frequency of the fork. These currents were transmitted to a second tuning-fork apparatus. The fork in this apparatus was placed between an electromagnet activated by the incoming current-pulses. To get the best results the lowest prime tones were required. This entailed using forks whose tones were barely audible. These tones were amplified by placing a resonator tuned to the proper frequency near the fork. By placing these resonator devices in connection, Helmholtz was able to combine numerous partial tones into tones indistinguishable from tones produced by musical instruments.

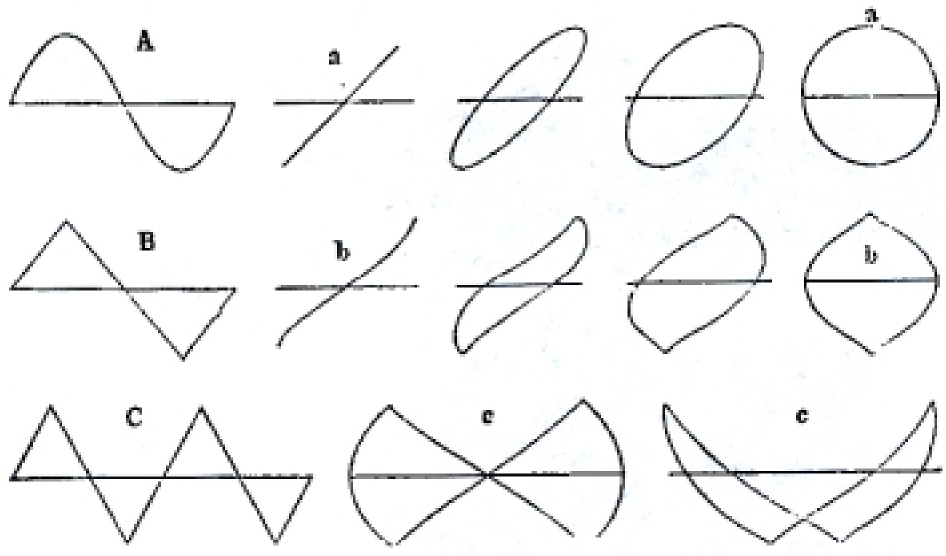


[Figure 7 Helmholtz's tuning fork interrupter]



[Figure 8 tuning-fork resonator]

Helmholtz described several methods for making auditory vibrations visible. The first of these was what he termed the "graphic method": "To render the law of such motions more comprehensible to the eye than is possible by lengthy verbal descriptions."[85] He illustrated the graphic method with the phonautograph, which consisted of a tuning fork with a stylus on one prong of the fork. The vibrating fork produced a curve on paper blackened with lampblack and attached to a rotating drum, the same arrangement as in Ludwig's kymograph or in Helmholtz's own myograph. The most dramatic of these visualization devices was the so-called vibrational microscope, an instrument embodying methods described first by Lissajous for observing compounded vibrational motions. The microscope was constructed so that the objective lens was mounted in one of the prongs of a tuning fork. The eyepiece of the microscope was mounted on a plate so that the tube of the microscope was attached to the backing of the bracket holding the tuning fork. The prongs of the fork were set in vibration by two electromagnets just as in the interrupter and resonator described above. When the tuning fork was set in motion, the object lens would vibrate vertically in a line. When the microscope was focused on a stationary grain of white starch and the forks set in motion, a white vertical line would be seen. By placing the grain of starch on a vertical string so that the grain was vibrating horizontally while the lens was moving vertically, the image viewed in the field of the microscope would be a line compounded of both motions inclined at 45°. As the phase of the resonator fork altered, the line visible in the field of the microscope shifted from a straight line inclined by 45° (when the two forks were in unison) through various oblique ellipses until the phase difference reached one quarter of the period, and then passed through a series of oblique ellipses to a straight line inclined 45° in the other direction from the vertical when the phase difference reached half a period.[86]



[Figure 9: Lissajou Figures in the Vibration Microscope]

The principal use Helmholtz made of these instruments was to manipulate phases of primary tones generated in his interrupter and resonator in order to demonstrate that phase differences in primary tones making up a compound tone—made visibly evident in the vibration microscope—had no effect on the perceived quality of the tone. According to Helmholtz, the quality of the tone was determined solely by the force of the impression on the ear, that is, on the amplitudes and primary tones entering the composition of the tone. As long as the relative intensities of the partial tones compounded into a musical tone remained the same, the tone would sound the same to the ear no matter how the alteration of phases of the partial tones affected the form of the wave. Helmholtz concluded from these experiments that the quality of the musical portion of a compound tone depends solely on the number and relative strength of its partial simple tones, and in no respect on their differences of phase.[87]

These experiments and the crucial role of the vibrating microscope in particular were the basis of a direct comparison between the eye and ear. As the imaging device revealed, the eye is capable of detecting differences, even relatively minute differences, between wave forms. The ear is not. “The ear, on the other hand, does not distinguish every different form of vibration, but only such as when resolved into pendular vibrations, give different constituents.”[88] With this conclusion several of the differences as well as the fundamental similarities in the mechanism of color vision and hearing came into view. The reason the ear is able to distinguish the partial tones in a compound tone is that among the 4500 or so different nervous fibers in the arches of Corti specific nerve fibers resonate in sympathetic vibration with the spectrum of primary tones composing musical tones. Simple tones of determinate pitch will be felt only by nerve fibers connected to the elastic bodies within the cochlear membrane which have a proper pitch corresponding to the various individual simple tones. The fact that amateurs, with minimal attention are able to distinguish the partial tones in a compound tone while trained musicians can distinguish differences of pitch amounting to half a vibration per second in a doubly accented octave would thus be explained by the size of interval between the pitches of two fibers. Similarly the fact that changes in pitch can take place continuously rather than in jumps found its explanation in the sympathetic vibration of arches with proper tones most nearly identical, while the elastic bodies in the membrane with more distantly separated proper tones were incapable of vibrating in resonance.

The physiological organization Helmholtz envisioned as the basis of tone sensation was modeled directly by the tuning fork and resonator apparatus. Indeed, the full set of resonators and connected tuning forks was a material model of the ear in reverse. The resonator apparatus was used to produce compound tones artificially out of the simple tones generated by the tuning-fork interrupter. But this transmitting device could also be imagined to run in reverse as a recording device. In this sense it was a material representation of the functioning ear, its resonators being the material analogs of the fibers in the Corti membrane, its tuning fork and acoustical interrupter being the device for translating, encoding, and telegraphing the component primary tones of the incoming sound wave analyzed by the resonators.

Moreover, the model also provided a resource for understanding the differences between the eye and ear as well as a suggestion for further development in the theory of color vision. Helmholtz' apparatus enabled him to conclude that the sensation of different pitches would be a sensation in different nerve fibers. The sensation of tone quality would depend upon the power of a given compound tone to set in vibration not only those of Corti's arches which correspond to its prime tone, but also a series of other arches, and hence to excite sensation in several different groups of nerve fibers. Thus some groups of nerves would be simulated through shared resonance, whereas other nerves would remain silent. Helmholtz jumped directly from this conclusion to the analogy with Young's hypothesis concerning the eye:

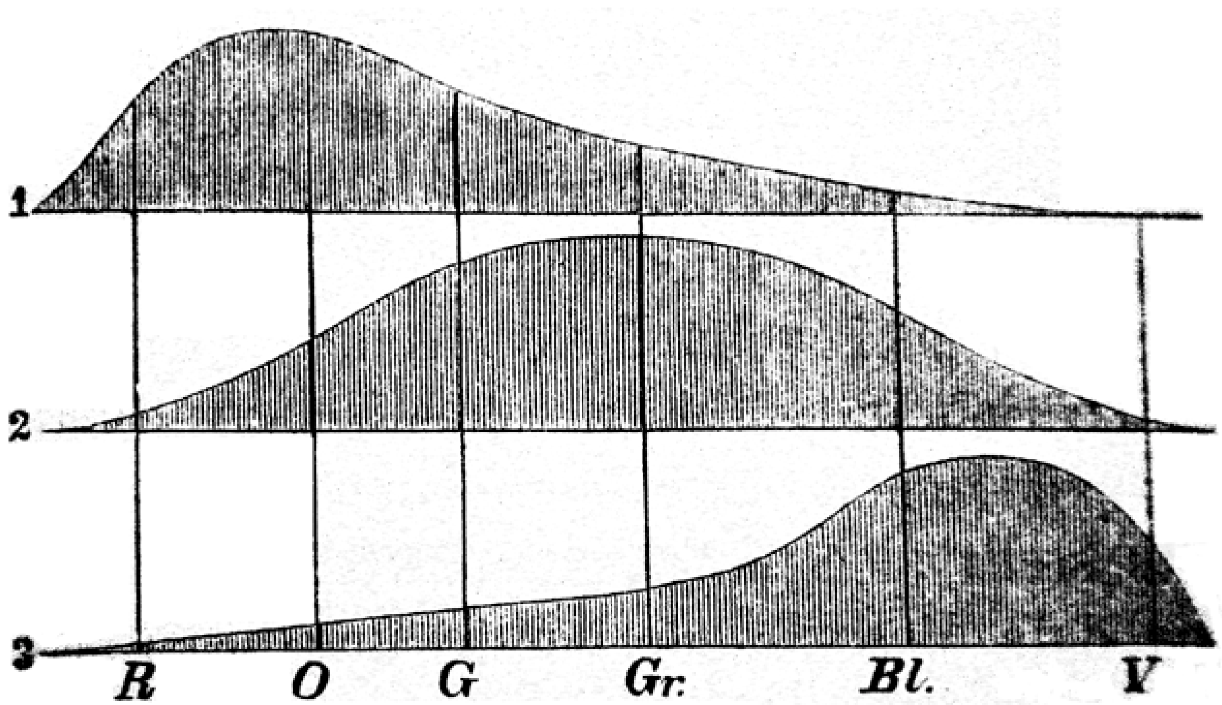
> Just as the ear apprehends vibrations of different periodic time as tones of different pitch, so does the eye perceive luminiferous vibrations of different periodic time as different colors, the quickest giving violet and blue, the mean green and yellow, the slowest red. The laws of the mixture of colors led Thomas Young to the hypothesis that there were three kinds of nerve fibers in the eye, with different powers of sensation, for feeling red, for feeling green, and for feeling violet. In reality this assumption gives a very simple and perfectly consistent explanation of all the optical phenomena depending on color. And by this means the qualitative differences of the sensations of sight are reduced to differences in the nerves which receive the sensations. For the sensations of each individual fiber of the optic nerve there remains only the quantitative differences of greater or lesser irritation.
>
> The same result is obtained for hearing by the hypothesis to which the investigation of quality of tone has led us. The qualitative difference of pitch and quality of tone is reduced to a difference in the fibers of the nerve receptive to the sensation, and for each individual fiber of the nerve there remains only the quantitative difference in the amount of excitement.[89]

The analysis of hearing seemed to suggest that just as the myriad of musical tones capable of being distinguished is rooted in an organic Fourier analyzer which yields specific elemental sensations, so the eye in similar fashion could be conceived as generating color from a primary set of sensations rooted in specific nerve fibers. The delicate power of discrimination the ear is capable of was explained by the large number of specific nerve fibers and related elastic resonating bodies in the Corti membrane. The inability of the eye similarly to resolve colors into elemental sensations would be explained accordingly as a result of the small number of different types of sensitive nerve fiber—three rather than roughly 1000—and by the assumption that all three nerve types respond in different degrees to light stimulation. The assumption that fibers predominantly sensitive to red, green, or violet light nonetheless respond weakly to light of other wavelengths would explain the continuity of transitions in the sensation of color as well as the inability of the attentive mind to analyze compound light into its elements. There is no music to the eye, because the eye has only three rather than the 1000 "resonator" types of the Corti membrane.

Young's Three-Receptor Theory Revisited

In order to provide evidence for the three-receptor hypothesis suggested by his acoustical researches, Helmholtz embarked upon an extensive series of investigations connected with after images. These researches expanded the analysis of the 1855 paper on color mixtures and led to a further refinement of the physiological model of the color space by establishing that it is not the hyperbola shaped figure arrived at in that paper but rather an equilateral triangle with red, violet, and green as its vertices, and that these subjective colors are all more saturated than the spectral colors. Helmholtz introduced the three-receptor theory for the first time in the *Handbuch der physiologischen Optik*, where it formed the centerpiece of section 20 on color vision and was elaborated throughout sections 21-23.

Helmholtz was led to reconsider the possible fundamental character of Young's theory by the fact that in constructing a color space he had to position complementary pairs of colors at different distances in order to accommodate the differences in degree of saturation of the colors forming the sensation of white.[90] This suggested that from a physiological point of view the spectral colors are not the most saturated or pure colors. This idea was further supported by his experiments on acoustical resonance, where resonators with similar proper tones would join in a compound tone. Young's hypothesis gave an immediate physiological ground for a similar expectation in the case of the eye. Young assumed that the physiological basis for color vision is the stimulation of three different types of nerve in the retina, and Helmholtz's experiments supported nerve types sensitive respectively to the red, green and violet ranges of the spectrum as the appropriate choices. With these choices one nerve type would be primarily sensitive to the lower frequencies, another to the middle frequencies, and a third to the upper frequencies.



[Figure 10: Response Curves of Three Receptors]

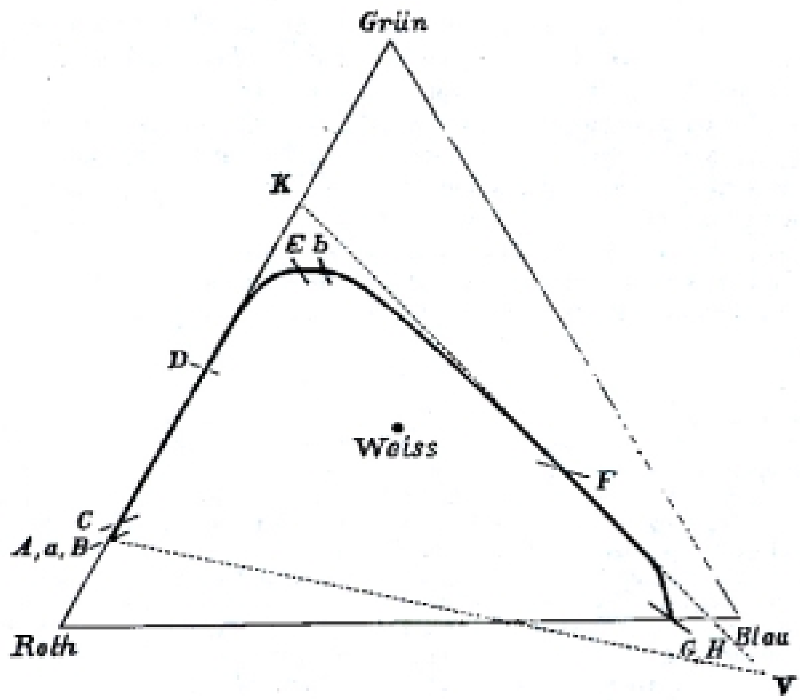
To Young's Hypothesis Helmholtz added several further auxiliary assumptions designed to provide a full physiological interpretation of the color space model. He assumed, for instance, that while the stimulation of each nerve type produces a response chiefly in the range of its primary color, it also produces a weak response in the neighboring ranges. Thus the stimulation of the violet-sensitive nerve endings also generates responses in the red- and green-sensitive nerves. According to this condition, therefore, with light of medium intensity, the stimulation of the green-sensitive receptor predominates initially producing the sensation of green but then shifting into yellow and red. Helmholtz also assumed that the strength of the nerve response or sensation in each nerve type is a function of the intensity of the light stimulating it. When stimulated by white light, for instance, the violet-sensitive nerves [reading the graph from right to left] are activated initially at low intensity; they rise more rapidly than the green- or red-sensitive receptors, reaching their maximum strength at lower intensities of light than the green- or red-sensitive nerves. Since the violet-sensitive nerves are activated at the lowest intensities, the sensation of violet will predominate at low intensity light. When stimulated by light of low intensity, for instance, the violet sensitive receptors generate sensations which begin with violet and then pass through blue over into green.[91]

Helmholtz's modification of the Young Hypothesis brought with it the assumption that the sensation of one color, such as red, is always accompanied by the weak stimulation of nerve activity in the other ranges. This was particularly evident in Helmholtz's depiction of the middle ranges between yellow and blue. An immediate consequence of this assumption, therefore, was that the sensation of spectral green, for instance, is not the most saturated green possible; for it is always accompanied by stimulation of the red- and violet-sensitive nerves. If the activity of these nerves could be suppressed while at the same time leaving the capacity of the green-sensitive nerves undiminished, it would be possible to have the sensation of a saturated green lying well outside the boundaries of the color space constructed for the spectral colors. The same would be true for red and violet. Against the sensation of these colors, the pure spectral colors would appear whitish. Here was a remarkable feature of the Helmholtz-Young Hypothesis which was possible to test empirically. From the point of view of Grassmann's theory of color mixtures, any three colors which could combine to produce white would be satisfactory choices for producing the color space. From the perspective of the assumed physiological mechanisms of the Helmholtz-Young hypothesis the most saturated colors should be those generated by the primary receptors. Experiments aimed at diminishing or eliminating the activity of one or two of the color-sensitive nerve endings in the retina would thus provide support for the distinction between three different nerve types as well as dramatic evidence for the choice of red, green, and violet as the three primary physiological colors.[92]

Clinical investigations of color-blindness provided support for the three-receptor hypothesis, but the most interesting experimental evidence for the theory resulted from the investigation of after-images formed by fixing the eyes upon either a bright object against a dark background or a dark object against a bright background. Helmholtz distinguished between positive and negative after-images. Similar to photographic images, positive after-images were those in which bright areas in the object are also bright areas in the image. Negative after-images reversed the relationship between bright and dark areas in the object, so that bright areas appeared dark in the after-image. Colored after-images follow similar patterns; only here the negative image is the complementary color. To form colored after-images Helmholtz recommended fixing a colored object, such as a colored square or cross, and then observing the after-image against either a very dark or against a white background of different brightness. The negative after-image of a colored object is best achieved by fixating for a long period upon, for instance, a colored piece of paper viewed against a gray background. If the colored paper is suddenly removed, the after-image appears as a negative in the color complementary to the color of the fixated object: the after-image of a red object will appear blue-green, that of a yellow object will appear blue, that of a green object red, and vice versa in each case.

The explanation for these phenomena followed directly from Thomas Young's assumption of three sensitive nerve types for the different colors. For since colored light does not excite all of the nerves with equal strength, the different degrees of excitation must afterward be followed by different degrees of fatigue. If the eye has seen red, the red-sensitive nerves are strongly excited and fatigued, the green- and violet-sensitive nerves are only weakly stimulated and only slightly fatigued. If in this condition white light falls on the eye, the green- and violet-sensitive nerves will be relatively more strongly affected than the red-sensitive. The impression of blue-green, the complementary color to red, will therefore predominate.[93] Continuing this line of experiment further, if the square section of the retina eye has been rendered insensitive to blue-green by fixating on a bright blue-green square against a red background, the square afterimage will be an even more saturated red than the background red itself. Helmholtz went on to devise a method for doing this experiment with pure spectral colors by repeating this experiment with in the prism and color-slit apparatus he used for establishing complementary colors. From these experiments Helmholtz concluded "*the most saturated objective colors which exist, namely the pure spectral colors, do not yet generate the sensation of the most saturated colors possible in the unfatigued eye*, but rather that we arrive at these colors only by rendering the eye insensitive to the complementary colors."[94]

With this result, Helmholtz had established the empirical basis for supporting his modified version of the Young Hypothesis. The color-space of possible subjective colors was not the curve **RGV** worked out in terms of the construction of white from complementary colors. Rather it was best represented as a triangle **VAR**, in which the spectral colors violet and red were displaced to $\mathbf{V_1}$ and $\mathbf{R_1}$. In this scheme, therefore, the surface $\mathbf{V_1ICGrGR_1}$ would contain the possible colors obtainable by mixing objective colors, whereas **R**, **V**, and **A** represented the pure primary colors. The basis of these physiological colors was the tripartite physical constitution of the light-sensitive nerves and their specific nerve energies.

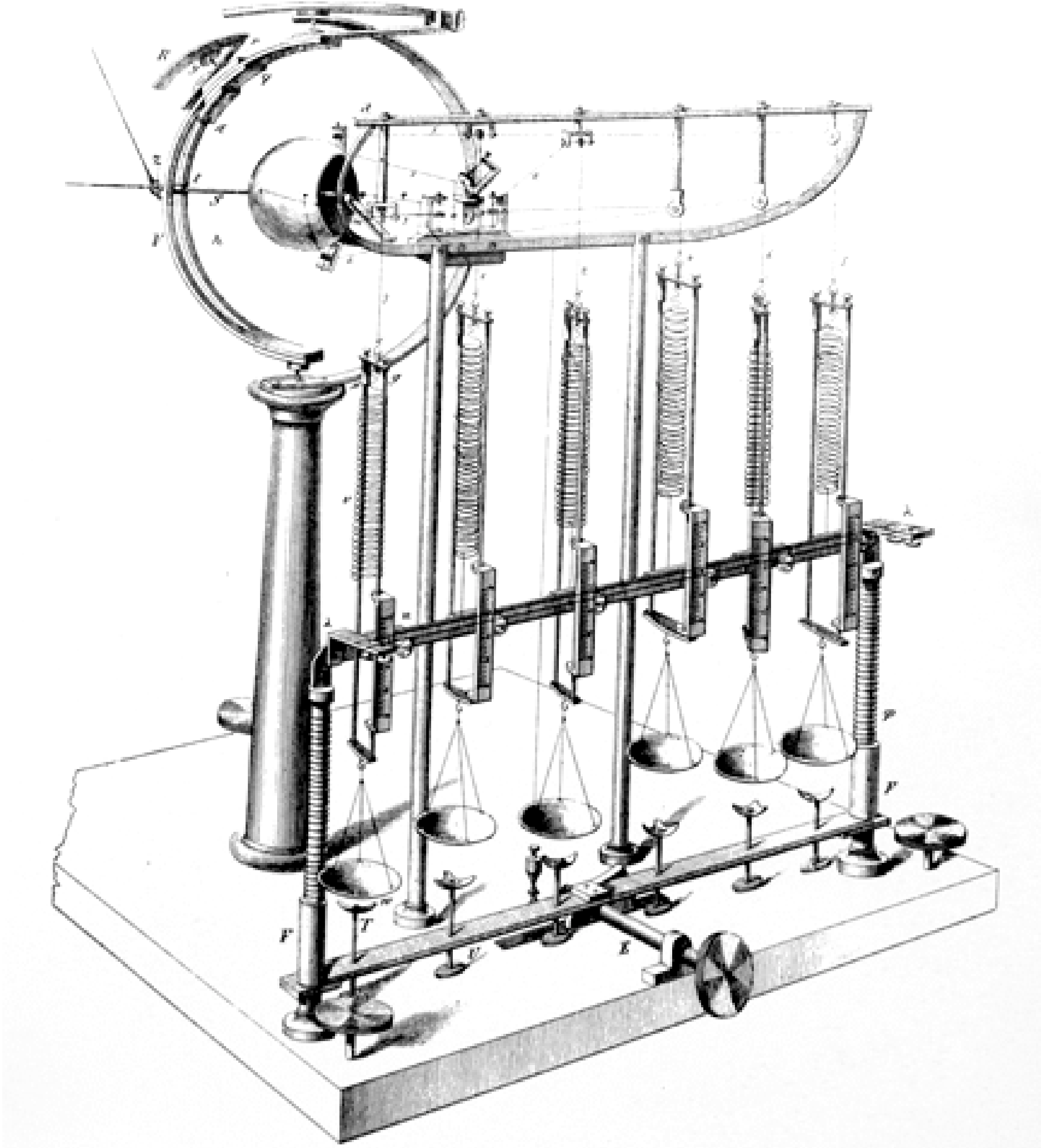


[Figure 11: Helmholtz's Color Triangle for Representing Subjective Colors]

Eye Movements and the Principle of Easiest Orientation

In discussing Helmholtz's work on color vision and physiological acoustics I have emphasized his effort to construct physical models that incorporated measurements directly analogous to the neurophysiological processes of measurement being performed by the eye and ear. I have also pointed to the way in which Helmholtz's models were intended to generate measurements that could plausibly enter as parameters specifying the n-dimensional manifolds he, like Herbart, believed the mind uses in building experiences of space and time out of sense data. In my view the most spectacular example of this strategy of relating models, instruments, and theory was Helmholtz's treatment of eye movements in connection with the construction of the 3D visual field.

Helmholtz's work in this area built on the researches of Adolf Fick and Wilhelm Wundt. Fick, a student of Helmholtz's friends Carl Ludwig and Emil du Bois-Reymond, was the first person to attempt an exact physical treatment of eye movements based on Helmholtz's work on the conservation of force. Fick's proposal was to treat eye movements as a problem in statics in which the moment of rotation of the eye at rest is the sum of the moments of rotation produced by the six muscles holding it in equilibrium and that in moving from one position to another the sum of muscle contractions is made with the least total expenditure of force.[95] Unfortunately, Fick could not provide a general solution to this problem, and was forced to proceed on a case by case basis.



[Figure 12 : Wundt’s Ophthalmotrope]

Wilhelm Wundt, Helmholtz’s assistant in Heidelberg at the time, attempted to improve on Fick's model in 1862. Wundt's approach was to develop his theory directly on a physical model of the eye known as the ophthalmotrope, first introduced by Reute in 1847 with a much improved version in 1857.[96] The idea was to construct the device in such a manner that springs would completely replicate the activity of the eye muscles. Wundt emphasized the need to develop an equation which incorporated a measure of the forces actually exerted by contracting muscles at any given point rather than a general rule capable of being fulfilled in infinitely many ways such as Fick’s least expenditure of force approach had done. It was crucial to take the contractile and elastic forces of muscles into account, the opposition between agonist and antagonist muscles in a given movement, as well as the resistance to motion offered by external obstacles, such as the optic nerve, fat, etc. Wundt called this the principle of least resistance.

Wundt's principle was stated as: In every voluntarily determined position of the visual axis, the eye always adopts that position in which the opposition to its muscles is least.[97]

Moreover, Wundt was keen on incorporating a psychological component into his treatment of eye movements. As Wundt saw it, the problem of eye movements is not just a problem in mechanics but also part of the intentional mental act of achieving visual orientation. Wundt argued that whatever solution works for the eye as a mechanical instrument must be compatible with the psychological regulative mechanism the mind uses in operating and guiding the eye. As we shall see, Helmholtz thought the psychological component postulated by Wundt fell short.

Drawing on work done by the Weber brothers on muscles while still a student in the lab of du Bois-Reymond, Wundt had established experimentally that when a muscle shortens or lengthens by a magnitude **e**, the resistance it experiences in each small part of the path **de** is proportional to **e.de**. The form of the equation describing the muscle contraction was, in short, similar to that used in describing the motion of springs, thus justifying the use of the ophthalmotrope as an appropriate physical model of the eye. The earlier work had also established that the resistance is directly proportional to the diameter of the muscle and inversely proportional to the length of the muscle. Where q is the diameter, l the length of the muscle and μ is the coefficent of muscle elasticity, Wundt obtained:

$$\mathrm{dw} = \mu \frac{\mathrm{q}}{\mathrm{l}} \mathrm{e}de$$

Integration of this equation gives the total resistance **w** experiened by the muscle contracting through distance e:

$$\mathrm{w} = \mu \frac{\mathrm{q}}{\mathrm{l}} \int_0^e \mathrm{e}de$$

If the resistances, lengths, cross-sectional diameters, and displacements are similarly represented for each of the six muscles, we then have:

$$\mathrm{W}_1 = \mu \frac{q_1}{l_1} \frac{\mathrm{e}_1^2}{2}, \qquad \mathrm{W}_2 = \mu \frac{q_2}{l_2} \frac{\mathrm{e}_2^2}{2},$$

Wundt supposed the eye to be a system in which all of the muscles acting like elastic springs bring the visual axis to a particular position according to the condition that the sum of the resistances is the smallest possible for that position; i.e., $W_1 + W_2 + W_3 + ... + W_6$ is a minimum. Neglecting the constant $\mu/2$ Wundt expressed the condition as:

$$\frac{q_1}{l_1}\mathrm{e}_1^2 + \frac{q_2}{l_2}\mathrm{e}_2^2 + ... + \frac{q_6}{l_6}\mathrm{e}_6^2 = \text{minimum}$$

or

$$\sum_{i=1}^{1=6} \frac{q_i}{l_i}\mathrm{e}_\mathrm{i}^2 = \text{minimum}$$

In my view the most interesting aspect of Wundt's analysis for Helmholtz came in his discussion of an alternative interpretation of the minimum equation and its significance as a regulative principle for guiding the visual axis. Wundt noted that equation (4) expressing the condition of equilibrium for the eye agrees in form with the general equation expressing the least squares of the observational error for six measurements, each weighted differently:

> If we were to suppose e_1 , e_2 , e_3 ,..., e_6 the observational errors for any phenomenon, q_1/l_1 , q_2/l_2 , ..., q_6/l_6 , those magnitudes which are assigned as weights of observation, our formula would be nothing other than the fundamental equation of the method of least squares, which asserts that the sum of the products of the squares of the observational errors multiplied by the weighted observations must be a minimum. Therefore, whenever we move the visual axis into a new position the eye proceeds just like a mathematician when he compensates for errors according to the rules of the probability calculus. The individual muscles behave like the individual observations, the lengthening and shortening which they experience in the transition to the new position behave like the unavoidable errors in observation, and the coefficients of resistance of the muscles behave exactly like the observational weightings.
>
> Indeed, it is easy to see that it would have been possible for us to have grounded the principle in a completely different way than we have pursued. We could have deduced it directly from the fundamental postulates of the probability calculus in the same way that the method of least squares is grounded.[98]

Wundt's source for this idea was undoubtedly a paper published in *Crell's Journal für die reine und angewandte Mathematik* by Carl Friedrich Gauss in 1829, entitled, "Über ein neues allgemeines Grundgesetz der Mechanik." The general law referred to in Gauss's title was the principle of least constraint. The principle stated that each mass in any system of masses connected to one another through any sort of external constraints moves at every instant in closest agreement possible with free movement, or under the movement each would follow under least possible constraint, the measure of the constraint which the entire system experiences at each instance being considered as the sum of the products of the masses and the squared deviation from the free path. Gauss had concluded this brief but powerful paper as follows:

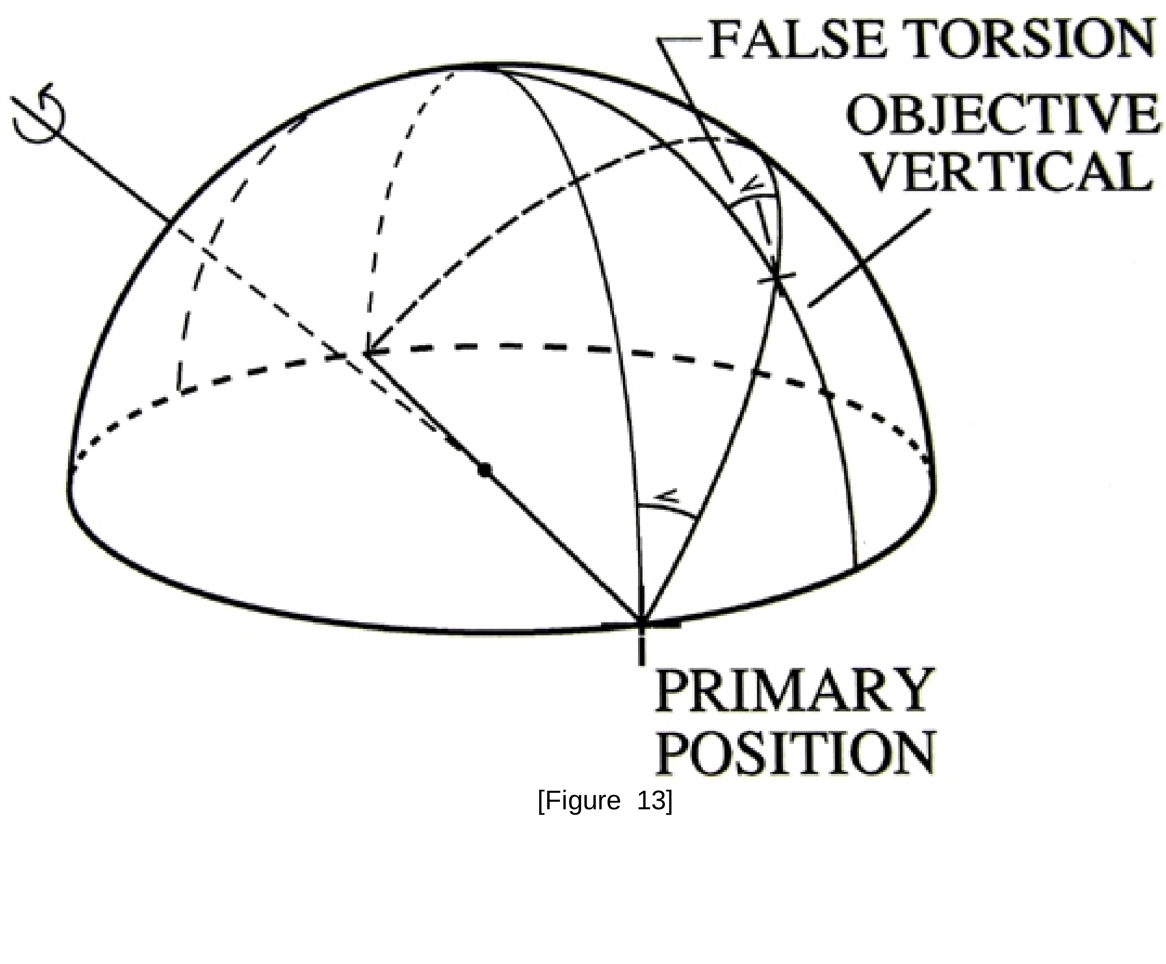
> It is very remarkable that free motions, if they cannot exist with the necessary conditions, are modified by nature in exactly the same manner that the mathematician calculating according to the method of least squares compensates experiences which are related to one another through the necessary dependence of connected magnitudes. This analogy could be followed further in still other directions, but this does not accord with my present purpose.[99]

Wundt's conclusion relating the probability calculus to his own principle of least resistance in eye movements was almost a literal quotation from Gauss's conclusion to the paper on the principle of least constraint. The comparison was completely appropriate; for the eye and the six attached muscles operates as a system of constrained motion closely analogous to the situation described by Gauss in deriving his law.

By calling the eye a mathematician operating in terms of the probability calculus Wundt signaled his intent to incorporate a psychological component in vision. By emphasizing least resistance as the psychological regulative principle guiding the eye, however, Wundt had reduced the problem of vision to one of mechanics. Although Wundt claimed to invoke a psychological process, that process seemed more directly related to resistance the eye encounters in moving rather than to any conscious acts related to vision. As Helmholtz would point out, it was unclear, for example, how the principle of least resistance could illuminate how single vision comes about. Moreover, Wundt seemed to imply in the passage quoted above that it would be possible to start from the probability calculus and arrive at constraints on eye movements, but he did not provide this derivation. The derivation of the conditions of eye movement from a psychophysical principle was Helmholtz's goal in his paper published in *von Graefe's Archiv* the following year (1863).

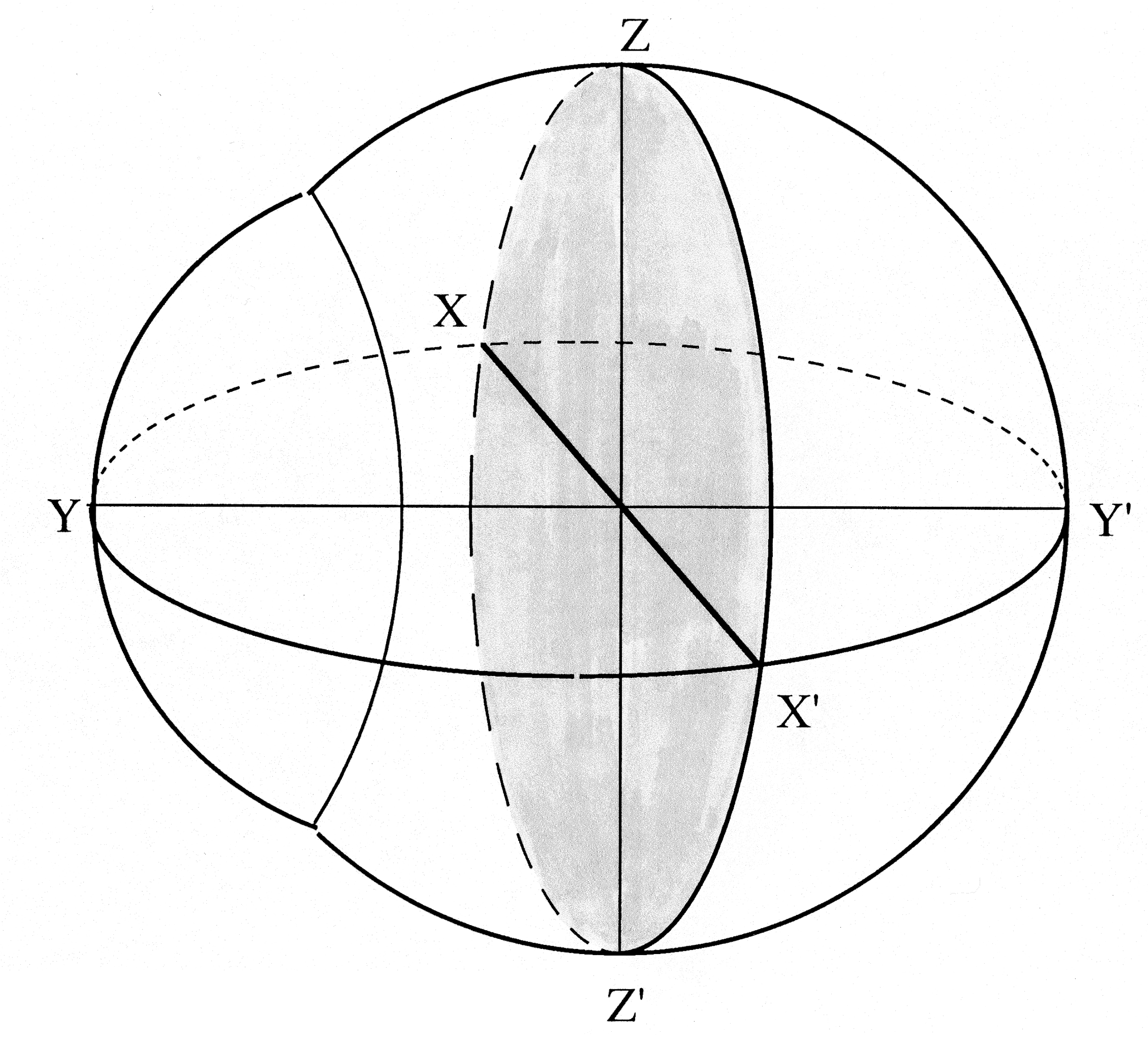
Helmholtz introduced his treatment of eye movements by stating that the conclusions Wundt and Fick had reached regarding the role of least constraint were probably correct. But he noted, "even if the principle...should prove to be completely applicable, it would not follow that an optical principle is not actually decisive."[100] By "optical" Helmholtz had in mind some principle directly and primarily concerned with visual perception but which in being executed would have the same effect in coordinating muscle movements as Wundt's principle of least resistance. According to Helmholtz the primary goal in vision is to avoid double images by fusing the images in both eyes into a single visual object. In order to bring this about, the eyes must be properly oriented. Given that all sorts of variation in the strengths of the eye muscles would be compatible with Wundt's principle of least resistance, it seemed more reasonable to consider the purposes of achieving singular vision as primary in the selection among the miriad muscle actions compatible with Wundt's law. Indeed, Helmholtz could cite numerous experimental examples where voluntary exertion produces the position of the eye best suited to vision. The principle actively guiding this process Helmholtz called the principle of easiest orientation. At first glance this might seem like a semantic variation of Fick's and Wundt's proposals, but as we shall see, Helmholtz took Wundt's idea that the eye is a mathematician as much more than a metaphor.

Crucial to Helmholtz's development of the principle of easiest orientation were the laws of Donders and especially of Listing. The eye has three primary degrees of freedom. It performs abductions and adductions around a central vertical axis through the eye; it performs elevations and depressions around a central horizontal axis; and finally it can perform torsions, cyclorotations, or wheel-like rotations, around the visual axis itself, the anterior-posterior axis of the eye. [See Figure 12] Helmholtz considered that the work of achieving single vision would be greatly simplified if, in order to maintain the orientation of the eye as it moves from one position to another the mind did not have to attend to elevations, abductions, as well as cyclorotations, but rather that cyclorotations would be excluded, thus leaving only two variables to be attended to. This would be accomplished if the cyclorotations made by the eye were completely dependent upon and determined by abductions and elevations. The laws of Donders and Listing, were they to be confirmed, would guarantee this dependence of eye movements on two variables alone and at the same time exclude cyclorotations.



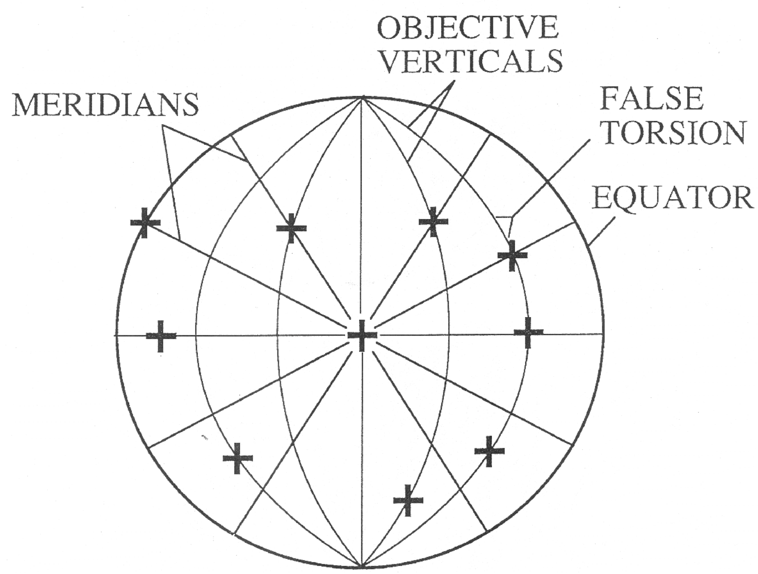
[Figure 13]

The law of Donders states that there is one and only one orientation for each position of the line of fixation (the visual axis). Helmholtz pointed out that this fact alone would greatly facilitate orientation, but it would not insure optimal conditions for a stable visual field during eye movements. For it would not guarantee a smooth transition from one configuration of points to an adjacent or remote configuration of points. This latter problem would be resolved, however, if in moving from one position to an adjacent position, the rotation of the line of fixation takes place around a single axis. Listing's Law states that every movement of the eye from the primary position to any other position takes place by rotation around an axis perpendicular to the meridian plane that passes through the final position of the line of fixation. It can easily be shown that each of these axes of rotation lies in an equatorial plane perpendicular to the visual axis. This is Listing's Plane. (See Figure 14)



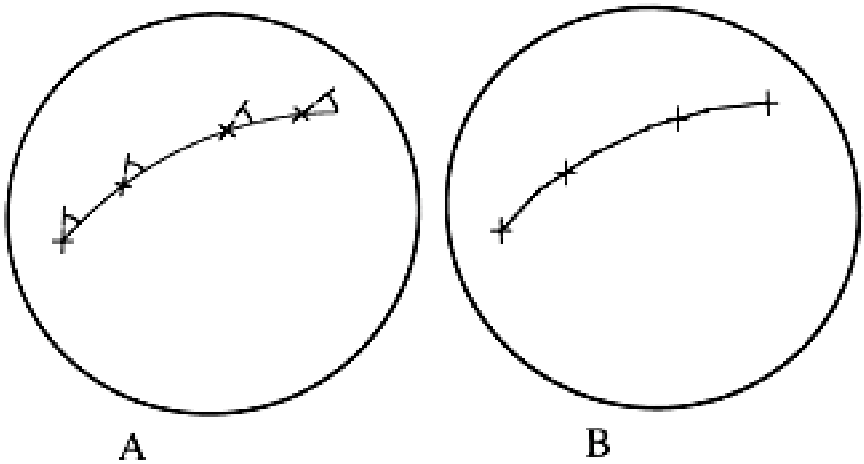
[Figure 14]

Helmholtz's principle of easiest orientation is a rule that restricts eye movements as closely as possible to axes in Listing's Plane. In order to discuss motions limited to rotational axes in Listing's plane, Helmholtz introduced what he termed the *atropic* or non-rotated line. The atropic line was a line perpendicular to the plane of the rotational axes of the eye, i.e., the XZ plane in Figure 13, or Listing's Plane. In the primary position the atropic line coincides with the visual axis. In treating the motion of the eye from a primary to secondary position Helmholtz showed it is possible to resolve the motion into two rotations by the parallelogram of motions. Thus a secondary position could be reached by first executing an elevation, **α**, around the horizontal axis, XX′ in the XY plane and then an abduction **ω** around the vertical axis, ZZ′ Alternatively, the motion could be represented as a single rotation around an axis in the equatorial XZ plane (Listing's Plane) intermediate between the horizontal and vertical axes and perpendicular to the plane containing the primary and secondary point. The direction and magnitude of that rotation could be determined by the parallelogram of motions according to familiar rules of projections in terms of direction cosines — our vector product. In terms of the atropic line, this meant that in general, for motions occuring in Listing's plane, any motion of the visual axis could be decomposed into rotations around axes perpendicular to the atropic line. If, however, the motion of the visual axis was generated around an axis not in Listing's plane, Helmholtz showed it would always be possible to decompose that displacement into a rotation around an axis perpendicular to the atropic line (i.e., an axis in Listing's plane) and a rotation, **ρ**, around the atropic line itself. The principle of easiest orientation demanded that the components of rotational motion having the atropic line as axis must be zero.



[Figure 15]

Helmholtz confirmed this approach experimentally by mapping the trajectory of the visual axis on a wall-grid placed opposite his experimental subjects. He projected the expected paths according to Listing's Law in selected angles of 10° (represented in Figure 15) Those lines consistent with motion around axes in Listing's Plane, Helmholtz went on to show, are the right projections onto the wall of great circles passing through the point Y′ in Figure 14, which he called the occipital point. For reasons which will become apparent in their connection with visual orientation Helmholtz designated these "direction circles." These are in a sense, the eye's own internal coordinate system. Also represented in Figure 15 are the objective verticals (that is lines perpendicular to the "horizon line" on the wall-grid) as projected from the occipital point. Except for the primary vertical, which is also a meridian passing through the occipital point, the projections of these lines onto the wall are not straight lines but arcs as indicated in the diagram. In practical terms, the implication of the grid construction was that if a cross is fixated by the eye in the primary position — indicated by the center of Figure 15 — the afterimage of the cross will retain the same orientation with respect to the primary position.



[Figure 16]

Helmholtz used this grid for experiments to investigate whether in fact the visual axis can move freely from one point to any other in the visual field — and not simply from a primary to a secondary position — without producing a cyclorotation. He showed that, for the most general case, if eye movements are restricted to a very small section of the visual field, the principle is satisfied. This results from the fact that for infinitesimally small movements, the sine of the angle of rotation is nearly identical with the rotational arc, so that the axes of rotation all lie virtually in the same plane. For larger displacements, however, this was not generally the case. The upshot was that for a continuous movement of the visual axis over a large angle of the visual field the succession of axes for each of the infinetesimally small rotations making it up could not all lie in the same plane. Hence, cyclorotations are unavoidable. Helmholtz did show, however, that there are some paths which are torsion free; namely along the direction circles. A cross moving along one of these lines always remains upright. Motion along the direction circles can be executed around axes in Listing's plane, so that cyclorotation will not occur. Thus, the direction circles are crucial to visual orientation. In any given part of the field the path between two points not connected by a meridian can be approximated as a succession of small paths along neighboring direction circles (See Figure 16). Any deviation from a directional circle would be regarded as an error. The task in visual orientation as the visual axis moves from point to point is to keep the sum of these errors a minimum; and this can be accomplished by the process of successive approximations along direction circles:

> If we consider, therefore, every rotation of the eye around the atropic line as an error, we can reduce the principle of easiest orientation to the following demand: The law of eye movements must be so determined that the sum of the squares of the errors for the total of all the possible infinitesimally small movements of the eye taken together shall be a minimum.[101]

Helmholtz went on in the next paragraph to reclaim Wundt's metaphor, only now recast as a psychophysical principle:

> The squares of the errors have to be understood here in the same sense as the well-known principle of least squares in the probability calculus.[102]

Wundt probably felt he had been upstaged by Helmholtz, his mentor and lab director, whose own work on the principle of easiest orientation was so clearly dependent on Wundt's work on the ophtalmotrope and discussion of least constraint. But Helmholtz had superseded Wundt's attempt to construct a psychophysical account by showing that the principles the mind employs in making judgments in pursuit of the fundamental aim of visual orientation, single vision, have as their physical consequence motion according to Listing's and Donders' laws, both of which are the optimal conditions for eye movements in a mechanical system operating under least constraint. The psychological and physical aspects were intimately linked in Helmholtz's solution like two sides of the same coin.

This work on eye movements was a crucial part of Helmholtz's empiricist theory of vision and provided a key element of his sign theory of perception with which I began. In Helmholtz's empiricist theory visual sensations involve psychological acts of judgment. The collection of points we judge to be a stable object is generated by stimulation of the retina. But that judgment is based on confirmation of a sensory hypothesis: To see a collection of points in a visual image as a stationary object, the points in the collection must retain their identical configuration as a group with respect to one another when the eye makes successive passes over the supposed object; Or, in the case of a moving object, the collection of points must retain its character as a connected group as it moves across the visual field. In effect, the mind is performing a series of experiments with the eye, testing the hypothesis of an object; and the outcome is judged a success if the squares of the errors after several passes or in different parts of the field are an acceptable minimum. The comparison base with respect to which we make this judgment of "fit" is the distribution of departures from Listing's law in different parts of the visual field as indicated by the cross experiments discussed above. Listing's law is crucial, for in a mechanical system such as the eye, it is the simplest method for approximating a distortion-free mapping of connected points on a sphere as the sphere adopts different successive orientations. As infants we may not have made such controlled experiments with horizontal and vertical lines, but we undoubtedly did some sort of eye exercises with the similar outcome of our becoming intimately familiar with the direction lines of our visual system, those distortion-free lines in the visual field which obey Listing's law perfectly. This measuring device embedded in the eye is in fact sensory knowledge, part of the system of local signs we carry around with us at all times, active but below the level of consciousness, like the violin player's skill, ready to be activated whenever we view external objects.

Conclusion

In his 1878 lecture, "The Facts of Perception," Helmholtz pointed to the frequent references in his previous work to the agreement between the findings of contemporary sensory physiology, including his own physiological researches, and the teachings of Kant. While cautioning that he would not "swear in *verba magistri* to all his more minor points," Helmholtz went on to note:

> I believe that the most fundamental advance of recent times must be judged to be the analysis of the concept of intuition into the elementary processes of thought. Kant failed to carry out this analysis; this is one reason why he considered the axioms of geometry to be transcendental propositions. It has been the physiological investigations of sense perception which have led us to elementary processes of understanding inexpressible in words which remained unknown and inaccessible to philosophers as long as they inquired only into knowledge expressed in language.[103]

Kant and his followers had assumed that the contents of intuition are given immediately in sensation.[104] As I have attempted to show in this study, a central organizing theme of all of Helmholtz's physiological researches was the persistent pursuit of the question, What is given in intuition? And as we have seen, central to Helmholtz's empiricist theory of spatial perception was the rejection of the claim that anything is given immediately in perception. Perception is a process constantly involving active measurement, hypothesis, and learning. It may transpire incredibly quickly, so as to give the appearance of immediacy; but a clever physiologist will be able to trick the senses and tease apart the stages of construction.

Nevertheless, Helmholtz agreed that Kant's theory of the apriori forms of intuition was a clear and important point, but these forms had to be completely general, empty of content and free from all restrictions in order to accommodate any content whatsoever that could enter a particular form of perception.[105] Where Helmholtz parted company with Kant and his followers was in their assertion of a pure geometry which has its foundations, including its axioms, in a transcendental intuition having nothing to do with physical bodies and their movement. In this geometry it is possible, according to Kant and his followers, to have a transcendental intuition of the congruence and equality of geometrical figures unmediated by experience. Helmholtz emphatically rejected Kant's argument for a pure geometry given in intuition, and in doing so he criticized Kant for not being critical enough in the *Critique*: What Kant should have claimed was that the form of space can be transcendental without its axioms being so.

What did Helmholtz have in mind by this notion of space as transcendental in partial agreement with Kant? Helmholtz's theory of space as transcendental was set out in other sections of the essay on the facts of perception and in his papers on the origin and meaning of the geometrical axioms. In those papers Helmholtz makes it clear that what he had in mind was the general, abstract notion of space explored by the new field of "metamathematics." Unavailable in Kant's own day, the new metamathematics explored the logical consistency and logical foundations of spatial forms as a pure analytical science—Helmholtz characterized it sporadically as "scientific geometry" and "algebraic geometry" — careful to exclude any unwarranted inferences drawn from "constructive intuition" inadvertently entangling everyday experience with the necessities of thought. The new metamathematics, Helmholtz wrote, had been prepared by the work of Gauss, Lobachevsky, Beltrami, and others, but especially by Bernhard Riemann.[106]

The key to the new approach Helmholtz characterized as "scientific geometry" is the observation that all spatial relations are measurable; even the axioms make reference to magnitudes, such as a straight line being the *shortest* distance between two points. Riemann, in Helmholtz's view, had called attention to the possibility of treating geometry as a particular application of the more general analytical treatment of relations among measurable quantities, an approach "which obviates entirely the danger of habitual perceptions being taken for necessities of thought."[107] Thus, following Riemann, a system in which one individual, a point for instance, can be determined by *n* measurements is called an *n-fold* extended aggregate, or an aggregate of *n* -dimensions. Casting his own researches in these terms, Helmholtz explained, "Thus the space in which we live is a threefold, a surface is a twofold, and a line is a simple extended aggregate of points. Time also is an aggregate of one dimension. The system of colors is an aggregate of three dimensions, inasmuch as each color, according to the investigations of Thomas Young and Clerk Maxwell, may be represented as a mixture of three primary colors in definite quantities. In the same way, we may consider the system of tones as an aggregate of two dimensions, if we distinguish only pitch and intensity and leave out of account differences of timbre."[108] Thus space as considered by geometry is just one application of a general aggregate of three dimensions, and as Riemann showed, by adding the special conditions of free mobility of solid bodies without change of form in translations and rotations, and postulating the special value of zero as the measure of curvature, we arrive at Euclidean space as we know it.

The relevance of the work on n-fold aggregates extended well beyond questions of geometry for Helmholtz. I have argued that for Helmholtz, just as for Herbart before him, the work on n-fold aggregates was relevant to solving a fundamental problem of psychophysics; namely, how to construct the mental machinery for relationships that are necessary, valid, prior to all experience, and at the same time related to the world of outer sense. This was the problem of the relation of ideas and the schematisms of the categories to the forms of perception Kant had bequeathed his followers. Helmholtz was initially led to the power of n-fold aggregates in treating these problems through his effort to sort out the relationship of the mind and the sensory apparatus of the eye in color perception and 3-D vision.

> While Riemann entered upon this new field from the side of the most general and fundamental questions of analytic geometry, I myself arrived at similar conclusions, partly through seeking to represent in space the system of colors (involving the comparison of one threefold extended aggregate with another) and partly through inquiries into the origin of our ocular measure for distances in the field of vision. Riemann starts by assuming the above-mentioned algebraic

> expression, which represents the most general form the distance between two infinitely close points, and deduces therefrom the conditions of mobility of rigid figures. I, on the other hand, starting from the observed fact that the movement of rigid figures is possible in our space, with the degree of freedom that we know, deduce the necessity of the algebraic expression taken by Riemann as an axiom.[109]

From the perspective of Helmholtz's Kant-inspired empiricist theory of perception the organs of sense played a role exactly analogous to the specifications of rigid bodies, constancy under rotations and translations, and curvature provided in Riemann's geometry. The sensory organs are measuring devices that specify a given n-fold aggregate as color, tone, or geometrical surface. As Helmholtz commented, "Here our own body with its organs is the instrument we carry about in space. Now it is the hand, now the leg that serves for a compass, while the eye turning in all directions is our theodolite for measuing arcs and angles in the visual field."[110] Helmholtz does not name Grassmann or Herbart in commenting on his path to these ideas. But it is difficult not to see them in the picture, perhaps as I have argued even crucially so.

I have suggested that in understanding how Helmholtz arrived at these results we not view him as relentlessly pursuing a Kantian research program. Rather I have proposed that we see him as attempting to unravel the complex problems of color vision and hearing with the tools of experimental physiology and physics. In this process the dialog among his instruments and physical models of sensory apparatus provided the primary impetus, and in the course of working out mechanisms for sensory perception he was led both to dissent from the Kantian orthodoxy and operationalize many of the elements of outer sense Kant had taken to be given immediately in the act of perception. In working out these details Helmholtz's selection of telegraphic apparatus was particularly crucial. The telegraph was certainly more than a useful metaphor. Telegraphic apparatus and its modification in various ways to achieve the ends of experiment had a significance beyond the fact that these devices were readily available and familiar objects of investigation for Helmholtz. Telegraphic devices were not only important as means for constructing models and experiments; telegraphy embodied a system of signification which was central to Helmholtz's views about mental representations and their relationship to the world, the sign theory of perception. I have suggested that the adaptation of Grassmann's approach to the specific requirements of physiological modeling was crucial. Grassmann's proposal of a formalism which operated in terms of three quantifiable measures was important and timely because it was a system, based on the notion of n-dimensional manifolds (or n-fold aggregates as Riemann would refer to them), for constructing representations which operated similarly to the way messages were encoded in the telegraph. Viewed in this light the telegraphic system with which Helmholtz was familiar in his daily experience and upon which he and his friends worked intensively was important in suggesting a direction for the development of his own ideas about representation:

> Nerves have been often and not unsuitably compared to telegraph wires. Such a wire conducts one kind of electric current and no other; it may be stronger, it may be weaker, it may move in either direction; it has no other qualitative differences. Nevertheless, according to the different kinds of apparatus with which we provide its terminations, we can send telegraphic dispatches, ring bells, explode mines, decompose water, move magnets, magnetize iron, develop light, and so on. So with the nerves. The condition of excitement which can be produced in them, and is conducted by them, is, so far as it can be recognized in isolated fibers of a nerve, everywhere the same, but when it is brought to various parts of the brain, or the body, it produces motion, secretions of glands, increase and decrease of the quantity of blood, of redness and of warmth of individual organs, and also sensations of light, of hearing, and so forth. Supposing that every qualitatively different action is produced in an organ of a different kind, to which also separate fibers of nerve must proceed, then the actual process of irritation in individual nerves may always be precisely the same, just as the electrical current in the telegraph wires remains one and the same notwithstanding the various kinds of effects which it produces at its extremities. On the other hand, if we assume that the same fiber of a nerve is capable of conducting different kinds of sensation, we should have to assume that it admits of various kinds of processes of irritation, and this we have been hitherto unable to establish.

> In this respect then the view here proposed, like Young's hypothesis for the difference of colors, has still a wider signification for the physiology of the nerves in general.[111]

Telegraphic apparatus was central to Helmholtz's work in physiological acoustics. relied on the representation of the ear as a tuning-fork resonator. The juxtaposition and comparison of differences between media, between hearing and vision, was a positive resource for revisiting Young's hypothesis. Indeed, as the above passage implies, the analogy between the Young hypothesis for color vision and Helmholtz's model for hearing and the assimilability of both to the telegraph as apparently generalizable for sensory physiology seemed to provide further convincing support for the approach. In the years between 1858-1860 when Helmholtz returned to the elaboration of the Young hypothesis through work on after-images[112] and color blindness[113] culminating in his general presentation in the second part of the *Physiological Optics* (1860), the search for a material embodiment of his models once again served as a resource for advancing theory. In his earlier work on subjective color Helmholtz had used the center-of-gravity method as a tool for constructing a color chart. As noted above, in response to Grassmann Helmholtz argued that a graphic representation which embodied relationships based on actual measurements of brightness of complementary colors forming the most intense white light should have the shape of a hyperbola. But Helmholtz decided to go even further and map the space corresponding to the combination of complementary spectral colors the eye actually sees.[114] The crucial resources enabling Helmholtz (and independently Maxwell) to generate such a color chart were studies of color blindness. In these studies the methods employed in constructing the graphical representation for combinations of spectral colors were turned to graphing the colors perceived by color blind individuals, persons presumably suffering from the absence or malfunction of one of the three receptors. A color chart could be constructed for persons suffering the loss of the red receptors, for example, completely from mixtures of yellow and blue.[115] Helmholtz, as I have shown, took this line of work a step further in his experiments on after-images by demonstrating that colors more saturated than spectral red, green, or violet can be generated by fatiguing one or two of the hypothesized three receptors through intense exposure to spectral light of the complementary color. For Helmholtz the construction of the color chart as a graphical representation embodied the very principles used by the eye in encoding signals perceived as color in the brain. In his studies on tone sensation Helmholtz had constructed a mechanical simulacrum for advancing his theory. In the final stages of his work on color vision the graphic trace itself became both the material embodiment of theory and the source of its improvement. Through his work on physiological color and acoustics Helmholtz became convinced of the power of n-fold aggregates as a general abstract calculus the mind uses in processing sense data. That idea was even more powerfully confirmed through Helmholtz's work on eye movements, where the mapping of connected groups of local retinal signs by means of rotations around axes in Listing's plane served as the physiological analog in optical space played by the assumption of rigid bodies in Riemann's geometry. When understood as the abstract space of n-dimensional manifolds emptied of all content, Kant's notion of space as the transcendental form of intuition was a powerful stimulus both scientific geometry and to psychophysical research.

Works Cited

Boring, Edwin G. Sensation and Perception in the History of Experimental Psychology. New York: D. Appleton, 1942.

Couturat, Louis. La logique de Leibniz d'après des documents inédits. Paris: F. Alcan, 1901.
Crowe, Michael J. A History of Vector Analysis. South Bend, IN: University of Notre Dame Press, 1967.

Dirichlet, G. Lejeune. "Über die Reduction der positiven quadratischen Formen mit drei unbestimmten ganzen Zahlen." Journal für die reine und angewandte Mathematik 40 (1849): 209-27.

Drobisch, Moritz Wilhelm. Erste Grundlehren der mathematischen Psychologie. Leipzig: Leopold Voss, 1850.

Engel, Friedrich. "Grassmanns Leben." Hermann Grassmanns gesammelte mathematische und physikalische Werke. Ed. Friedrich Engel. Vol. 3. Leipzig: Sächsische Akademie der Wissenschaften zu Leipzig (Teubner), 1894-1911.

Fick, Adolf. "Die Bewegungen des menschlichen Augafpels." Zeitschrift für rationelle Medicin 4 (1854): 801.

Friedman, Michael. Kant and the Exact Sciences. Cambridge, MA: Harvard University Press, 1992.

Gauß, Carl Friedrich. "Review of Ludwig August Seeber's *Untersuchungen über die Eigenschaften der positiven ternären quadratischen Formen*, Freiburg im Breisgau, 1831 (originally published in the *Göttingische gelehrte Anzeigen*, 9 July, 1831)." Carl Friedrich Gauß: Werke. Vol. 2 (1876). Göttingen: Königlichen Gesellschaft der Wissenschaften zu Göttingen (W. F. Kaestner), 1870-. 188-96.

---. "Über ein neues allgemeines Grundgesetz der Mechanik." Carl Friedrich Gauss: Werke. Vol. 5 (1877). Göttingen: Königlichen Gessellschaft der Wissenschaften, Dieterichschen Universitätsdruckerei (W. F. Kaestner)], 1870-. 5-28.

Grassmann, Hermann. Die Wissenschaft der extensiven Grössen oder die Ausdehnungslehre, eine neue mathematische Disciplin. in *Hermann Grassmann's Gesammelte mathematische and physikalische Werke*, edited by Friedrich Engel, Teubner; Leipzig, 1894 (3 volumes): Vol 1, Part 1: 47-49 ed. Leipzig: Otto Wigand, 1844.

---. "Zur Theorie der Farbenmischung." Annalen der Physik and der Chemie 89 (1853): 69-84.
Hatfield, Gary. The Natural and the Normative: Theories of Spatial Perception from Kant to Helmholtz. Cambridge, MA: MIT Press, 1990.

Heath, A.E. "Hermann Grassmann. The Neglect of his Work. The Geometric Analysis and its Connection with Leibniz' Characteristic." The Monist 27 (1917): 1-56.

Helmholtz, Hermann. Die Lehre von den Tonempfindungen als physiologische Grundlage für die Theorie der Musik. Braunschweig: Vieweg, 1863.

---. "Die neuren Fortschritte in der Theorie des Sehens (1868)." Vorträge and Reden. 5th ed. Vol. 2. Braunschweig: F. Vieweg, 1903. 265-365.

---. "Die Tatsachen in der Wahrnehmung (1878)." Vorträge und Reden von Hermann Helmholtz, ed. 2 vols. Vol. 1, 85-117. 5th ed. Vol. 2. Braunschweig: F. Vieweg, 1903. 213-47; 387-406.

---. Handbuch der physiologischen Optik. 2nd ed. Hamburg and Leipzig: Leopold Voss, 1896.
---. "Über Combinationstöne." Annalen der Physik und Chemie 99 (1856): 526.

---. "Über Combinationstöne." Monatsbericht der Königlichen Akademie der Wissenschaften zu Berlin. (1856): 279-85.

---. "Über das Sehen des Menschen (1855)." Vorträge und Reden von Hermann Helmholtz. 5th ed. Vol. 1. Braunschweig: F. Vieweg, 1903. 85-117.

---. "Über den Ursprung und die Bedeutung der geometrischen Axiome (1870)." Vorträge und Reden von Hermann Helmholtz. 5th ed. Vol. 2. Braunschweig: F. Vieweg, 1903. 1-31; 381-83.

---. "Über die Methoden, kleinste Zeittheile zu messen, and ihre Anwendung fur physiologische Zwecke." Wissenschaftliche Abhandlungen. Vol. 2 (1883). Leipzig: Barth, 1882-1895. 862-80.

---. "Über die normalen Bewegungen des menschlichen Auges." Archiv für Ophthalmologie 9 (1863): 153-214.

---. "Über die physiologischen Ursachen der musikalischen Harmonie (1857)." Vorträge und Reden von Hermann Helmholtz. 5th ed. Vol. 1. Braunschweig: F. Vieweg, 1903. 119-55.

---. "Über die subjectiven Nachbilder im Auge (1858)." Wissenschaftliche Abhandlungen. Vol. 3 (1895), 1882-1895. 13-15.

---. "Über die Theorie der zusammengesetzten Farben." Annalen der Physik and der Chemie 87 (1852): 45-66.

---. "Über die Theorie der zusammengesetzten Farben." Wissenschaftliche Abhandlungen. Vol. 2. Leipzig: Barth, 1883. 1-23.

---. "Über die Zusammensetzung von Spectralfarben." Annalen der Physik and der Chemie 89 (1855): 1-28.

---. "Über Farbenblindheit (1859), in , Vol. , pp." Wissenschaftliche Abhandlungen. Vol. 2 (1883), 1882-1895. 346-49.

---. "Über Farbenblindheit (1859)." Wissenschaftliche Abhandlungen. Vol. 2. Braunschweig: F. Vieweg, 1903. 346-49.

Helmholtz, Hermann von. Epistemological Writings: The Paul Hertz/Moritz Schlick Centenary Edition of 1921 with Notes and Commentary by the Editors. Trans. edited Malcolm F. Lowe, with an introdion and bibliography, by Robert S. Cohen and Yehuda Elkana. Dordrecht, Holland: D. Reidel Pub. Co., 1977.

Herbart, Johann Friedrich. Lehrbuch zur Einleitung in die Philosophie. Sämmtliche Werke. Ed. Gustav Hartenstein. Vol. 1. 6 vols. Leipzig: Leopold Voss, 1850-1851.

---. Sämmtliche Werke. Ed. Gustav Hartenstein. Vol. 1. 6 vols. Leipzig: Leopold Voss, 1850-1851.

Jacobi, C.G.J. "Über die Reduction der quadratischen Formen auf die kleinste Anzahl der Glieder." Journal für die reine und angewandte Mathematik 39 (1848): 290-92.

Jacobi, C.G.J. "Über die Reduction der quadratischen Formen auf die kleinste Anzahl der Glieder." C.G.J. Jacobi's gesammelte Werke. Ed. Karl Weierstrass. Vol. 6. Berlin: Königliche Preußische Akademie der Wissenschaften (Georg Reimer), 1881-1891. 318-21.

Kahl, Russell, ed. Selected Writings of Hermann von Helmholtz. Middletown, Connecticut: Wesleyan University Press, 1971.

Kant, Immanuel. Critique of Pure Reason. Trans. Norman Kemp Smith. New York: St. Martins, 1929.

Koenigsberger, Leo. Herman von Helmholtz. 3 vols. Braunschweig: Friedrich Vieweg und Sohn, 1902-1903.

Kremer, Richard L. "Innovation through Synthesis: Helmholtz and Color Research." Hermann von Helmholtz and the Foundations of Nineteenth Century Science. Ed. David Cahan. Berkeley: University of California Press, 1993. 205-58.

Kronecker, Leopold, and Leopold Fuchs, eds. G. Lejeune Dirichlet's Werke. Vol. 2. 2 vols. Berlin: Königliche Preußische Akademie der Wissenschaften (Georg Reimer), 1889-1897.

Lenoir, Timothy. "The Eye as Mathematician: Clinical Practice, Instrumentation, and helmholtz's Construction of an Empiricist Theory of Vision." Hermann Helmholtz: Philosopher and Scientist. Ed. David Cahan. Berkeley: University of California Press, 1992. 109-53.

Loemker, Leroy, ed. Gottfried Wilhelm Leibniz, Philosopical Papers and Lettters. Chicago: University of Chicago Press, 1956.

McKendrick, John G. Hermann Ludwig Ferdinand von Helmholtz. New York: Longmans, 1899.
Millington, E.C. "History of the Young-Helmholtz Theory of Colour Vision." Annals of Science 5 (1941): 167-76.

Reute, C.G. Theodor. "Das Ophthalmotrop, dessen Bau und Gebrauch." Göttinger Studien 1 (1845): 128-29.

---. Ein neues Opthalmotrop. Leipzig: Otto Wigand, 1857.

Riemann, Bernhard. "Über die Hypothesen, welche der Geometrie zu Grunde liegen (1854)." Bernhard Riemann's gesammelte mathematische Werke und wissenschaftlicher Nachlaß. Ed. Heinrich Weber. Leipzig: Teubner, 1876.

Seebeck, August. "Bemerkungen über Resonanz und über Helligkeit der Farben im Spectrum." Annalen der Physik und Chemie 62.viii (1844): 571-76.

Sherman, Paul. Colour Vision in the Nineteenth Century: The Young-Helmholtz-Maxwell Theory. Bristol: Adam Hilger Ltd., 1981.

Turner, R. Steven. In the Mind's Eye: Vision and the Helmholtz-Hering Controversy. Princeton: Princeton University Press, 1994.

Ulyenbroek, ed. Christi. Hugenii aliorumque seculi XVII. virorum celebrium exercitationes mathematicae et philosophiae.: Hagae comitum, 1833.

Vogel, Stephan. "Sensation of Tone, Perception of Sound, and Empiricism." Hermann von Helmholtz and the Foundations of Nineteenth Century Science. Ed. David Cahan. Berkeley: University of California Press, 1993. 259-87.

Weber, Heinrich, ed. Bernhard Riemann's gesammelte mathematische Werke und wissenschaftlicher Nachlaß. Leipzig: Teubner, 1876.

Wundt, Wilhelm. "Beschreibung eines künstlichen Augenmuskelsystems zur Untersuchung der Bewegungsgesetze des menschlichen Auges im gesunden und kranken Zustanden: Part I." Archiv für Ophthalmologie 8 (1862): 88-114.

---. "Über die Bewegung der Augen: Part I." Archiv für Ophthalmologie 8 (1862): 1-87.

Equation insert for note 95:

In an accompanying appendix to the paper, Helmholtz went on to give a mathematical derivation of this proposition. He developed an expression for a small rotation around the atropic line as a function of the angles of abduction **α**, elevation **θ**, and cyclorotation **ω**, in the primary coordinate axes of the eye, postulating as a condition for motion that **ω** be a single-valued function of **α** and **θ**. He showed that when **α**, **θ**, and **ω** all undergo a small displacement, the resultant rotation **ρ** around the atropic line constructed in terms of the composition of small rotations can be represented as:

$$\rho = d\omega + \cos\alpha\theta = \frac{d\omega}{d\alpha}d\alpha + (\frac{d\omega}{d\theta} + \cos d\alpha)d\theta.$$

The sums of the squares of these magnitudes for all infinitely small displacements, **ds** of the atropic line over the entire visual field were supposed to be made a minimum. Carrying out this complex calculation over the next five pages of the article confirmed the result that a minumum value for this equation would indeed produce a rotation obeying Listing's Law.

Endnotes

[1] Herman Helmholtz, "Über das Sehen des Menschen, "Über das Sehen des Menschen (1855)." Vorträge Und Reden Von Hermann Helmholtz. 5th ed. Vol. 1. Braunschweig: F. Vieweg, 1903. 85-117. p. 116.

[2] Timothy Lenoir, "The Eye as Mathematician: Clinical Practice, Instrumentation, and helmholtz's Construction of an Empiricist Theory of Vision." Hermann Helmholtz: Philosopher and Scientist. Ed. David Cahan. Berkeley: University of California Press, 1992. 109-53.

[3] Hermann Hehnholtz, Handbuch der physiologischen Optik. 2nd ed. Hamburg and Leipzig: Leopold Voss, 1896, p. 443.

[4] *Ibid*.

[5] *Ibid*., p. 446.

[6] Hermann Helmholtz. "Die neuren Fortschritte in der Theorie des Sehens (1868)." Vorträge and Reden. 5th ed. Vol. 2. Braunschweig: F. Vieweg, 1903. 265-365, translated in Russell Kahl, ed*.,* Selected Writings of Hermann von Helmholtz. Middletown, Connecticut: Wesleyan University Press, 1971, pp. 144-222, especially pp. 185-186.

[7] *Ibid*., p. 187

[8] This point is a source of confusion for some treatments of Helmholtz's theory of sensation. The theory of color mixture is presented in Book II of the *Physiological Optics* in which the physiological components of vision are treated. We discover in Book III devoted to psychological factors in vision, however, that pure sensations of color unmixed with phenomena of contrast and compensatory judgements concerning expected states of affairs can occur only through careful experimental arrangements and with trained observers

[9] See "Die neueren Fortschritte in der Theorie des Sehens," in Kahl, Selected Writings, p. 192.

[10] Helmholtz concluded his lecture on the progress in vision research with the following statement concerning the correspondence between perceptions and the external world:

> Beyond these limits-for example, in the region of qualities-we can prove conclusively that in some instances there is no correspondence at all between sensations and their objects. Only the relations of time, space, and equality and those which are derived from them (number, size, and regularity of coexistence and of sequence)-mathematical relations, in short-are common to the outer and the inner worlds. Here we may indeed anticipate a complete correspondence between our conceptions and the objects which excite them. Quoted from Kahl, Selected Writings, p. 222.

[11] See Hermann Helmholtz, "Über die Methoden, kleinste Zeittheile zu messen, and ihre Anwendung fur physiologische Zwecke." Wissenschaftliche Abhandlungen. Vol. 2 (1883). Leipzig: Barth, 1882-1895. 862-80, especially p. 873.

[12] See "Die neueren Fortschritte in der Theorie des Sehens," Kahl, Selected Writings, p. 181

[13] Johann Friedrich Herbart, Sämmtliche Werke. Ed. Gustav Hartenstein. 6 vols. Leipzig: Leopold Voss, 1850-1851.

[14]Johann Friedrich Herbart, Lehrbuch zur Einleitung in die Philosophie. Sämmtliche Werke. Ed. Gustav Hartenstein. Vol. 1. 6 vols. Leipzig: Leopold Voss, 1850-1851: 258

> Kant lost sight of the true concept of causality. His concept of causality, as the rule for a temporal succession, belonged completely to the world of phenomena. And so must it be, if the succession of events is supposed to proceed directly out of the causal relation. But it need not happen this way if any particular occurrence, such as the origin of perceptions within us or free action, is to be deduced from an intelligible cause [i.e. a Ding an sich in Kant's sense]. The concept of causality cannot be restricted to phenomena. It remains indispensable for real--and especially for mental--passivity and activity [Leiden und Thun].

---

[15] Kant himself explicitly acknowledged this problem in his *pre*-critical philosophy in a letter to Marcus Herz of February 21, 1772:

> "In my dissertation I had contented myself with expressing the nature of intellectual representations merely negatively: namely, that they were not modifications of the soul by means of the object. But how then a representation is possible that could otherwise relate to an object without being in any way affected by it I passed over in silence. I had said: the sensible representations represent the things as they appear, the intellectual as they are. But through what are these things then given to us if they are not[so given] through the mode in which we are affected…" Quoted by Michael Friedman, Kant and the Exact Sciences. Cambridge, MA: Harvard University Press, 1992. p. 35, n. 58.

Herbart and his contemporaries felt this problem still remained even in Kant's writings of the critical period.

[16] Michael Friedman, Kant and the Exact Sciences. Cambridge, MA: Harvard University Press, 1992. See especially Introduction, pp. 1-52.

[17] *Ibid*. especially pp. 25-27.

[18] See Friedman, pp. 56-66 for a crystal clear discussion of this issue.

[19] Immanuel Kant, Critique of Pure Reason. Trans. Norman Kemp Smith. New York: St. Martins, 1929. B 203-204. Kant discussed this application of the understanding to sensibility elsewhere as follows:

> We can think no line without drawing it in thought, no circle without describing it. We can absolutely not represent the three dimensions of space without setting three lines at right angles to one another from the same point. And even time we cannot represent without attending in the drawing of a straight line (which is to be the outer figurative representation of time) merely to the action of synthesis of the manifold, by which we successively determine inner sense--and thereby to the succession of this determination in it. (B154-155)

[20] Thus in the Critique of Pure Reason, A25, Kant wrote:

> Space is represented as an infinite given magnitude. A general concept of space (which is common to both a foot and an ell alike) can determine nothing in regard to magnitude. Were there no limitlessness in the progression of intuition, no concept of relations could by itself, supply a principle of their infinitude.

[21] Herbart, Lehrbuch zur Einleitung in die Philosophie. Sämmtliche Werke. Ed. Gustav Hartenstein. Vol. 1. 6 vols. Leipzig: Leopold Voss, 1850-1851: 310.

[22]Herbart, Schriften zur Metaphysik, Part 2: Allgemeine Metaphysik nebst den Anfängen der philosophischen Naturlehre, in Sämmtliche Werke*,* Vol 4 (1851): 165-166. This text was written by Herbart in 1828 for his lectures in Göttingen.

[23]Herbart's discussion of lines and planes is in *ibid*., p. 187 ff.

[24]*Ibid.,* p. 118.

[25]*Ibid.*, p. 312. In a similar fashion a few pages earlier (p. 306) Herbart characterized the space in which real objects are placed as "the symbol of the possible community of things standing in causal relationship."

[26] Herbart establishes the straight line as the shortest path between two points by showing that by mentally tracing any other path between AC, such as AβC, we would have to make at least one superfluous motion. He assumes that AC is the hypotenuse of the triangle AβC; for if AC is the hypotenuse composed of directions Aβ, βC, and if each of these directions can be similarly conceived as compounded of other directions; then to get from A to C he imagines a perpendicular βf to AC, so that

$$A\beta = A\Phi + \Phi\beta,\ \ \beta C = \beta\Phi + \Phi C.\ \ \text{And } A\Phi + \Phi C = A\Phi + \Phi\beta + \Phi\beta + \Phi C.$$

But the path along the hypotenuse, $AC = A\Phi + \Phi C$. Hence, the path via β involves two extra motions, Φβ and βΦ.

[27]*Ibid.,* pp. 203-205.

[28]*Ibid.,* p. 204.

[29]Schriften zur Metaphysik, Part 2: Allgemeine Metaphysik nebst den Anfängen der philosophischen Naturlehre, in Sämmtliche Werke*,* Vol 4 (1851): pp. 212-218.

[30]*Ibid.*, p. 248 ff.

[31]*Ibid.,* pp. 228-231, pp. 249-250, and see especially pp. 252-253.

[32] Helmholtz's mentor, Johannes Müller, writing in the 1830s rejected a similar constructivist approach set forth by Steinbuch. Herbart noted that the problem was even more complex than might at first glance be imagined; for it was necessary that "thousands or millions of perceptions which are simultaneously in consciousness have to mutually inhibit one another and become stabilized....Instead of two or three tones...we have to consider a multitude of simple sensations which are infinitely near to one another and flow together [in the same unified experience]." Johann Freidrich Herbart, Psychologie als Wissenschaft, Part 1, in Sämmtliche Werke, Vol 5 (1850): 349-350.

[33]See Herbart, Psychologie als Wissenschaft, Part 1: 479-480.

[34]See Herbart, Psychologie als Wissenschaft, Part 2, in Sämmtliche Werke, Vol. 6: 120-121.

[35]Herbart, Psychologie als Wissenschaft, Part 1, in Sämmtliche Werke*,* Vol. 5: 322.

[36]*Ibid.,* 323-324. See also especially Herbart, *Lehrbuch zur Psychologie* (1834), in *Sämmtliche Werke*, Vol. 5: 57. See especially, p. 489 and 489n.

[37]Herbart, *Psychologie als Wissenschaft*, Part 1: 489.

[38]*Ibid.*, pp. 489-490.

[39]*Ibid.,* pp. 118-119.

[40]*Ibid.*, p. 126.

[41]*Ibid.,* p. 132.

[42]*Loc. cit.*

[43]See Moritz Wilhelm Drobisch, Erste Grundlehren der mathematischen Psychologie. Leipzig: Leopold Voss, 1850. iv-vii, and xii.

[44]See Bernhard Riemann, "Über die Hypothesen, welche der Geometrie zu Grunde liegen (1854)." Bernhard Riemann's gesammelte mathematische Werke und wissenschaftlicher Nachlaß. Ed. Heinrich Weber. Leipzig: Teubner, 1876. 255. Also see the draft of an uncompleted work entitled, "Versuch einer Lehre von den Grundbegriffen der Mathematik und Physik als Grundlage für die Naturerklärung," *Ibid.,* pp. 489-506, especially p. 490, where the goals of the program are related to Herbart.

[45]Helmholtz wrote that Herbart had initiated the task of providing the philosophical foundations for revising Kant and rejecting the nativist thesis in sensory physiology. See Handbuch der physiologischen Optik, Leopold Voss; Leipzig, 1867: pp. 496 and 595.

[46]Perhaps the most direct statement made by Gauß on this subject was in a review published in the *Göttingische gelehrte Anzeigen,* 9 July, 1831, of Ludwig August Seeber's Untersuchungen über die Eigenschaften der positiven ternären quadratischen Formen*,* Freiburg im Breisgau, 1831. In criticizing the prolixity of Seeber's methods, Gauß mentioned that one could arrive at some extremely powerful results which go well beyond problems in higher arithmetic by giving ternary quadratic forms a geometrical representation. Through this system points, lines, surfaces and solids are represented in spaces constructed from parallelograms and paralleopipedal solids and all the operations using determinants provide the means for solving problems of geometrical constructions. Gauß concluded his brief illustration of the methods with the statement that "It is at least possible to recognize what a rich field these researches could open up which would not only be of interest for high theory but they would also lead to convenient as well as general treatments of all the relations in which for example crystal forms can be used." Carl Friedrich Gauß, "Review of Ludwig August Seeber's Untersuchungen über die Eigenschaften der positiven ternären quadratischen Formen, Freiburg im Breisgau, 1831 (originally published in the Göttingische gelehrte Anzeigen, 9

July, 1831)." Carl Friedrich Gauß: Werke. Vol. 2 (1876). Göttingen: Königlichen Gesellschaft der Wissenschaften zu Göttingen (W. F. Kaestner), 1870-. 188-96, especially pp. 194-196.

[47]Two persons who took up Gauß's suggestive ideas and attempted to develop them further were C.G. Jacobi and G. Lejeune Dirichlet. Both men published papers in 1848/49 on quadratic forms and the powerful results which could be obtained from treating them as the structure for constructing a geometrical system of representation. See G. Lejeune Dirichlet, "Über die Reduction der positiven quadratischen Formen mit drei unbestimmten ganzen Zahlen." Journal für die reine und angewandte Mathematik 40 (1849): 209-27, reprinted in Leopold Kronecker and Leopold Fuchs, eds. G. Lejeune Dirichlet's Werke. Vol. 2. 2 vols. Berlin: Königliche Preußische Akademie der Wissenschaften (Georg Reimer), 1889-1897. 29-48. Also see C.G.J Jacobi, "Über die Reduction der quadratischen Formen auf die kleinste Anzahl der Glieder." Journal für die reine und angewandte Mathematik 39 (1848): 290-292, reprinted in C.G.J. Jacobi's gesammelte Werke. Ed. Karl Weierstrass. Vol. 6. Berlin: Königliche Preußische Akademie der Wissenschaften (Georg Reimer), 1881-1891. 318-21.

[48] Ulyenbroek, ed. Christi. Hugenii aliorumque seculi XVII. virorum celebrium exercitationes mathematicae et philosophiae: Hagae comitum, 1833.

[49]Christi. Hugenii aliorumque seculi XVII. virorum celebrium exercitationes mathematicae et philosophicae, fasc. II: 6. Quoted from Michael J. Crowe, A History of Vector Analysis. South Bend, IN: University of Notre Dame Press, 1967: 3-4, translation from Leroy Loemker, ed. Gottfried Wilhelm Leibniz, Philosopical Papers and Lettters. Chicago: University of Chicago Press, 1956. 384-385.

[50] *Ibid*.

[51]This episode is recounted from the archival sources of the Jablonskischen Gesellschaft der Wissenschaftenby Friedrich Engel in his biography of Hermann Grassmann, see: Friedrich Engel, "Grassmanns Leben." Hermann Grassmanns gesammelte mathematische und physikalische Werke. Ed. Friedrich Engel. Vol. 3. Leipzig: Sächsische Akademie der Wissenschaften zu Leipzig (Teubner), 1894-1911. Vol. III, 2: 110.

[52]Leipziger Zeitung, 9 March 1844: 877. The reference to the Göttinger Anzeigen concerned a review of the Uylenbroek edition and a discussion of Leibniz's characteristic.

[53]The most thorough treatment of Grassmann's work and the problems of its reception is in Michael J. Crowe, A History of Vector Analysis. 54-96. The relationship between Leibniz' analysis in situ and Grassmann's work is also discussed by Louis Couturat, La logique de Leibniz d'après des documents inédits. Paris: F. Alcan, 1901. Also see A.E. Heath, "Hermann Grassmann. The Neglect of his Work. The Geometric Analysis and its Connection with Leibniz' Characteristic." The Monist 27 (1917): 1-56.

[54]Quoted from Friedrich Engel, "Grassmanns Leben," in Werke 3: 113-114

[55]Quoted by Engel, *Ibid.,* p. 101.

[56]*Ibid.*, p. 101.

[57] For outstanding treatments of Helmholtz's developing theory of color vision see R. Steven Turner, In the Mind's Eye: Vision and the Helmholtz-Hering Controversy. Princeton: Princeton University Press, 1994. especially pp. 95-113; Richard L. Kremer, "Innovation through Synthesis: Helmholtz and Color Research." Hermann von Helmholtz and the Foundations of Nineteenth Century Science. Ed. David Cahan. Berkeley: University of California Press, 1993. 205-58. My account here differs from Turner's and Kremer's in emphasizing the controversy with Grassmann as a source for stimulating Helmholtz's interest in the theory of manifolds and in applying these ideas to geometry and the construction of sensory space. In my view, Herbart's emprirical psychology provided a blueprint for much of Helmholtz's approach to physiological psychology; while the encounter with Grassmann stimulated Helmholtz to pursue concrete strategies for implementing it.

For earlier accounts of Helmholtz's color theory see John G. McKendrick, Hermann Ludwig Ferdinand von Helmholtz. New York: Longmans, 1899; Leo Koenigsberger, Herman von Helmholtz, 3 vols. Braunschweig: Friedrich Vieweg und Sohn, 1902-03; E.C. Millington, "History of the Young-Helmholtz Theory of Colour Vision," *Annals of Science* 5 (1941). 167-76; Edwin G. Boring, *Sensation and Perception in the History of Experimental Psychology*, New York: D. Appleton, 1942; Paul Sherman, *Colour Vision in the Nineteenth Century: The Young-Helmholtz-Maxwell Theory*, Bristol: Adam Hilger Ltd., 1981.

[58] In their commentary on the epistemological writings of Helmholtz, Schlick and Hertz create the impression that Helmholtz, like everyone else at the time, did not appreciate the work of Grassmann. This must be qualified. The methods of the Ausdehnungstehre may not have been applied, but they could not have gone unnoticed, at least not by Helmholtz. Grassmann made such direct reference to the basic concepts of the methods in his paper on color mixtures that it would have been impossible for anyone who read that paper not to have been acquainted with the general strategy of Grassmann’s abstract approach. See the note of Schlick and Hertz in Helmholtz, Hermann von. Epistemological Writings: The Paul Hertz/Moritz Schlick Centenary Edition of 1921 with Notes and Commentary by the Editors. Translated and edited by Malcolm F. Lowe, with an introduction and bibliography, by Robert S. Cohen and Yehuda Elkana. Dordrecht, Holland: D. Reidel Pub. Co., 1977. 112, note 27.

[59] Hermann Grassmann, Die Wissenschaft der extensiven Grössen oder die Ausdehnungslehre, eine neue mathematische Disciplin. in Hermann Grassmann's Gesammelte mathematische and physikalische Werke, edited by Friedrich Engel, Teubner; Leipzig, 1894 (3 volumes): Vol 1, Part 1: 47-49 ed. Leipzig: Otto Wigand, 1844. 47-49.

[60] *Ibid*., 52.

[61] *Ibid*., pp. 12-13.

[62] Hermann Helmholtz, "Über die Theorie der zusammengesetzten Farben," Annalen der Physik and der Chemie 87 (1852). 45-66, reprinted in Wissenschaftliche Abhandlungen. Vol. 2. Leipzig: Barth, 1883. 1-23.

[63] *Ibid*., pp. 15-16.

[64] *Ibid*., p. 17.

[65] *Ibid*., pp. 12-13.

[66] *Ibid*., p. 21.

[67] Hermann Grassmann, "Zur Theorie der Farbenmischung," Annalen der Physik and der Chemie 89 (1853). 69-84, reprinted in Gesammelte Werke, Vol. 11, 2. 161-173. See especially pp. 161-162.

[68] *Ibid*., pp. 167-168

[69] *Ibid*., p. 168

[70] *Ibid*., pp. 169-170.

[71] Helmholtz, "Über die Theorie der zusammengestzten Farben," Wissenschaftliche Abhandlungen, Vol. 2. 13-14.

[72] Helmholtz, "Über die Zusammensetzung von Spectralfarben." Annalen der Physik and der Chemie 89 (1855): 1-28.Wissenschaftliche Abhandlungen, Vol. 2. 51, and Physiologischen Optik. 277.

[73] *Ibid*., p. 64. PO. p. 288.

[74] In his discussion of Grassmann's center of gravity method in the Physiologische Optik, Hehnholtz noted that according to the conditions of the method one begins with three quantities of three colors, none of which is a mixture of the others, and assigns them positions on the color plane such that they do not all lie in a straight line. The position of two of the colors on the plane must always remain arbitrary, so that four conditions remain which determine the form of the curve of the color space. One might demand, for instance-as Grassmann had done in his construction of the color table-that five primary colors should be placed equidistant from white. In this case the perimeter of the color space would be a circle except for the portion connecting red and violet, which would have to be a straight line as in Figure 8 below. See Handbuch der physiologischen Optik, p. 287.

[75] "Über die Zusammensetzung von Spectralfarben," p. 61. P0, pp. 288-289. If a thin object, such as a pencil or a wooden dowel, were placed in front of the screen upon which he projected his color mixtures, shadows were cast in each of the two colors entering the mixture. If the field of white were composed from violet and yellow-green, the violet shadow came out dark but sharply defined, whereas the yellow-green shadow appeared weak, distinguishable almost only by its color and not at all by its brightness. In a mixture of red and green-blue, the red was very weak in comparison; whereas in a mixture of orange and cyan blue, both shadows appeared equally bright.

---

[76] Helmholtz, "Über die Zusammensetzung von Spectralfarben," Wissenschaftliche Abhandlungen, Vol. 2. p. 70.

[77] For an excellent analysis of Helmholtz's physiological acoustics and its epistemological relevance, see Sephan Vogel, Vogel, Stephan. "Sensation of Tone, Perception of Sound, and Empiricism." Hermann von Helmholtz and the Foundations of Nineteenth Century Science. Ed. David Cahan. Berkeley: University of California Press, 1993. 259-87.

[78] "Die neueren Fortschritte in der Theorie des Sehens," in Kahl, ed., p. 181.

> I have myself subsequently found a similar hypothesis very convenient and well suited to explain in a most simple manner certain peculiarities which have been observed in the perception of musical tones, peculiarities as enigmatic as those we have been considering in the eye. In the cochlea of the internal ear, the ends of the nerve fibers, which lie spread out regularly side by side, are provided with minute elastic appendages (the rods of Corti) arranged like the keys and hammers of a piano. My hypothesis is that each of these separate nerve fibers is constructed so as to be sensitive to a definite tone, to which its elastic fiber vibrates in perfect consonance. This is not the place to describe the special characteristics of our sensations of musical tones which led me to frame this hypothesis. Its analogy with Young's theory of colors is obvious, and it explains the origin of overtones, the perception of the quality of sounds, the difference between consonance and dissonance, the formation of the musical scale, and other acoustic phenomena by as simple a principle as that of Young.

[79] "Über Combinationstöne," Annalen der Physik und Chemie. 99 (1856). 526, in Wissenschaftliche Abhandlungen. Vol. I. 290; also see, Helmholtz, "Über Combinationstöne," Monatsbericht der Königlichen Akademie der Wissenschaften zu Berlin, 1856. 279-285, in Wissenschaftlichen Abhandlungen. Vol I. 256-262, especially p. 257:

> In analogy to the primary colors of the spectrum we intend to call such tones simple tones in contrast to the compound tones of musical instruments, which are actually accords with a dominant fundamental tone.

[80] August Seebeck, "Bemerkungen über Resonanz und über Helligkeit der Farben im spectrum," Annalen der Physik und Chemie. 62. no. viii (1844). 571-576. This paper was a response to a paper by Melloni translated and included in the Annalen der Physik und Chemie. 56 (1842). 574-587, entitled "Beobachtung über die Färbung der Netzhaut und der Krystall-Linse." Melloni proposed treating the sensation of color as a resonance phenomenon analagous to acoustical resonance.

[81] Hermann Helmholtz, Die Lehre von den Tonempfindungen als physiologische Grundlage für die Theorie der Musik. Braunschweig: Vieweg, 1863. 35-36.

[82] Tonempfindung, p. 55.

[83] Hermann Helmholtz, "Über die physiologischen Ursachen der musikalischen Harmonie (1857)." Vorträge und Reden von Hermann Helmholtz. 5th ed. Vol. 1. Braunschweig: F. Vieweg, 1903. 119-55, in Russell Kahl, ed. Selected Writings of Hermann von Helmholtz. Middletown, Connecticut: Wesleyan University Press, 1971. 85

[84] *Ibid* .

[85] Tonempfindung, p. 20.

[86] *Ibid*., p. 126.

[87] Tonempfindung, p. 126. Italicized in original.

[88] *Ibid*., p. 128. Italics in the original.

[89] *Ibid*., p. 148.

[90] Handbuch der physiologischen Optik, p. 369.

[91] *Ibid*., p. 320.

[92] *Ibid*., pp. 292-293.

[93] *Ibid*., p. 367.

---

[94] *Ibid*., p. 370. Emphasis in the original.

[95] Adolf Fick, "Die Bewegungen des menschlichen Augafpels," Zeitschrift für rationelle Medicin. 4 (1854). 120.

[96] C.G. Theodor Reute. "Das Ophthalmotrop, dessen Bau und Gebrauch." Göttinger Studien 1 (1845): 128-29. The improved version appeared in C.G. Theodor Reute. Ein neues Opthalmotrop. Leipzig: Otto Wigand, 1857. Helmholtz discussed both the work of Reute and Wundt in his Handbuch der physiologischen Optik (1st edition), part III, section 27, p. 526.

[97] Wilhelm Wundt, "Beschreibung eines künstlichen Augenmuskelsystems zur Untersuchung der Bewegungsgesetze des menschlichen Auges im gesunden und kranken Zustanden: Part II." Archiv für Ophthalmologie 8 (1862): 88-114.

[98] Wilhelm Wundt, "Über die Bewegung der Augen: Part I." Archiv für Ophthalmologie 8 (1862): 1-87.

[99] Carl Friedrich Gauss. "Über ein neues allgemeines Grundgesetz der Mechanik." Carl Friedrich Gauss: Werke. Vol. 5 (1877). Göttingen: Königlichen Gessellschaft der Wissenschaften, Dieterichschen Universitätsdruckerei (W. F. Kaestner)], 1870-. 5-28, quoted from p. 28.

[100] Hermann Helmholtz. "Über die normalen Bewegungen des menschlichen Auges." Archiv für Ophthalmologie 9 (1863): 153-214, quoted from p. 160.

[101] Ibid., p. 169. Italicized in the original.

[102] Ibid., p. 169. In an accompanying appendix to the paper, Helmholtz went on to give a mathematical derivation of this proposition. He developed an expression for a small rotation around the atropic line as a function of the angles of abduction $\boldsymbol{\alpha}$, elevation $\boldsymbol{\theta}$, and cyclorotation $\boldsymbol{\omega}$, in the primary coordinate axes of the eye, postulating as a condition for motion that $\boldsymbol{\omega}$ be a single-valued function of $\boldsymbol{\alpha}$ and $\boldsymbol{\theta}$. He showed that when $\boldsymbol{\alpha}$, $\boldsymbol{\theta}$, and $\boldsymbol{\omega}$ all undergo a small displacement, the resultant rotation $\boldsymbol{\rho}$ around the atropic line constructed in terms of the composition of small rotations can be represented as:

[insert equation here nb: microsoft word doesn't allow images in endnotes]
The sums of the squares of these magnitudes for all infinitely small displacements, **ds** of the atropic line over the entire visual field were supposed to be made a minimum. Carrying out this complex calculation over the next five pages of the article confirmed the result that a minumum value for this equation would indeed produce a rotation obeying Listing's law.

[103] Hermann Helmholtz. "Die Tatsachen in der Wahrnehmung (1878)." Vorträge und Reden von Hermann Helmholtz. 2 vols. Vol. 2. Braunschweig: F. Vieweg, 1903. 213-47, quoted from p. 244, my translation. Kahl's translation in "The Facts of Perception," Selected Writings of Hermann von Helmholtz. Middletown, Connecticut: Wesleyan University Press, 1971. 390-91, is a bit more generous.

[104] Ibid., pp. 379-380.

[105] Ibid., p. 408.

[106] Helmholtz, “On the Origin and Meaning of Geometric Axioms,” in Kahl, ed., Selected Writings of Hermann von Helmholtz. 253.

[107] Ibid., p. 253.

[108] Ibid., pp. 253-254.

[109] Ibid., pp. 255-56.

[110] Ibid., p. 259.

[111] Tonempfindung, p. 149.

[112] Hermann Helmholtz. "Über die subjectiven Nachbilder im Auge (1858)." Wissenschaftliche Abhandlungen. Vol. 3 (1895), 1882-1895. 13-15.

[113] Hermann Helmholtz. "Über Farbenblindheit (1859)," in Wissenschaftliche Abhandlungen. Vol. 2 (1883), 1882-1895. 346-49.

[114] Paul D. Sherman has observed that it is surprising Helmholtz did not immediately perform the calculations necessary to produce such an "actual" color chart. James Clerk Maxwell was the first person to undertake the construction of a color chart representation based on quantitative data. See Paul D. Sherman. Colour Vision in the Nineteenth Century: The Young-Helmholtz-Maxwell Theory. Bristol: Adam Hilger Ltd., 1981.114-115; 153-183.

[115] Helmholtz. "Über Farbenblindheit," Wissenschaftliche Abhandlungen. Vol. 2. 348.